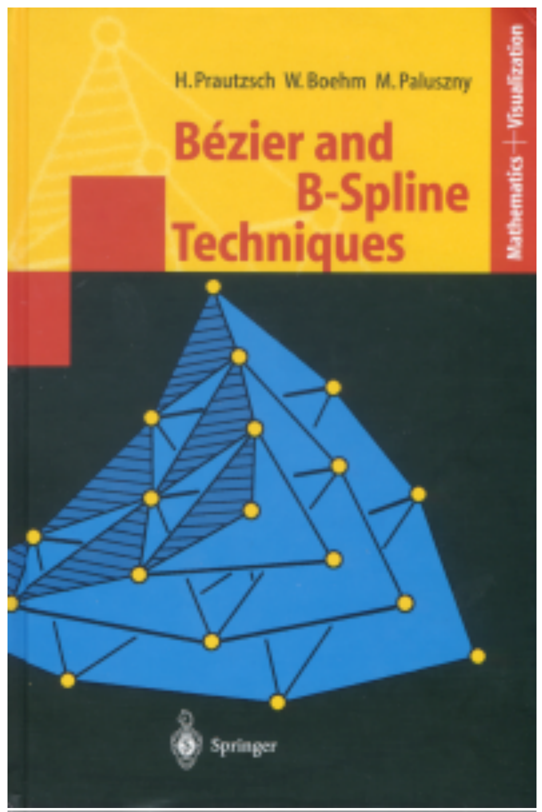
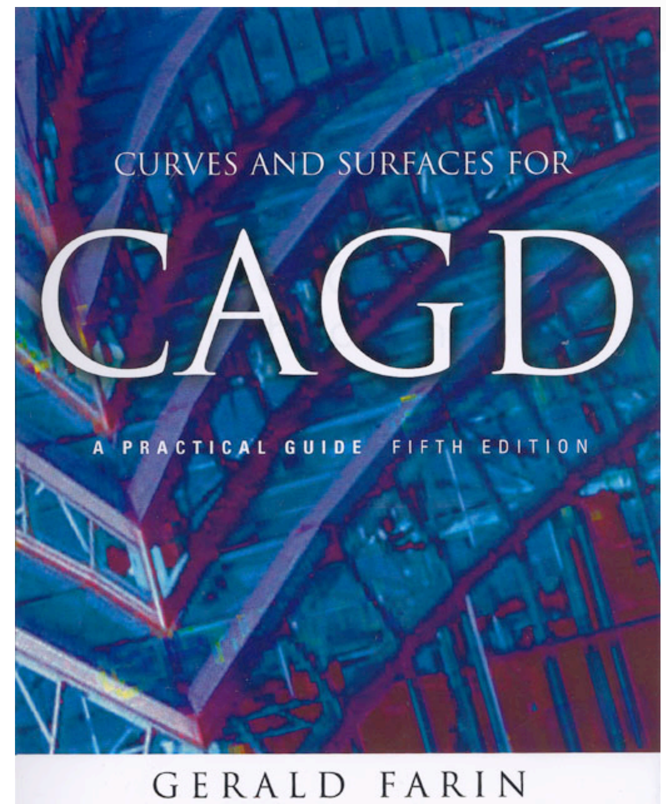
Bézier Curves

by Hao Li

Some literature...

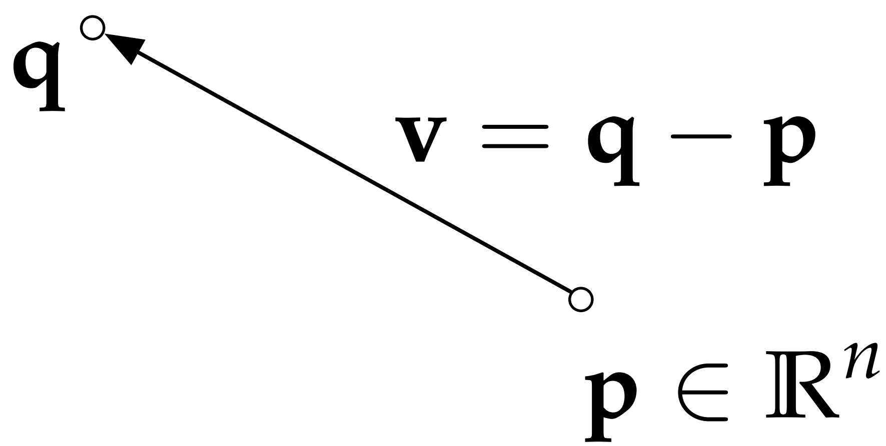


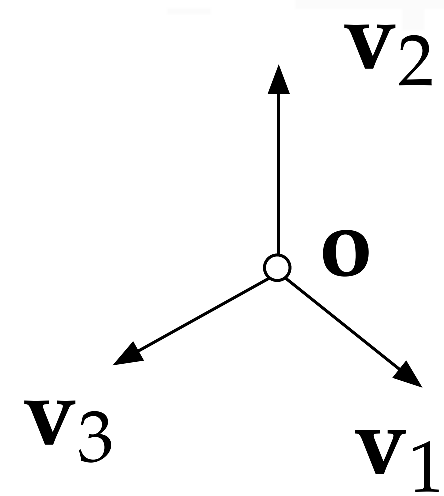


Back in 2001...


Der de Casteljau-Algorithmus und Bézier-Kurven
Vortrag von Hao Li

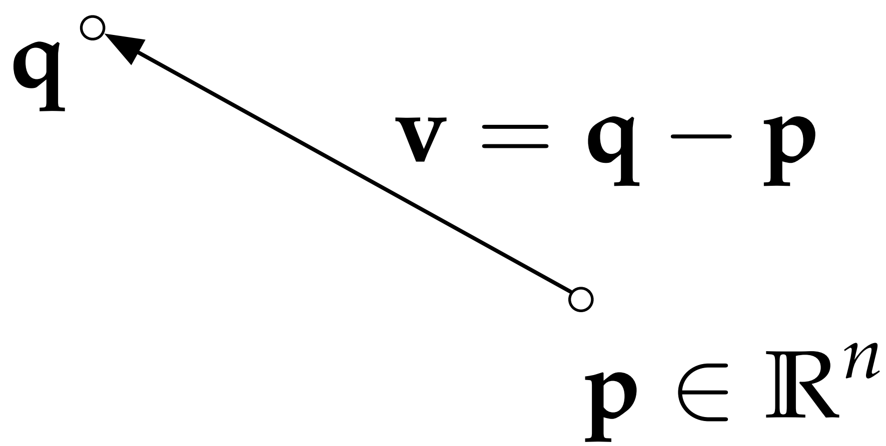
Affine Geometry

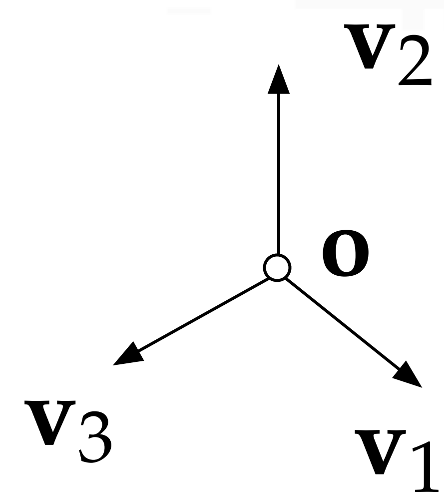




affine space $\mathcal{A}$ with underlying V as $\mathbb{R}^n$

Affine Geometry





affine space $\mathcal{A}$ with underlying V as $\mathbb{R}^n$

Affine Independence

$\mathbf{p}_0, \ldots, \mathbf{p}_m \in \mathcal{A}$ is affine independent

if $\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_m - \mathbf{p}_0$ is linearly independent

Affine Combinations

$\mathbf{q} \in \mathcal{A}$ $\qquad$ $\mathbf{p}_0, \ldots, \mathbf{p}_n$ affine independent

$$
\begin{aligned}
\mathbf{q} &= \mathbf{p}_0 + (\mathbf{p}_1 - \mathbf{p}_0)x_1 + \ldots + (\mathbf{p}_n - \mathbf{p}_0)x_n \\
&= \mathbf{p}_0 x_0 + \ldots + \mathbf{p}_n x_n
\end{aligned}
$$

x_i are the barycentric coordinates of $\mathbf{q}$ $\qquad$ $\sum_{i=1}^{n} x_i = 1$

Affine Combinations

affine
combination

$$\mathbf{a} = \sum \mathbf{a}_i \alpha_i$$

point $\sum \alpha_i = 1$

vector $\sum \alpha_i = 0$

$\alpha_i \geq 0$
convex
combination

Affine and Linear Maps

$$\begin{aligned} \Phi : \mathcal{A} &\to \mathcal{B} \\ \mathbf{x} &\mapsto \mathbf{y} = \mathbf{a} + A\mathbf{x} \end{aligned}$$

$$\begin{aligned} \phi : U &\to V \\ \mathbf{v} &\mapsto \mathbf{u} = A\mathbf{v} \end{aligned}$$

$\mathbf{a}$ is the image of the origin of $\mathcal{A}$

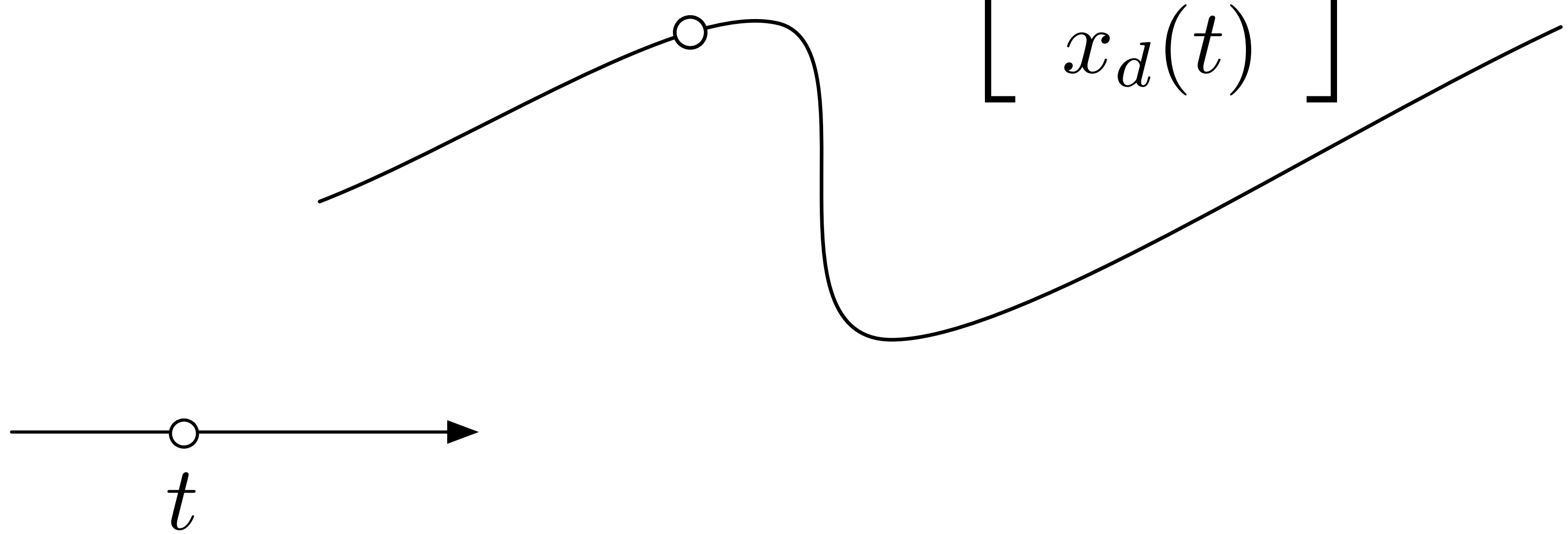
Direct Implication

$$\Phi\left(\sum \mathbf{a}_i \alpha_i\right) = \sum \Phi(\mathbf{a}_i)\alpha_i$$

affine maps commute with affine combinations

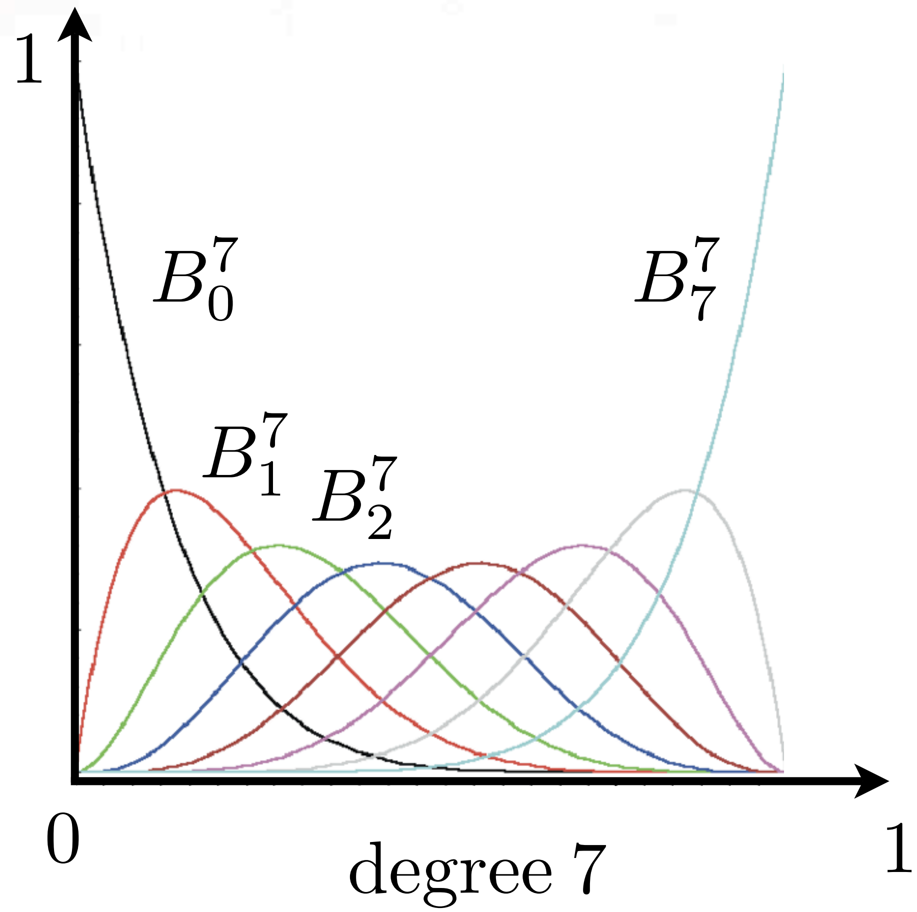
Parametric Curves

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{bmatrix} \in \mathbb{R}^d$$



$\mathbf{x}(t)$ polynomial curve if $x_i(t)$ polynomials

Bernstein Polynomials



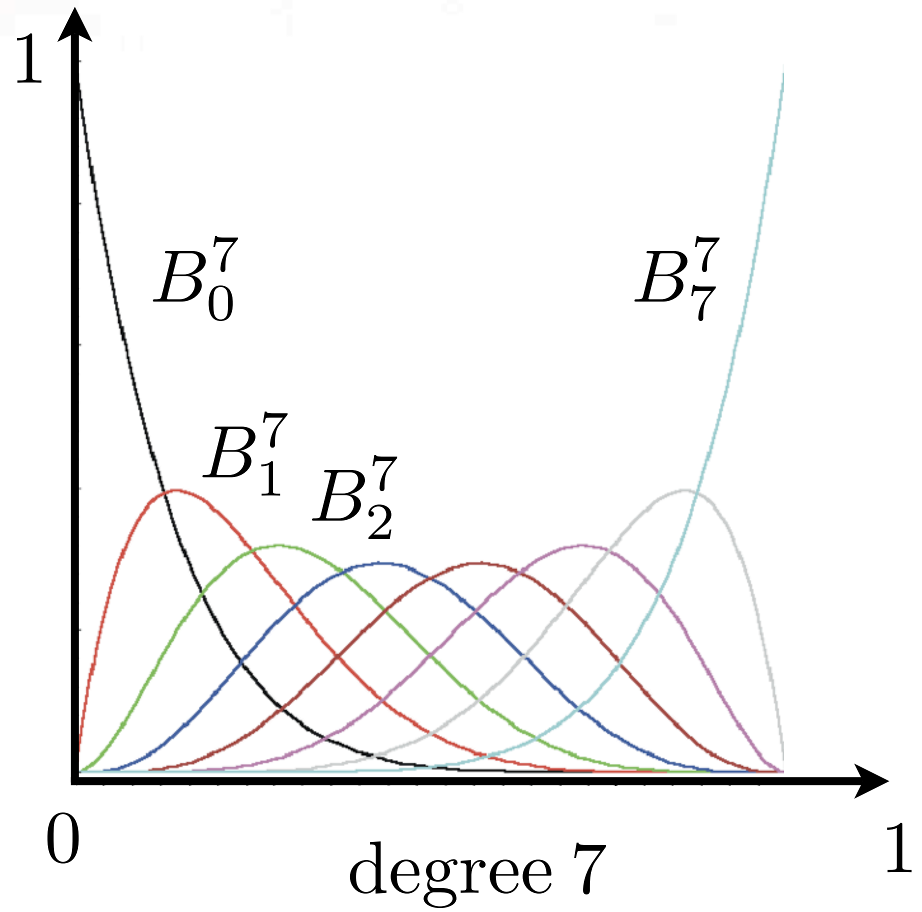
binomial expansion

$$1 = (t + (1-t))^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}$$

↓

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Bernstein Polynomials



$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- linear independent
- partition of unity
- roots at 0 and 1 only
- symmetric
- positive in 0 and 1

understand them as weights in linear combinations

n+1 linearly independent Bernstein polynomials form a basis of all polynomials of degree ≤ n

↓

every polynomial curve $\mathbf{b}(t)$ of degree ≤ n has a unique nth degree Bézier representation.

Bézier Representation

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{c}_i \alpha_i t^i = \sum_{i=0}^{n} \mathbf{c}_i B_i^n(t)$$

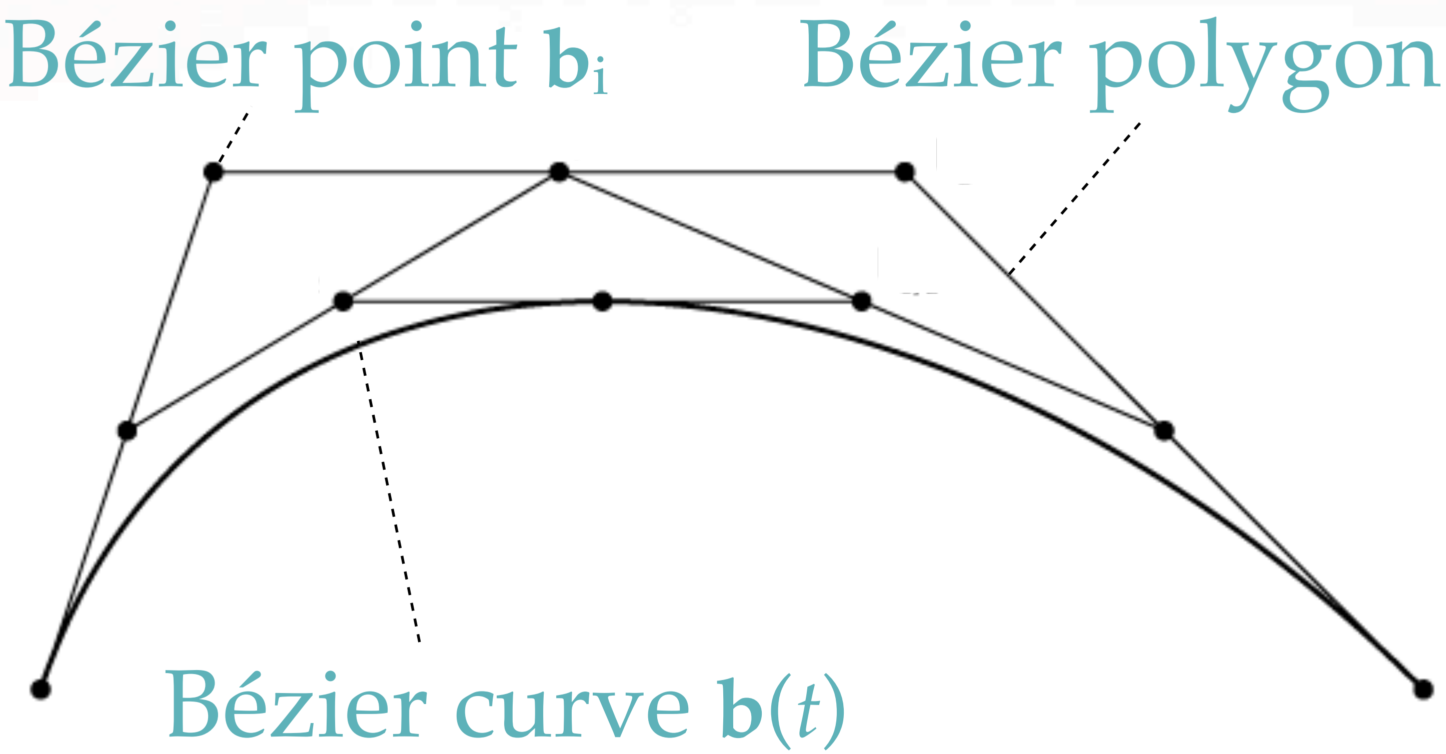
monomial　　　Bézier

Properties of Bernstein Polynomials

↓

Bézier "Curves"

Properties



- end point interpolation
- $\mathbf{b}(t)$ is affine combination $\mathbf{b}_i$
- affine invariance
- convex hull
- symmetry

- variation diminishing
- linear precision

The de Casteljau Algorithm

Bernstein polynomial recursion formula

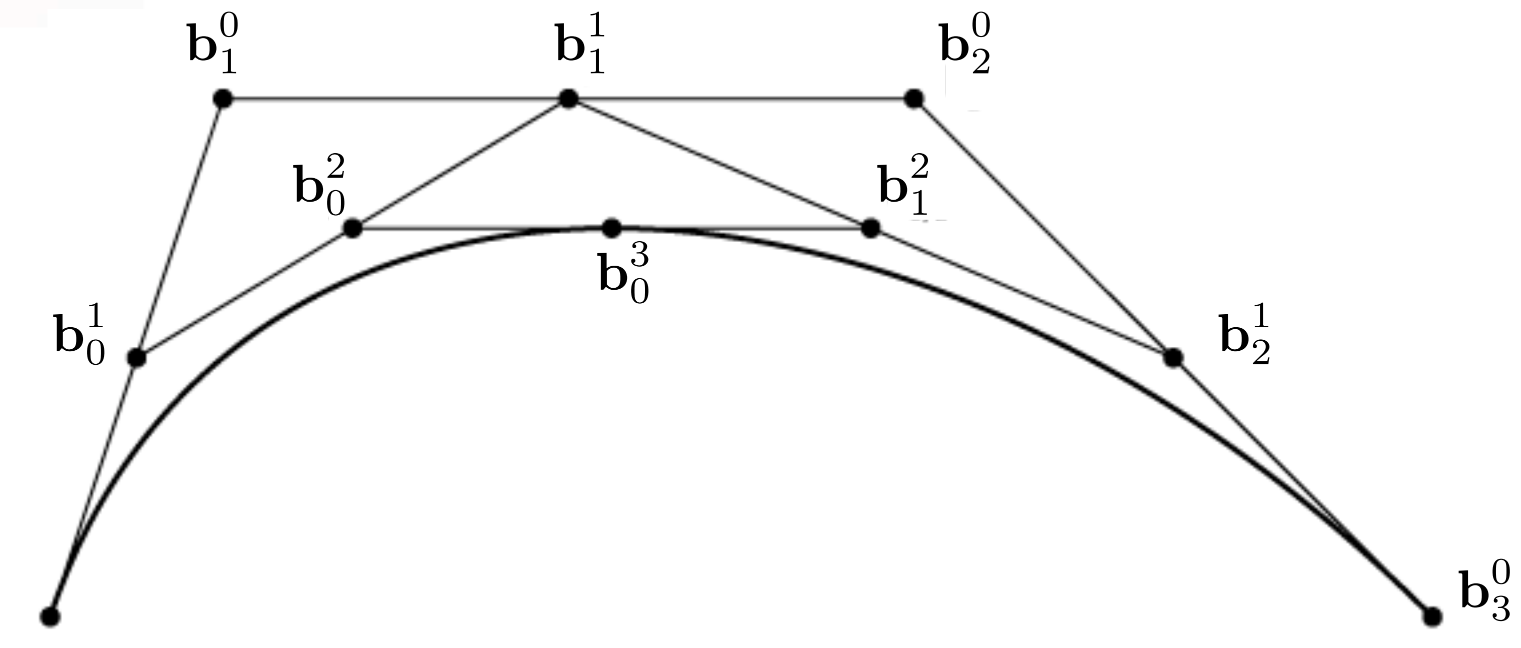
$$\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i} \quad \longrightarrow \quad B_i^{n+1}(t) = tB_{i-1}^{n}(t) + (1-t)B_i^n(t)$$

$$\downarrow$$

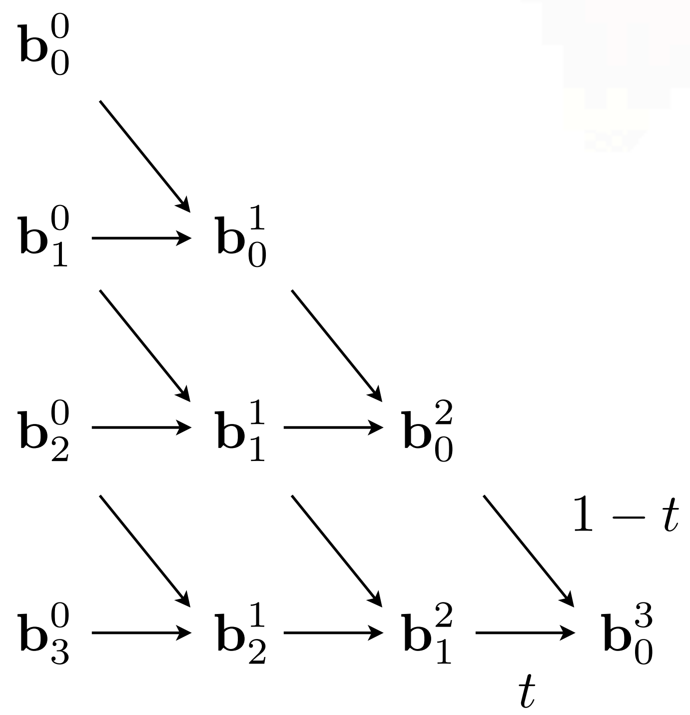
$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t) = \sum_{i=0}^{n-1} \mathbf{b}_i^1 B_i^{n-1}(t) = \ldots = \sum_{i=0}^{0} \mathbf{b}_i^n B_i^0(t) = \mathbf{b}_0^n$$

with $\mathbf{b}_i^{k+1} = (1-t)\mathbf{b}_i^k + t\mathbf{b}_{i+1}^k$

de Casteljau Scheme



cubic polynomial in Bézier form



Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} B_i^n(t) = n(B_{i-1}^{n-1} - B_i^{n-1}(t))$$

↓

first derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{b}(t) = n \sum_{i=0}^{n-1} \Delta \mathbf{b}_i B_i^{n-1}(t)$$

forward difference

$$\Delta \mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$$

↓

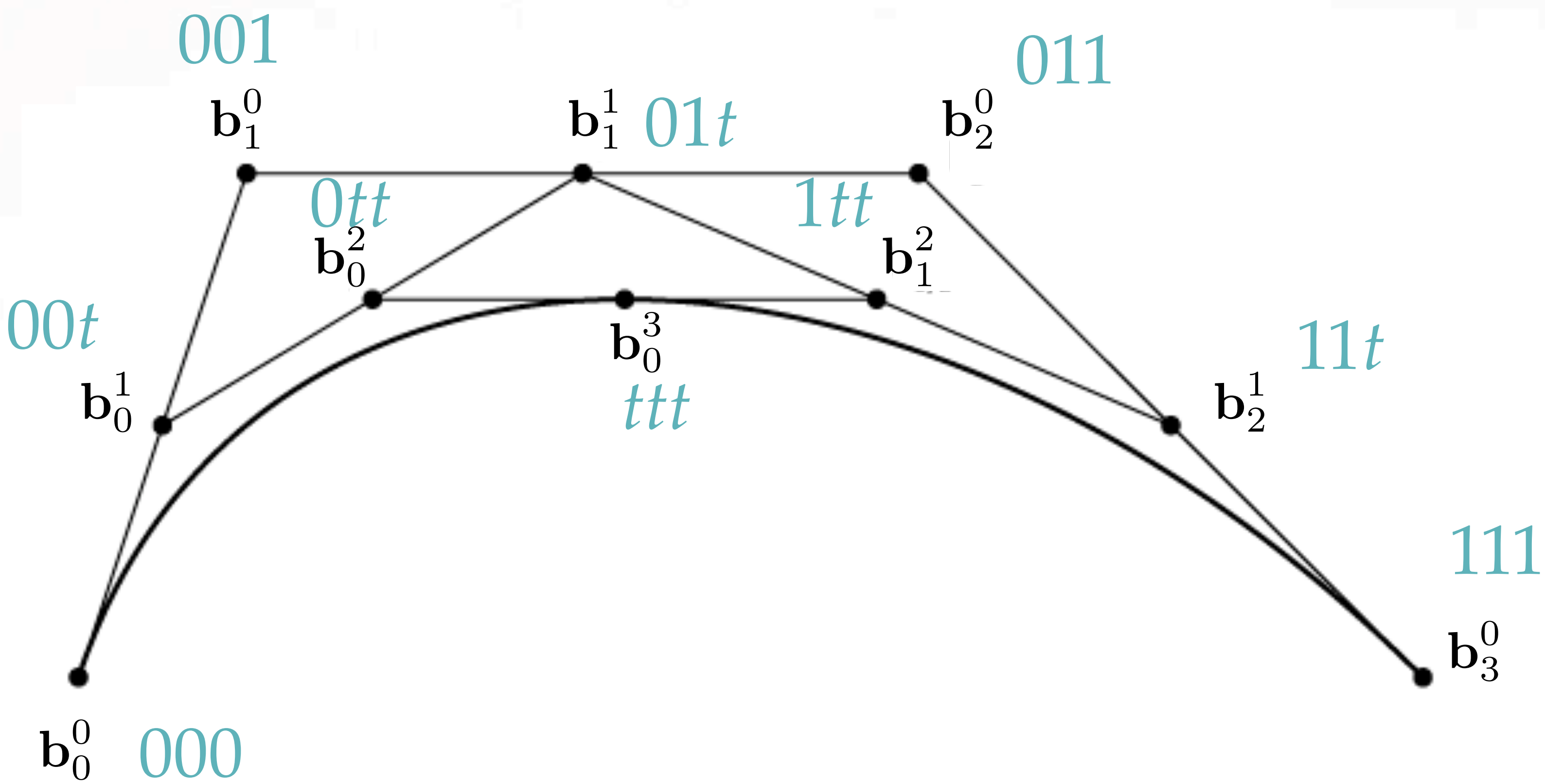
rth derivative

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} \mathbf{b}(t) = \frac{n!}{n-r!} \sum_{i=0}^{n-r} \Delta^r \mathbf{b}_i B_i^{n-r}(t)$$

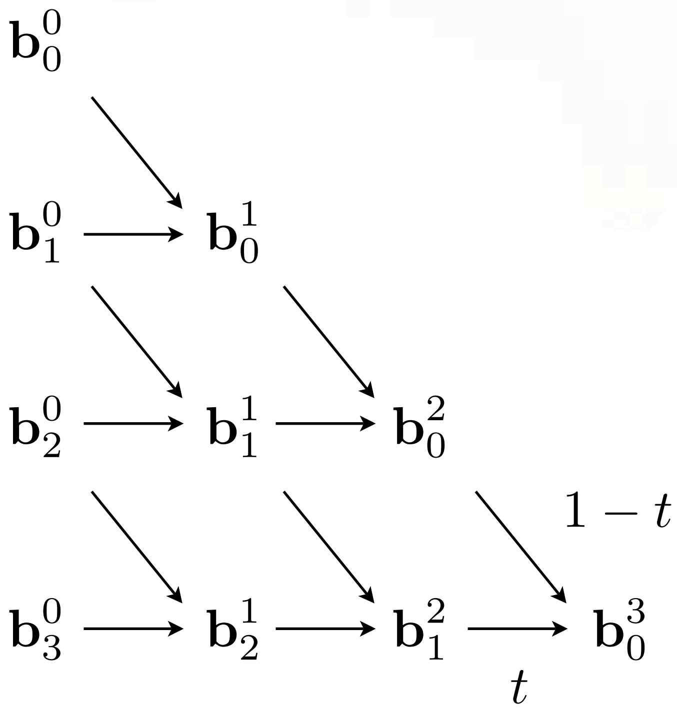
rth forward difference

$$\Delta^r \mathbf{b}_i = \Delta^{r-1} \mathbf{b}_{i+1} - \Delta^{r-1} \mathbf{b}_i$$

Blossoming



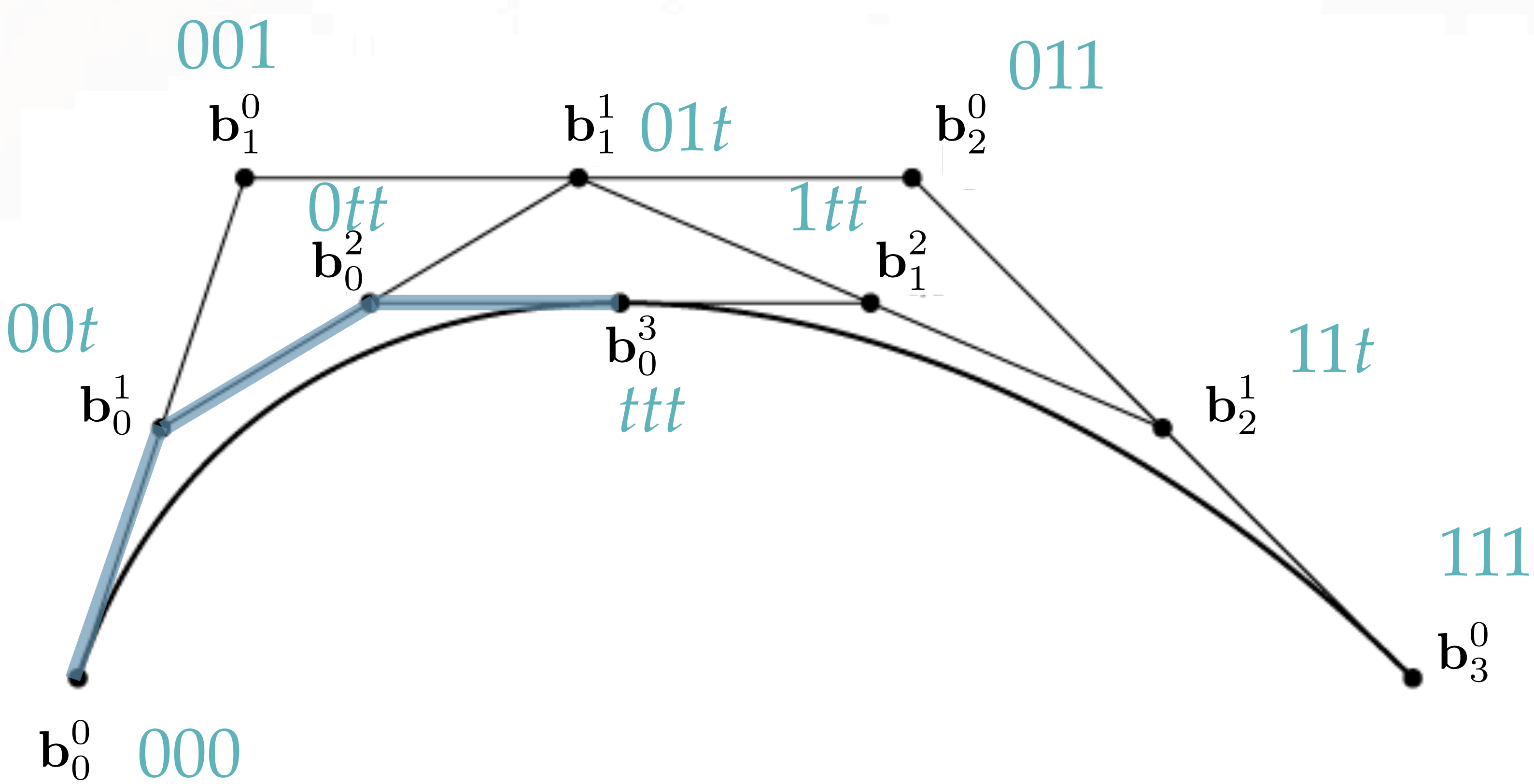
001
$\mathbf{b}_1^0$
$\mathbf{b}_1^1$ 01t
$\mathbf{b}_2^0$
011
0tt
1tt
$\mathbf{b}_0^2$
$\mathbf{b}_1^2$
00t
$\mathbf{b}_0^3$
ttt
11t
$\mathbf{b}_0^1$
$\mathbf{b}_2^1$
111
$\mathbf{b}_3^0$
$\mathbf{b}_0^0$ 000

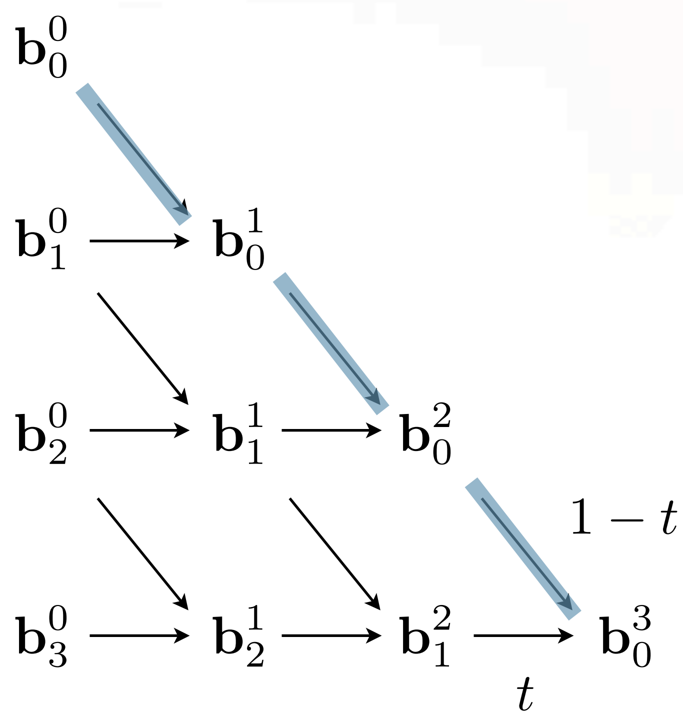


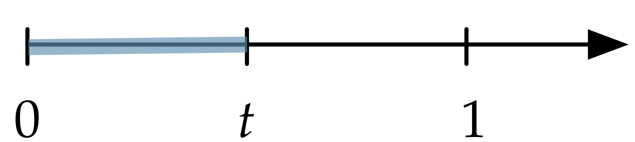
$\mathbf{b}_0^0$
$\mathbf{b}_1^0$ $\mathbf{b}_0^1$
$\mathbf{b}_2^0$ $\mathbf{b}_1^1$ $\mathbf{b}_0^2$
$1-t$
$\mathbf{b}_3^0$ $\mathbf{b}_2^1$ $\mathbf{b}_1^2$ $\mathbf{b}_0^3$
t

0
t
1

Subdivision







Convergence under Subdivision

- Subdivision properties often studied using symmetric polynomials

- Convergence of subdivision is quadratic with the size of subintervals (Proof via Taylor expansion)

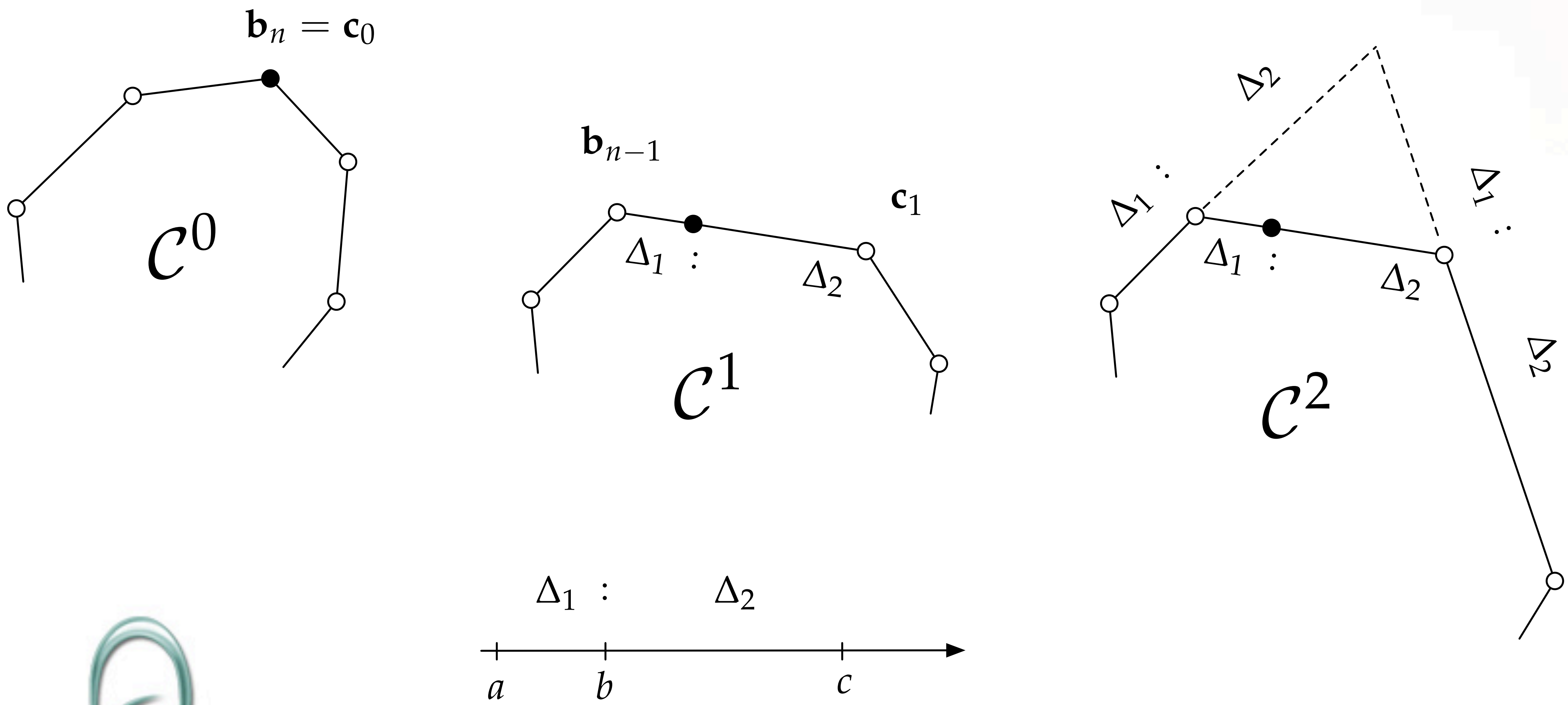
- The Bézier polygon of small curve segments are good approximations of this segment.

Applications of Subdivision

- Piecewise linear approximation of curve generation
- Theoreme Proving (e.g. variation diminishing)
- Intersection Test
- Differentiability analysis of composite Bézier curves (c.f. Stärk's theorem)

Simple C^r joints

(c.f. Stärk's construction)



Further Readings

- "Bézier and B-Splines Techniques" [Prautzsch '02]
- "Curves and Surfaces for CAGD A Practical Guide" [Farin '02]
- "Grundlagen der geometrischen Datenverarbeitung" [Hoschek & Lasser '92]
- "Differential Geometry of Curves and Surfaces" [Do Carmo '76]

- CAGD Applets: http://i33www.ibds.uni-karlsruhe.de/applets/

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