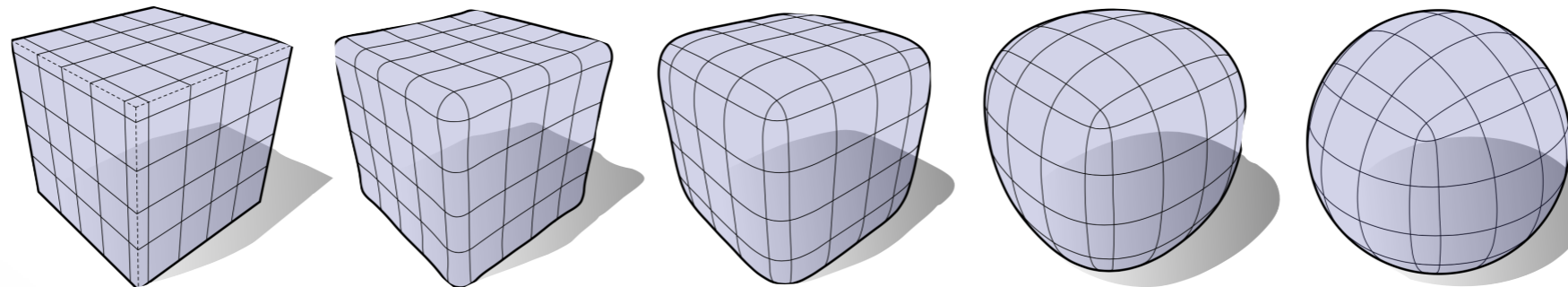


4.1 Classic Differential Geometry 2



Hao Li

<http://cs621.hao-li.com>

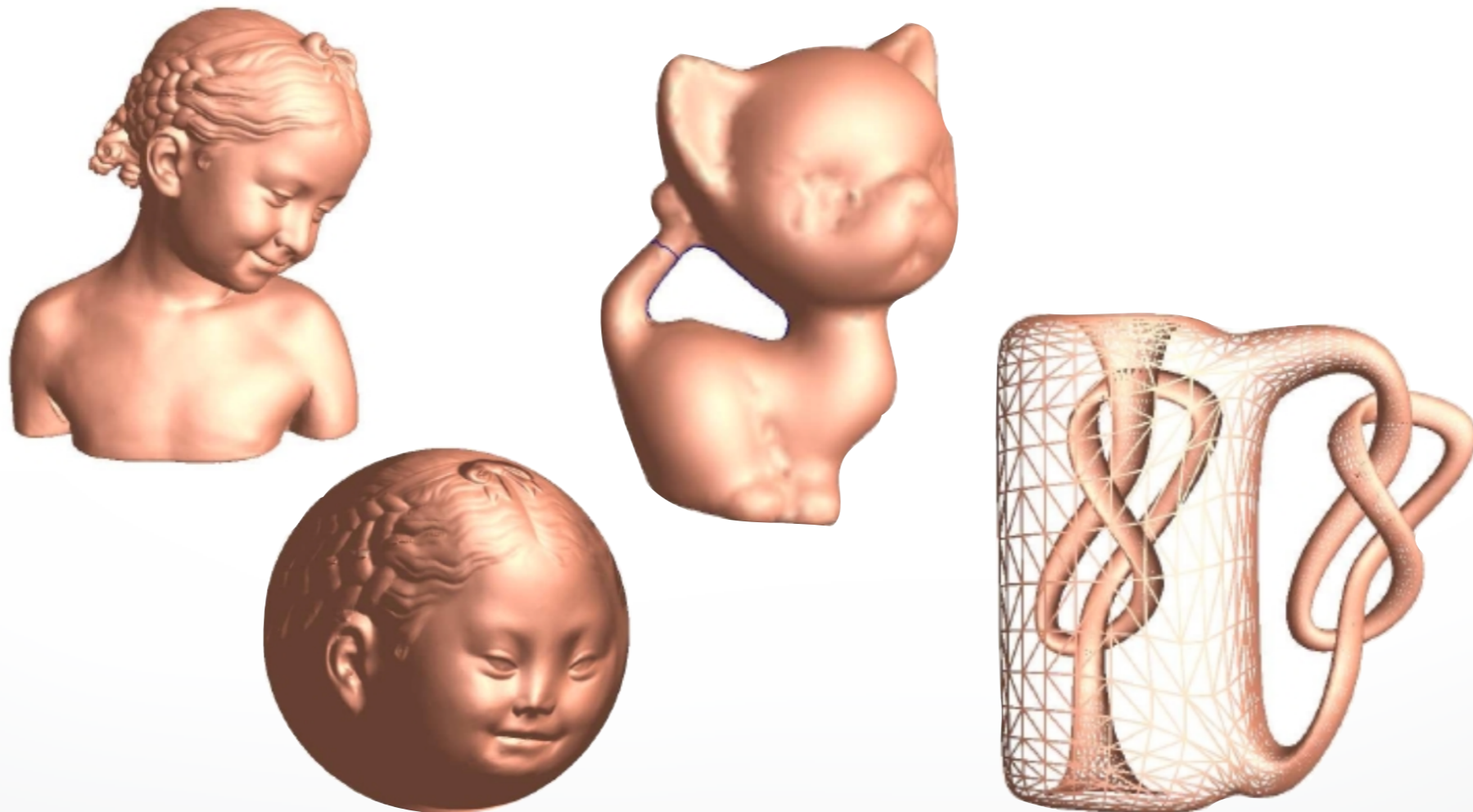
Outline

- Parametric Curves
- **Parametric Surfaces**

Surfaces

What characterizes shape?

- shape does not depend on Euclidean motions
 - metric and curvatures
- smooth continuous notions to discrete notions



Metric on Surfaces

Measure Stuff

- angle, length, area
 - requires an inner product
- we have:
 - Euclidean inner product in domain
- we want to turn this into:
 - inner product on surface

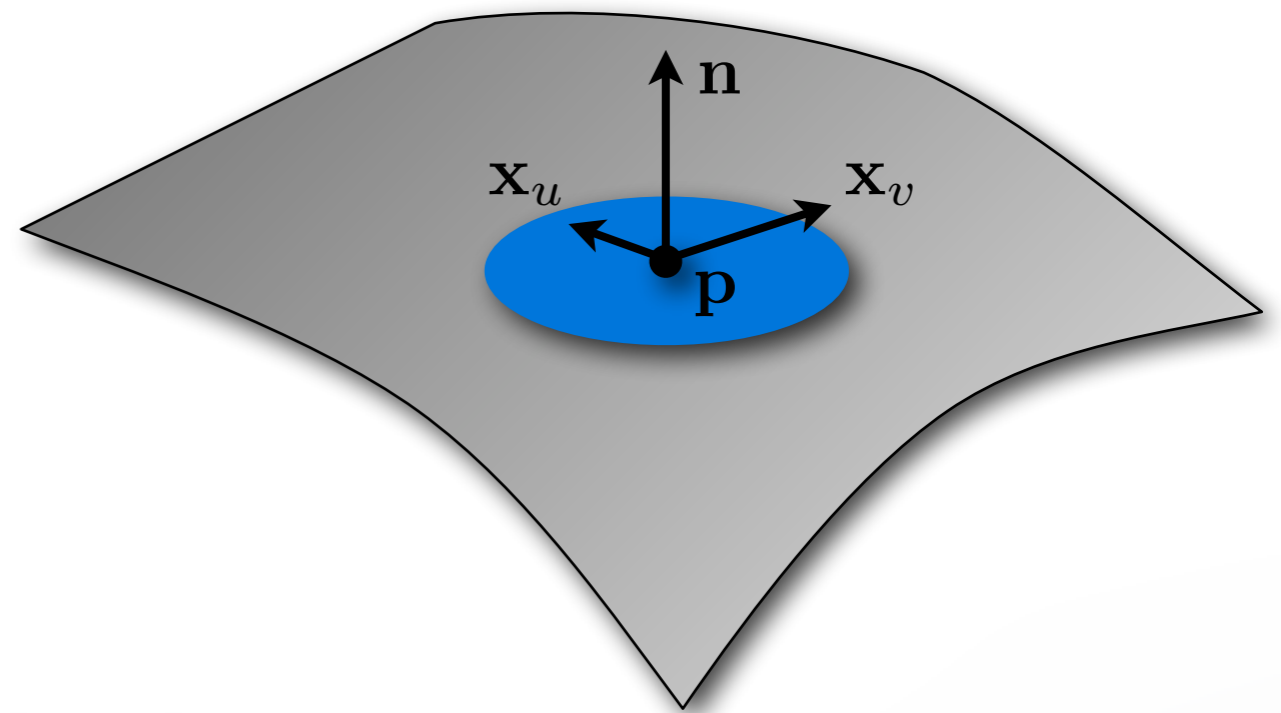
Parametric Surfaces

Continuous surface

$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0} \quad \text{normal exists}$$

Angles on Surface

Curve $[u(t), v(t)]$ in **uv-plane** defines **curve on the surface** $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

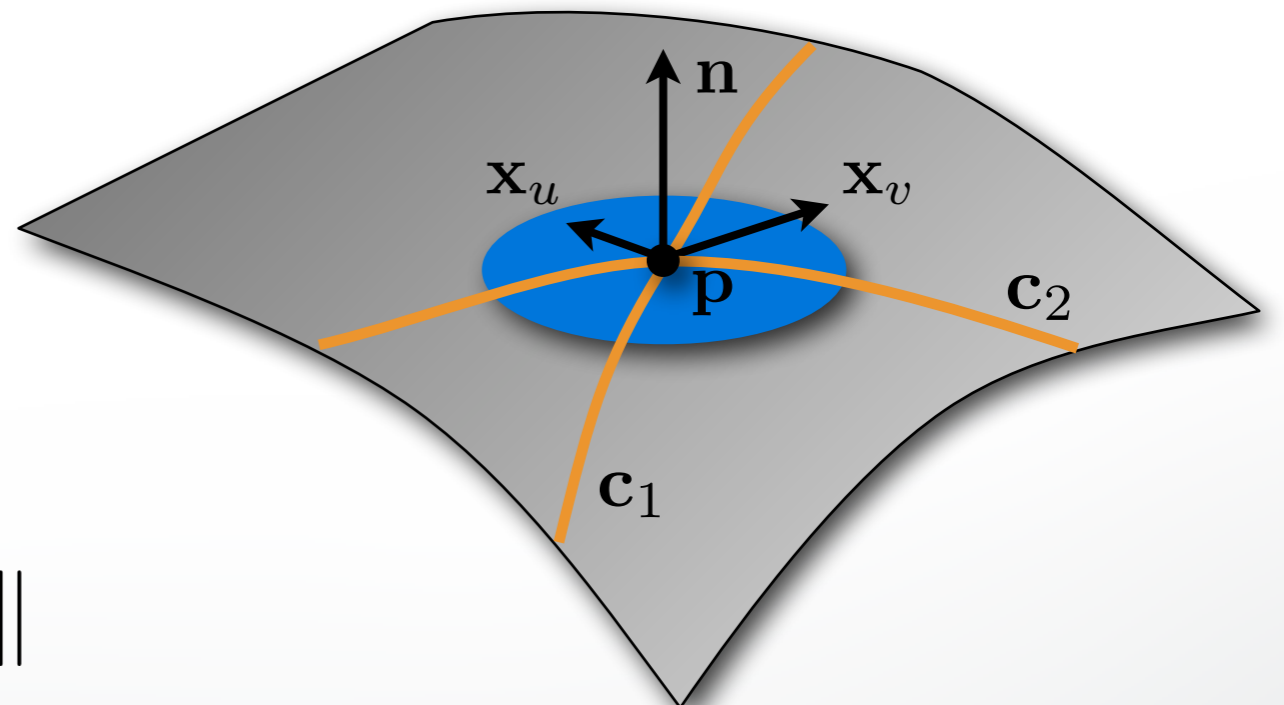
Two curves \mathbf{c}_1 and \mathbf{c}_2 **intersecting at p**

- angle of intersection?
- two tangents \mathbf{t}_1 and \mathbf{t}_2

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos \theta \|\mathbf{t}_1\| \|\mathbf{t}_2\|$$



Angles on Surface

Curve $[u(t), v(t)]$ in uv-plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

Two curves \mathbf{c}_1 and \mathbf{c}_2 intersecting at \mathbf{p}

$$\begin{aligned}\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\ &= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\ &= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}\end{aligned}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form **I** allows to measure
(w.r.t. surface metric)

Angles $\mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$

Length $ds^2 = \langle (du, dv), (du, dv) \rangle$
 $= Edu^2 + 2Fdudv + Gdv^2$

squared
infinitesimal
length

Area $dA = \|\mathbf{x}_u \times \mathbf{x}_v\| du dv$
 $= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} du dv$
 $= \sqrt{EG - F^2} du dv$

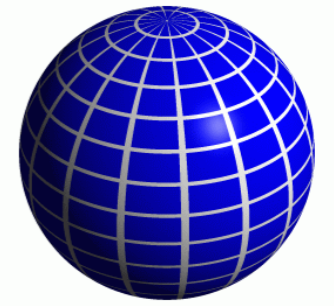
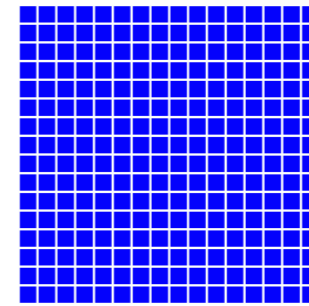
infinitesimal
Area

cross product \rightarrow determinant with unit vectors \rightarrow area

Sphere Example

Spherical parameterization

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u, v) \in [0, 2\pi) \times [0, \pi)$$



Tangent vectors

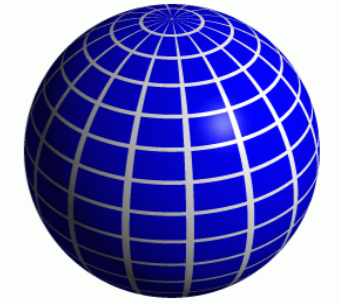
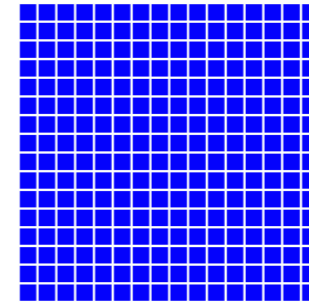
$$\mathbf{x}_u(u, v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u, v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Sphere Example

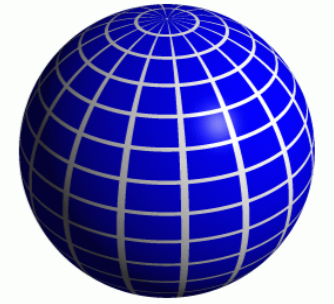
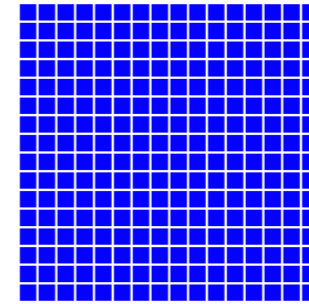
Length of equator $\mathbf{x}(t, \pi/2)$



$$\begin{aligned}\int_0^{2\pi} 1 \, ds &= \int_0^{2\pi} \sqrt{E (u_t)^2 + 2F u_t v_t + G (v_t)^2} \, dt \\ &= \int_0^{2\pi} \sin v \, dt \\ &= 2\pi \sin v = 2\pi\end{aligned}$$

Sphere Example

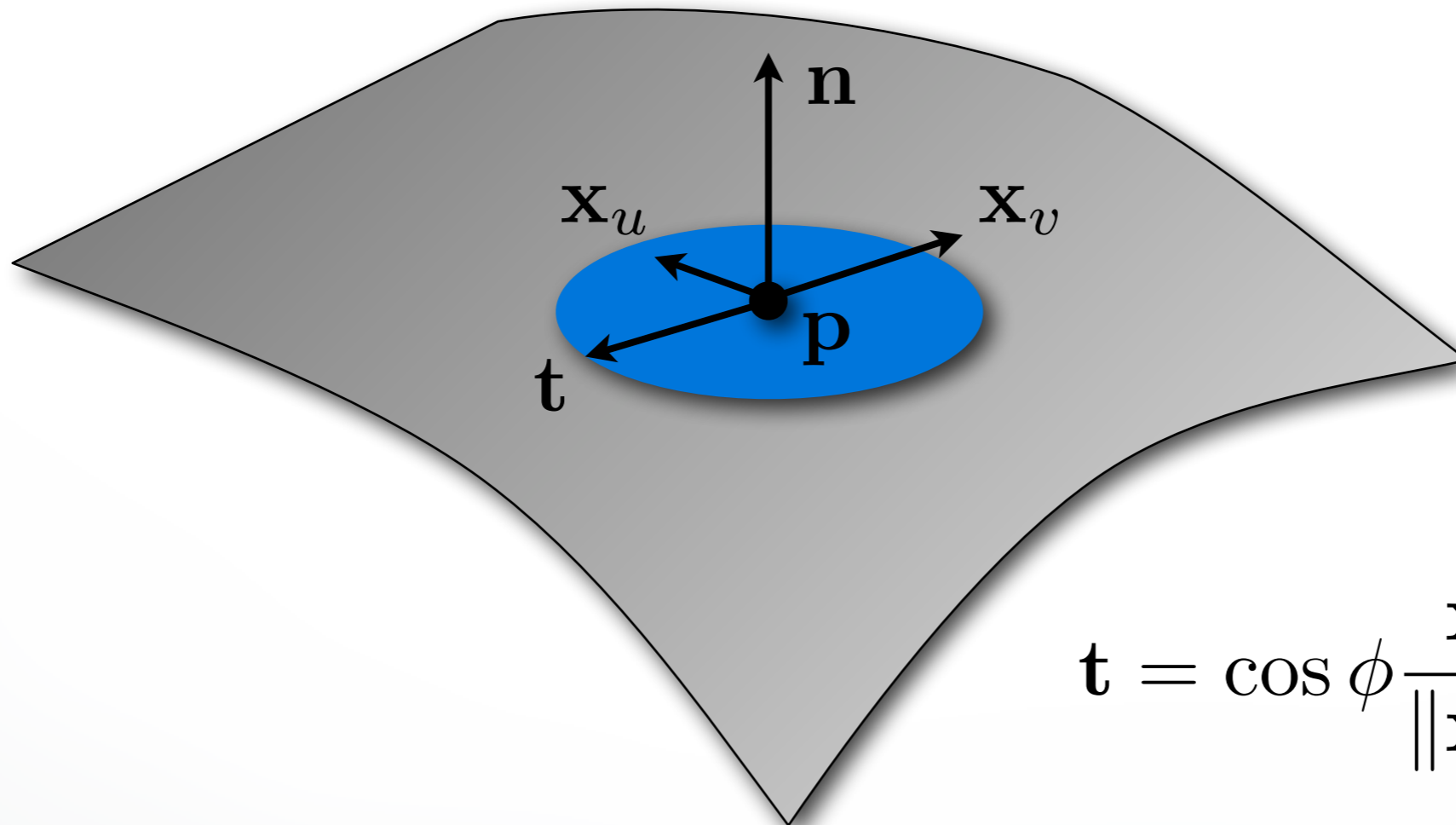
Area of a sphere



$$\begin{aligned}\int_0^\pi \int_0^{2\pi} 1 \, dA &= \int_0^\pi \int_0^{2\pi} \sqrt{EG - F^2} \, du \, dv \\ &= \int_0^\pi \int_0^{2\pi} \sin v \, du \, dv \\ &= 4\pi\end{aligned}$$

Normal Curvature

Tangent vector \mathbf{t} ...



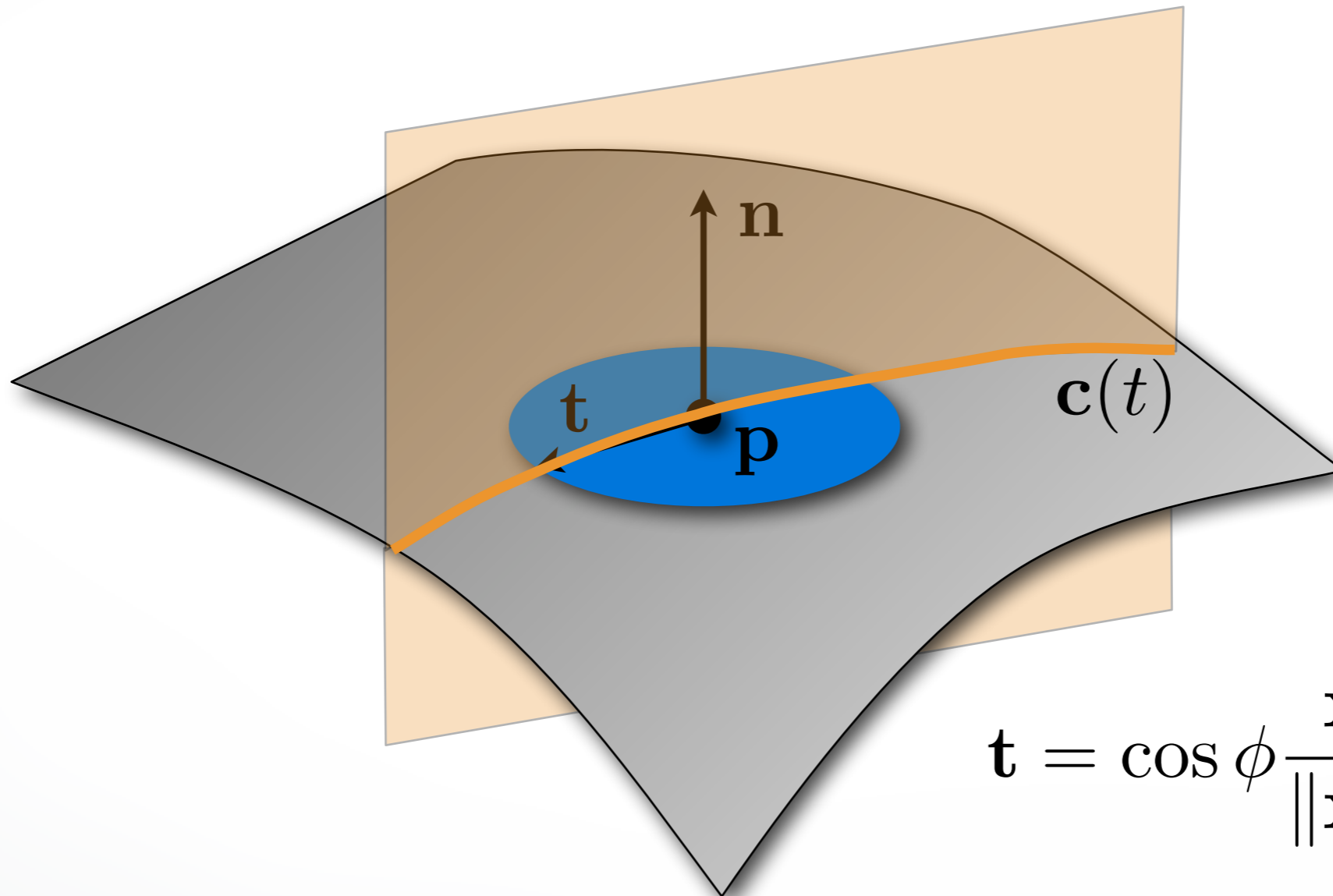
$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

unit vector

Normal Curvature

... defines intersection plane, yielding curve $\mathbf{c}(t)$

normal curve



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

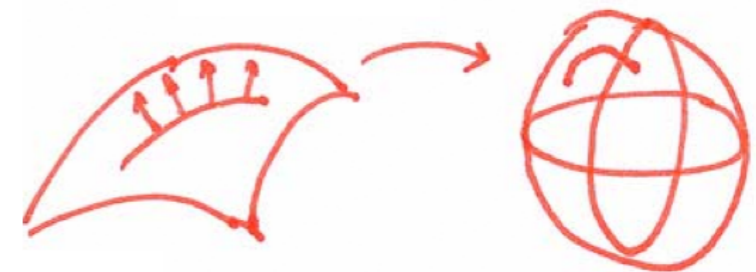
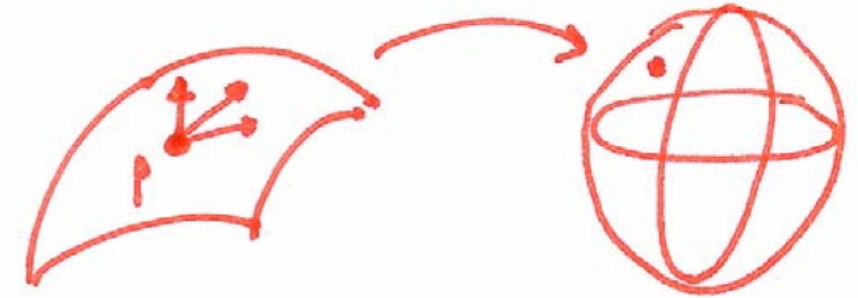
Geometry of the Normal

Gauss map

- normal at point

$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \quad N : S \rightarrow S^2$$

- consider curve in surface again
 - study its curvature at p
 - normal “tilts” along curve



Normal Curvature

Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p}(t) = \mathbf{x}(u, v)$

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

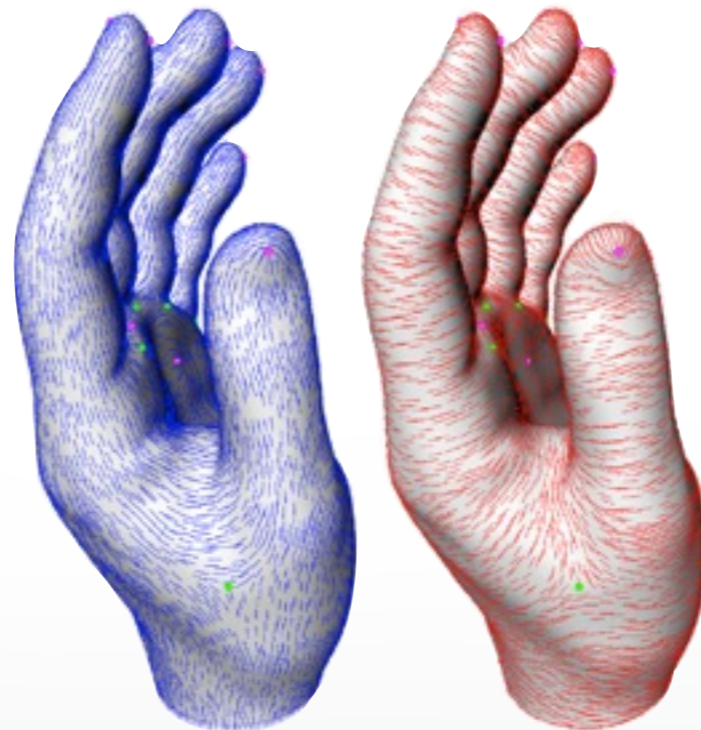
normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \quad \begin{array}{l} \mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} = (a, b) \end{array}$$

Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler theorem $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Corresponding principal directions \mathbf{e}_1 , \mathbf{e}_2 are orthogonal



Surface Curvature(s)

Principal curvatures

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- Corresponding principal directions \mathbf{e}_1 , \mathbf{e}_2 are orthogonal

Special curvatures

- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$ **extrinsic**
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ **intrinsic (only first FF)**

Invariants

Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$
$$\text{tr } dN_p = \kappa_1 + \kappa_2 = H$$

- eigenvalues and orthovectors

$$dN_p(e_1) = \kappa_1 e_1 \quad dN_p(e_2) = \kappa_2 e_2$$

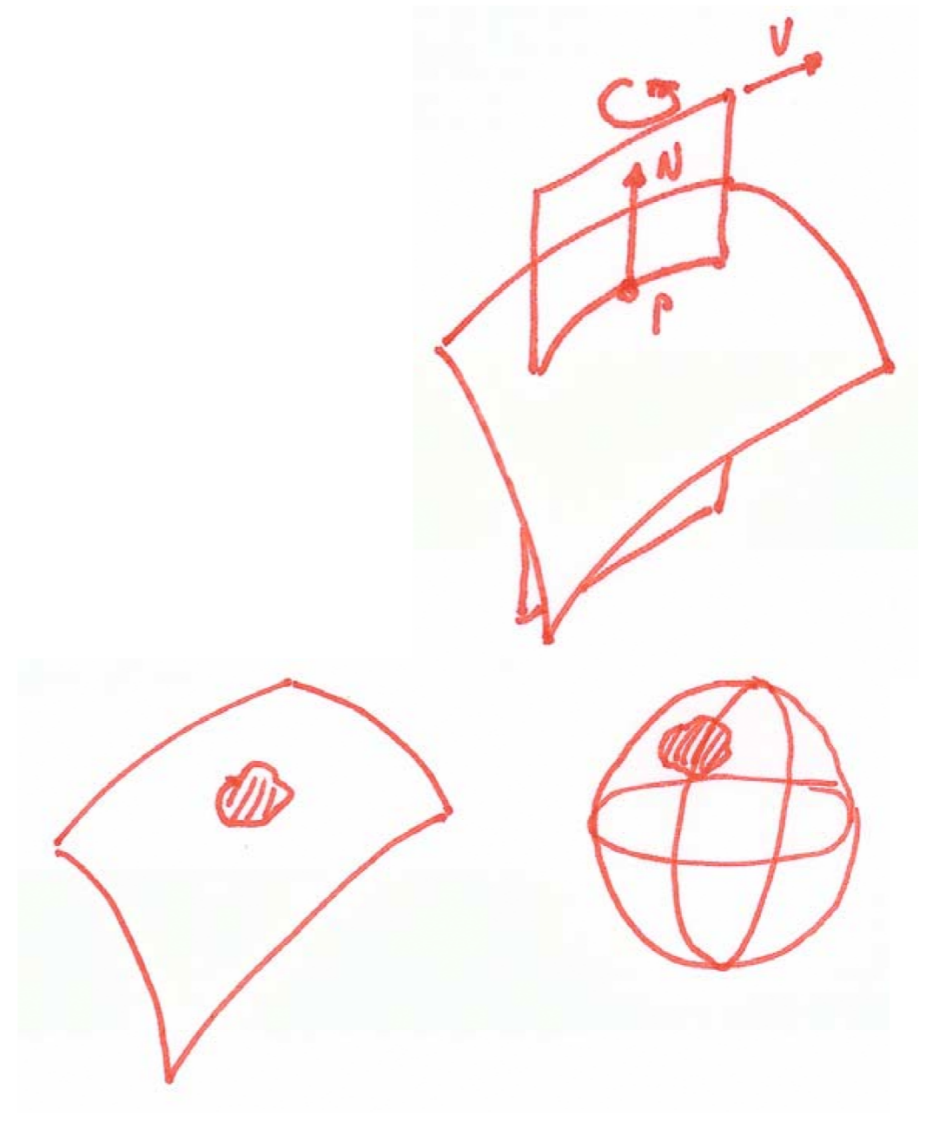
$$II_p|_{S \subset T_p S} \begin{cases} \text{max} \rightarrow \kappa_1 \\ \text{min} \rightarrow \kappa_2 \end{cases}$$

Mean and Gaussian Curvature

Integral representations

$$H_p/2 = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) d\theta$$

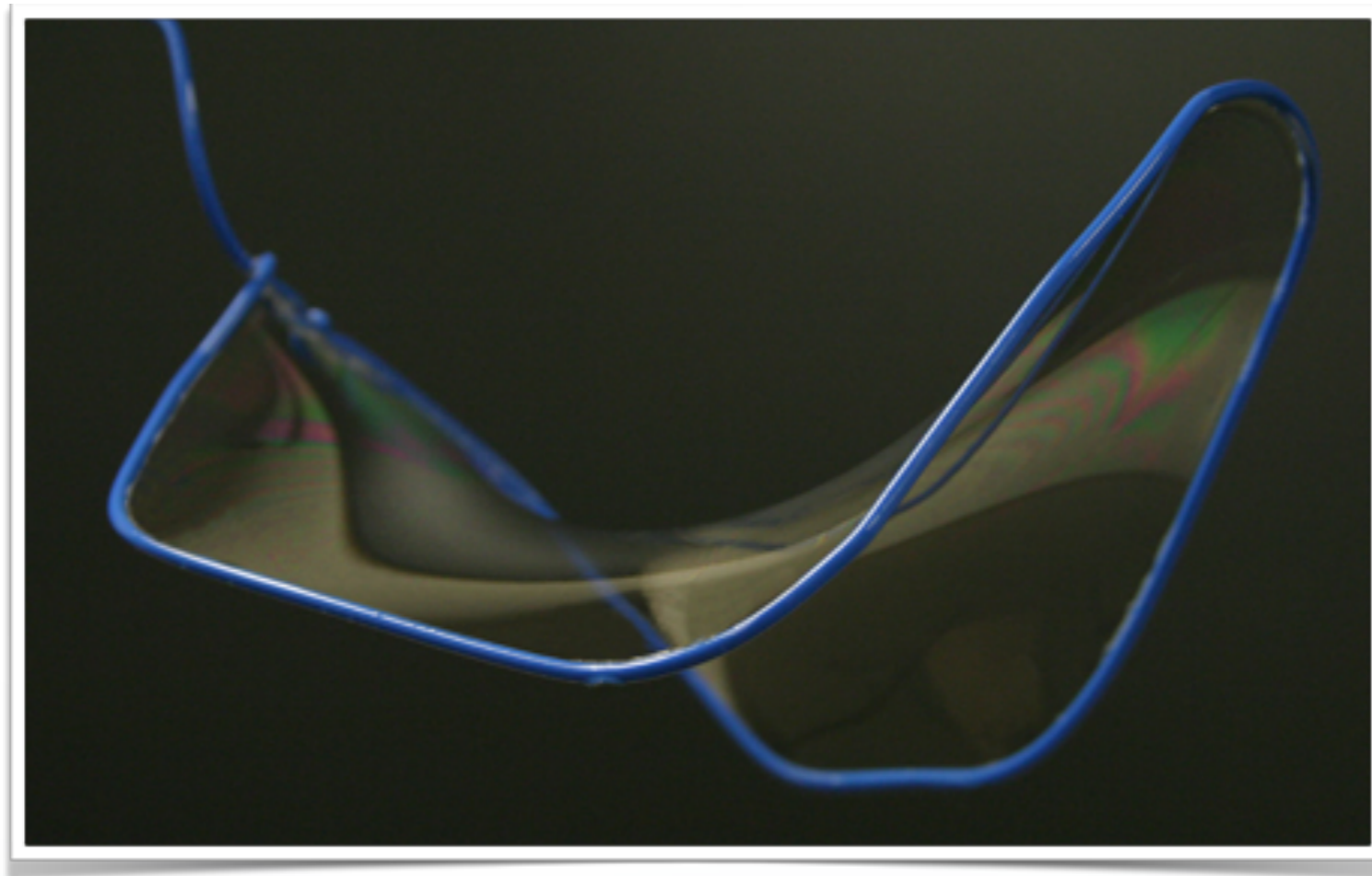
$$K_p = \lim_{A \rightarrow 0} \frac{A_G}{A}$$



Curvature of Surfaces

Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere \rightarrow minimal surface



soap film

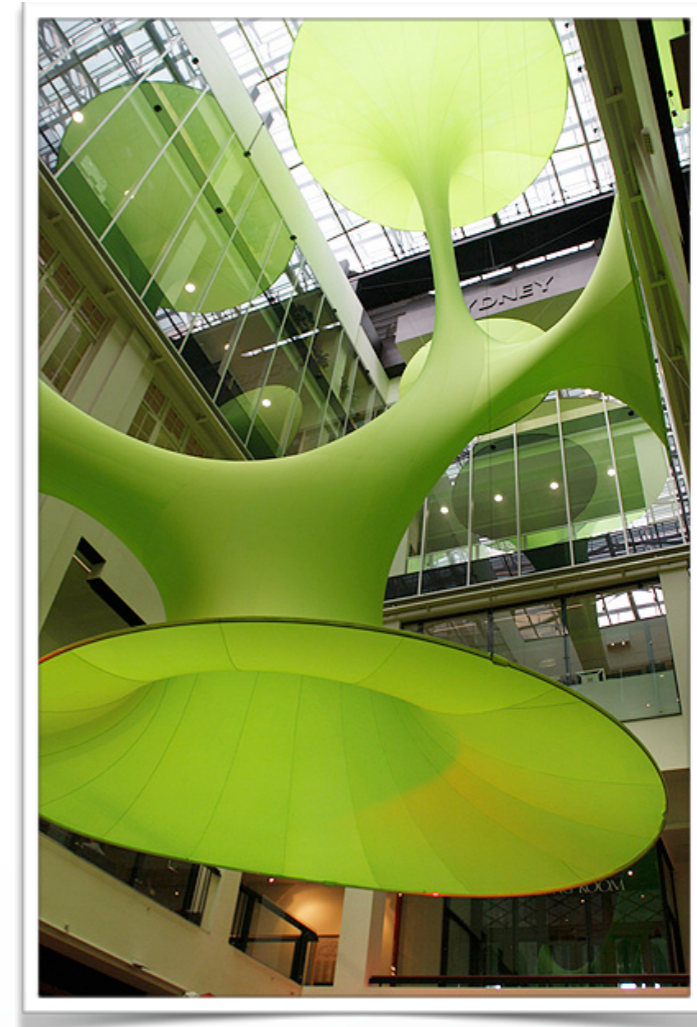
Curvature of Surfaces

Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere \rightarrow minimal surface



Green Void, Sydney
Architects: Lava



Curvature of Surfaces

Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

- $K = 0$ everywhere \rightarrow developable surface

surface that can be flattened to a plane without distortion (stretching or compression)



Disney, Concert Hall, L.A.
Architects: Gehry Partners



Timber Fabric
IBOIS, EPFL

Shape Operator

Derivative of Gauss map

- second fundamental form

$$II_p(\mathbf{v}) = \langle dN_p(\mathbf{v}), \mathbf{v} \rangle$$

- local coordinates

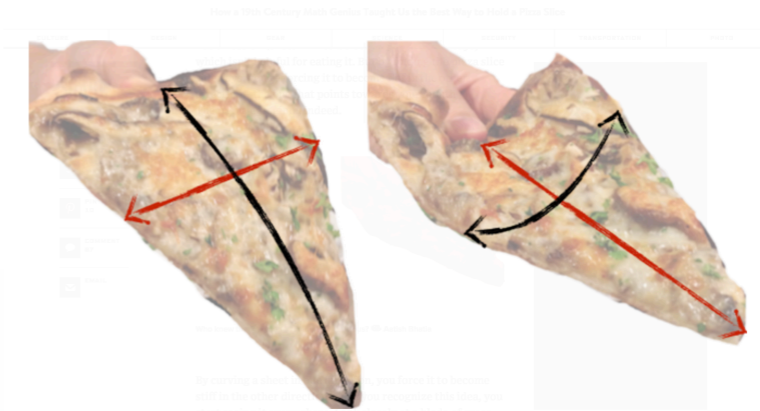
$$II_p = - \begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Intrinsic Geometry

Properties of the surface that only depend on the first fundamental form

- length
- angles
- Gaussian curvature (Theorema Egregium)
remarkable theorem (Gauss)

$$K = \lim_{r \rightarrow 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$



Gaussian curvature of a surface is invariant under local isometry

Classification

Point \mathbf{x} on the surface is called

- elliptic, if $K > 0$
- hyperbolic, if $K < 0$
- parabolic, if $K = 0$
- umbilic, if $\kappa_1 = \kappa_2$ **or isotropic**

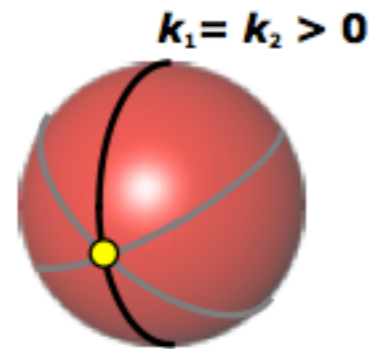
Gaussian curvature K

Classification

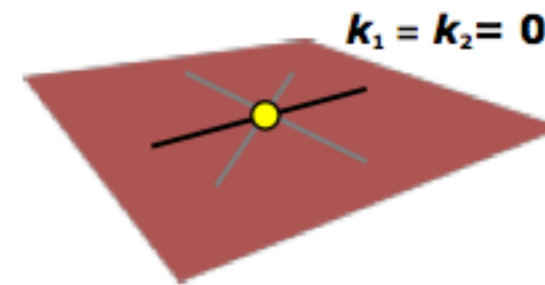
Point x on the surface is called

Isotropic

Equal in all directions



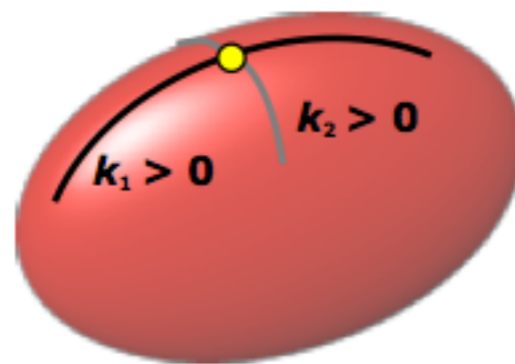
spherical



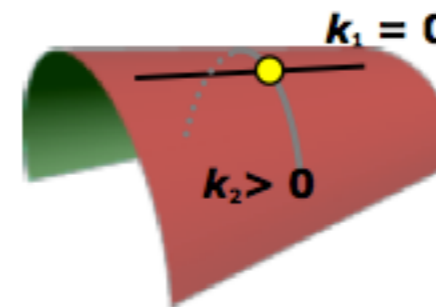
planar

Anisotropic

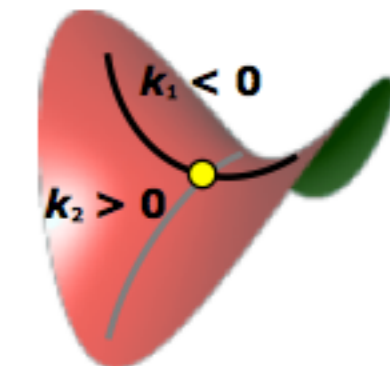
Distinct principal directions



elliptic
 $K > 0$



parabolic
 $K = 0$
developable



hyperbolic
 $K < 0$

Gauss-Bonnet Theorem

For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

$$\int K(\text{glove}) = \int K(\text{cow}) = \int K(\text{ball}) = 4\pi$$

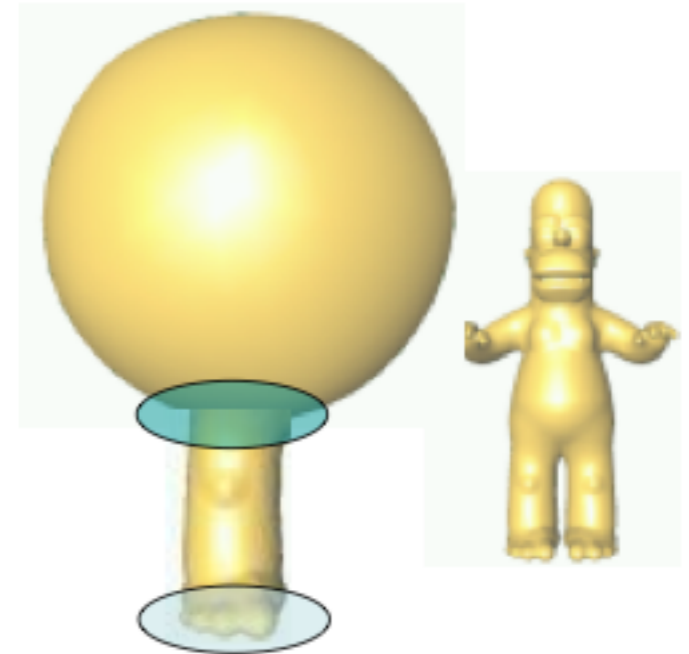
Gauss-Bonnet Theorem

Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



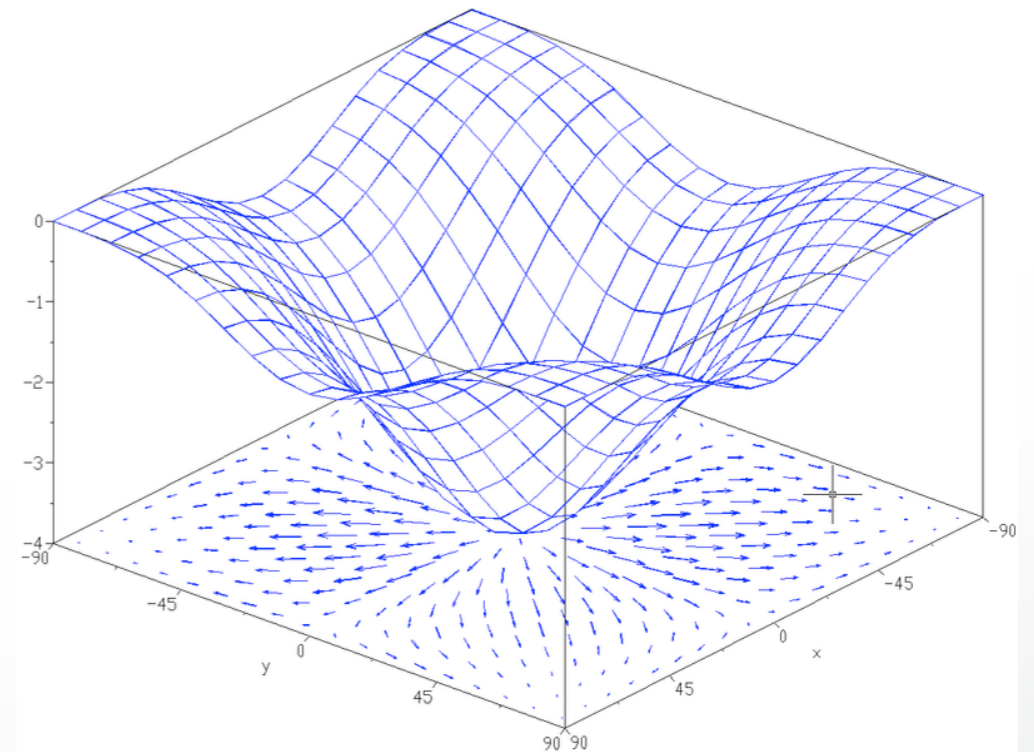
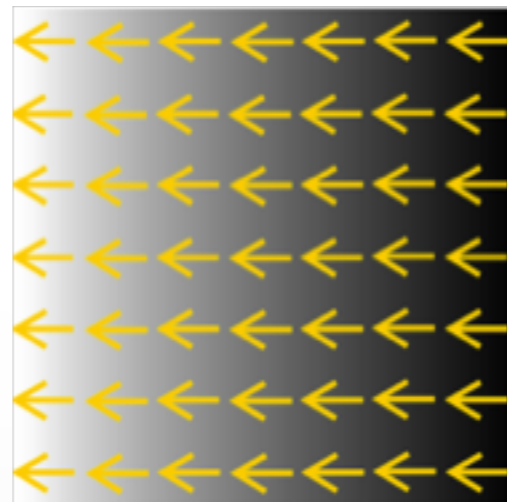
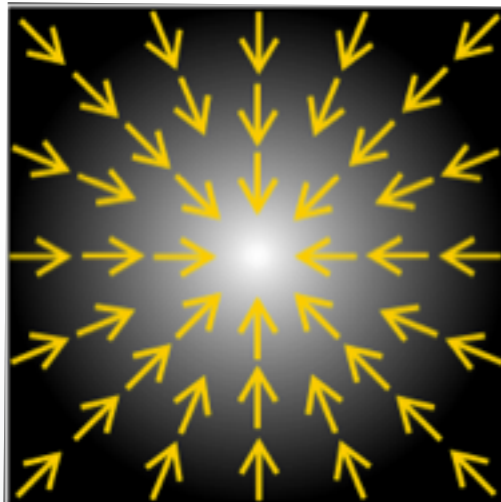
when sphere is deformed, new
positive and negative curvature cancel out

Differential Operators

Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend



Differential Operators

Divergence

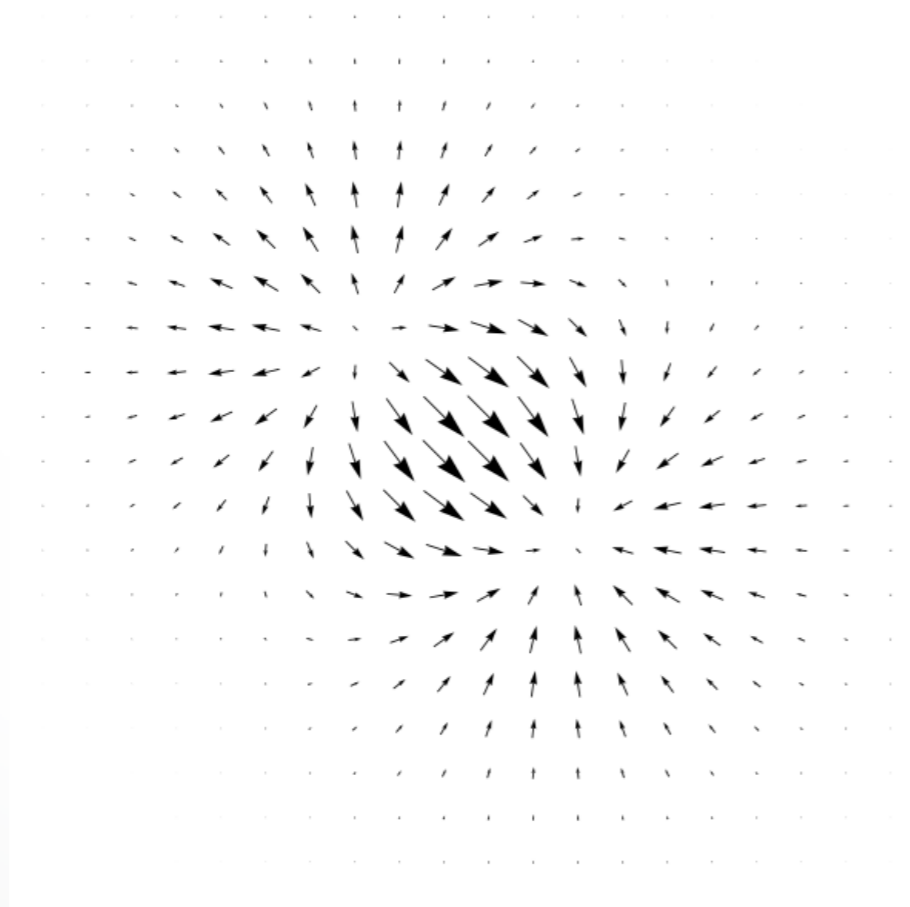
$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: incompressible fluid
 - velocity field is divergence-free

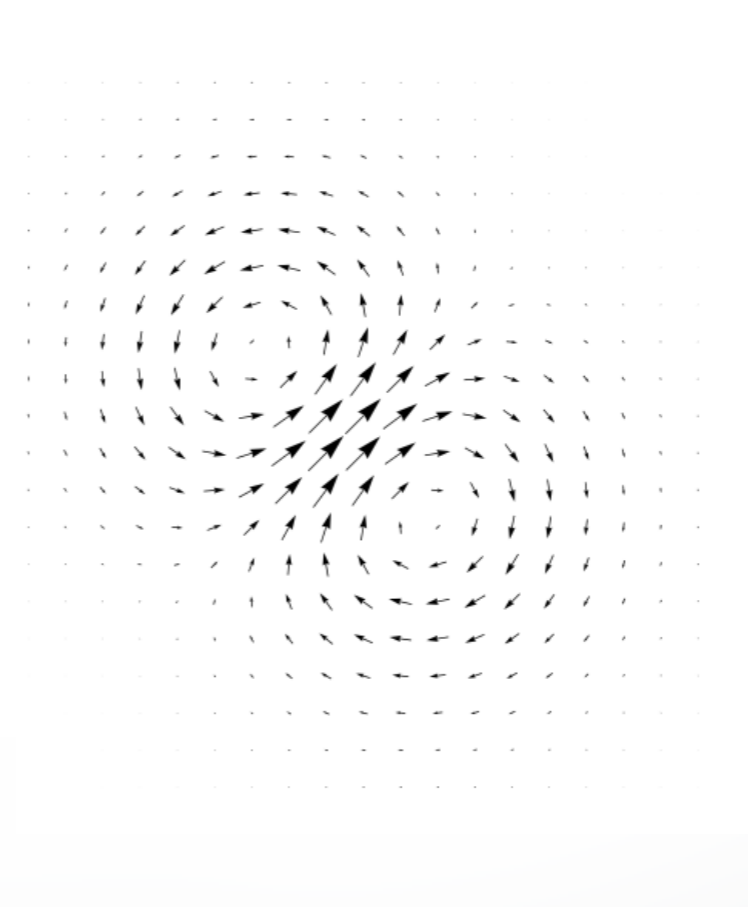
Differential Operators

Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

Laplace Operator

Laplace operator

function in Euclidean space

gradient operator

divergence operator

2nd partial derivatives

Cartesian coordinates

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Laplace-Beltrami Operator

Extension of Laplace of functions on manifolds

Laplace-
Beltrami

gradient
operator

...of the surface

$$\Delta_{\mathcal{S}} f = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on
manifold \mathcal{S}

divergence
operator

Laplace on the surface

Laplace-Beltrami Operator

Laplace-Beltrami

gradient operator

mean curvature

function on manifold \mathcal{S}

divergence operator

surface normal

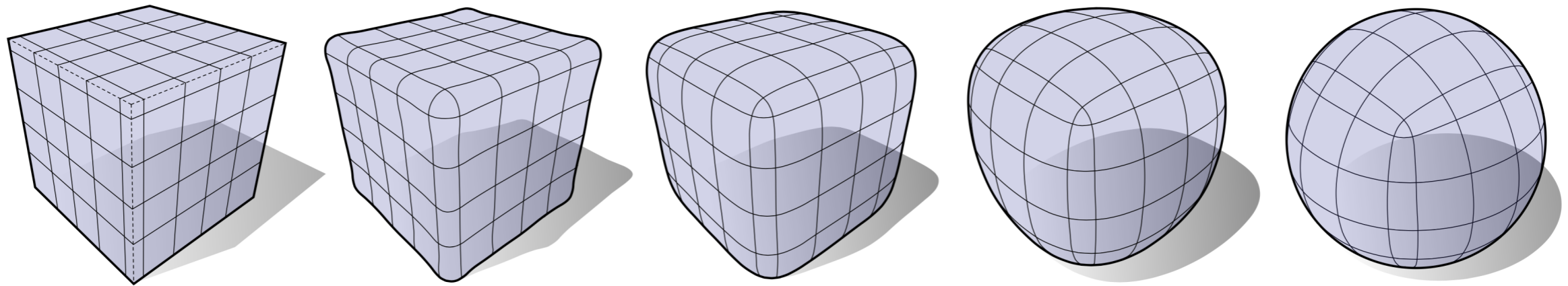
$$\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H \mathbf{n}$$

The diagram illustrates the Laplace-Beltrami operator equation $\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H \mathbf{n}$. It features six labels with arrows pointing to specific parts of the equation: 'Laplace-Beltrami' points to $\Delta_{\mathcal{S}}$; 'function on manifold \mathcal{S} ' points to \mathbf{x} ; 'divergence operator' points to $\operatorname{div}_{\mathcal{S}}$; 'gradient operator' points to $\nabla_{\mathcal{S}}$; 'mean curvature' points to H ; and 'surface normal' points to \mathbf{n} .

Literature

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- A. Pressley: **Elementary Differential Geometry**, Springer, 2010
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Next Time



Discrete Differential Geometry

<http://cs621.hao-li.com>

Thanks!

