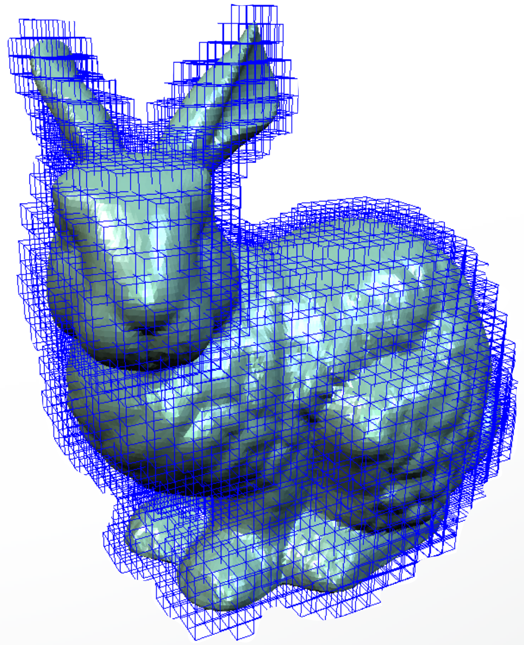
2.1 Explicit & Implicit Surfaces



Hao Li

http://cs621.hao-li.com

Administrative

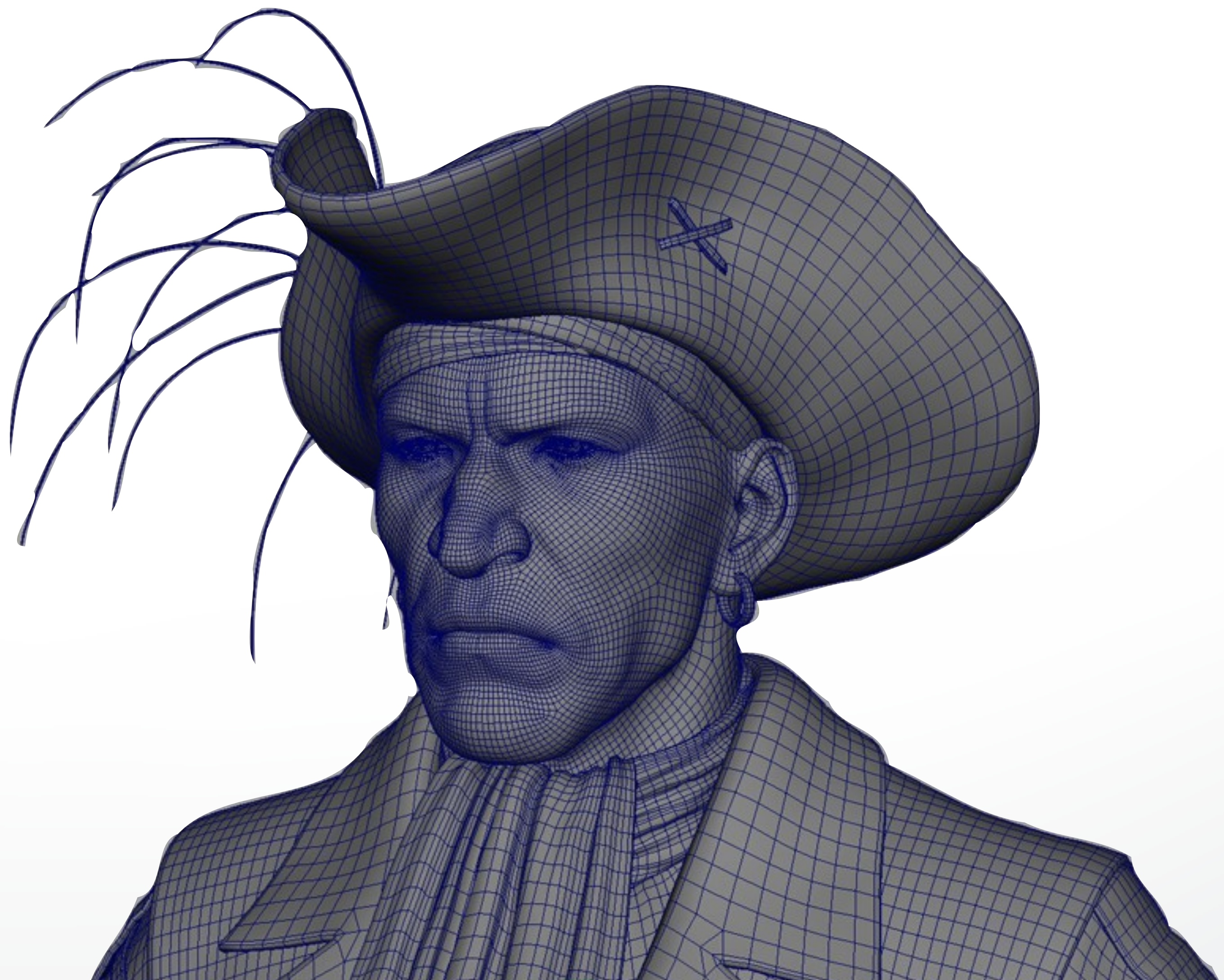
- **Exercise 1 discussion: Next Time!**
- **Hao Li (Instructor)**
  - Office Hour: Tue 12:30 PM - 1:30 PM, SAL 244

- **Zeng Huang (TA)**
  - Office Hour: TBD, PHE 108
  - zenghuan@usc.edu

Last Time

Polygonal meshes are

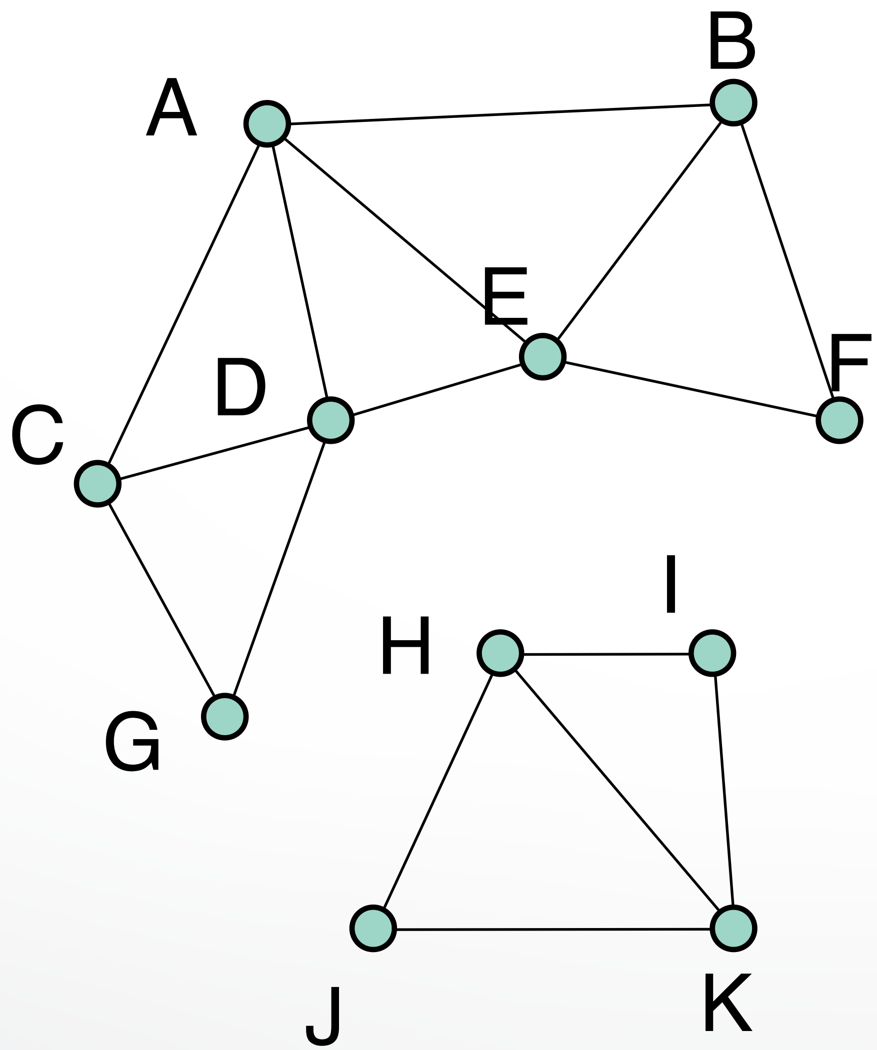
- Effective representations
- Flexible
- Efficient, simple, enables unified processing

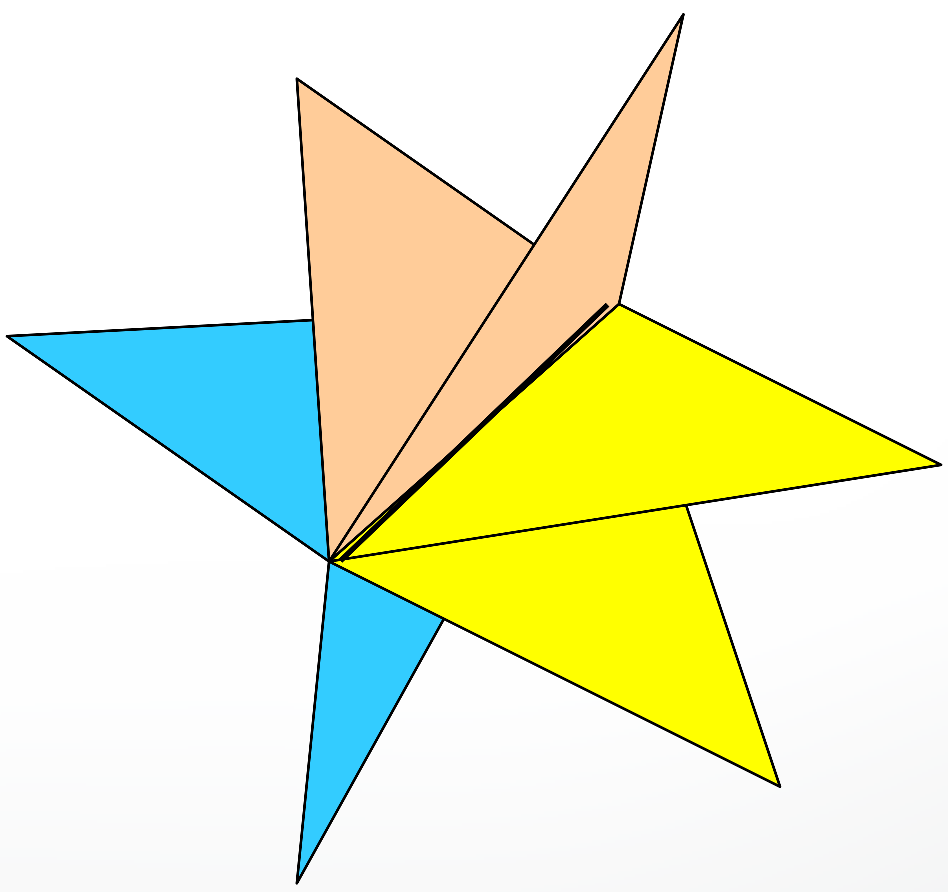


Last Time

Connection between Meshes and Graphs

- Formalism (valence, connections, subgraph, embedding…)
- Definitions (boundary, regular edge, singular edge, closed mesh)
- triangulation → triangle mesh

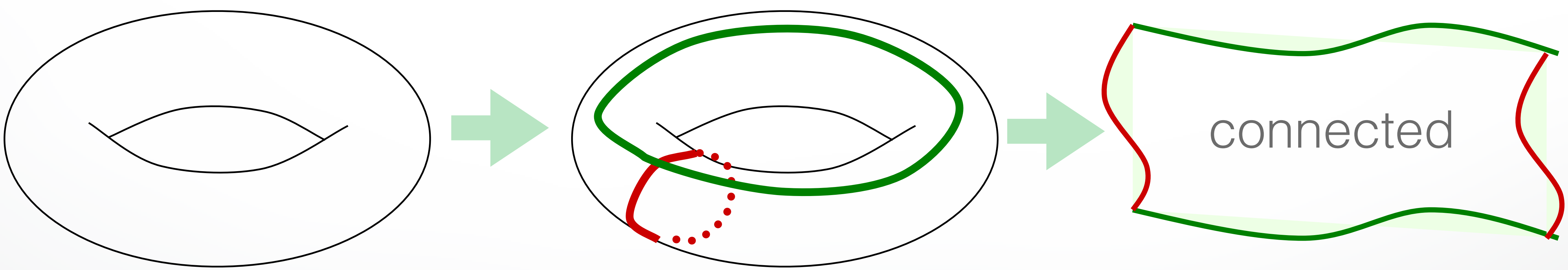




Last Time

Topology

- Genus, Euler characteristic
- Euler Poincaré formula $V - E + F = 2(1 - g)$
- Average valence of triangle mesh: 6
- Triangles: F = 2V, E = 3V
- Quads: F = V, E = 2V

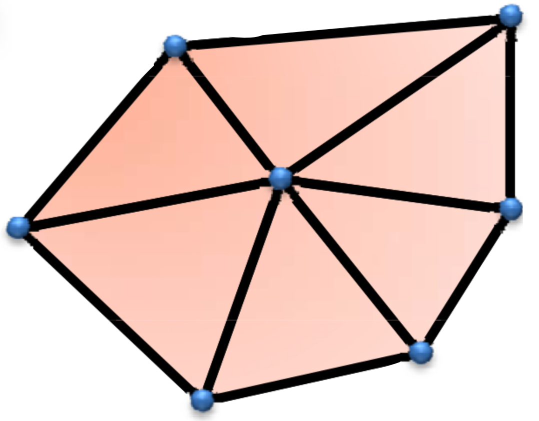


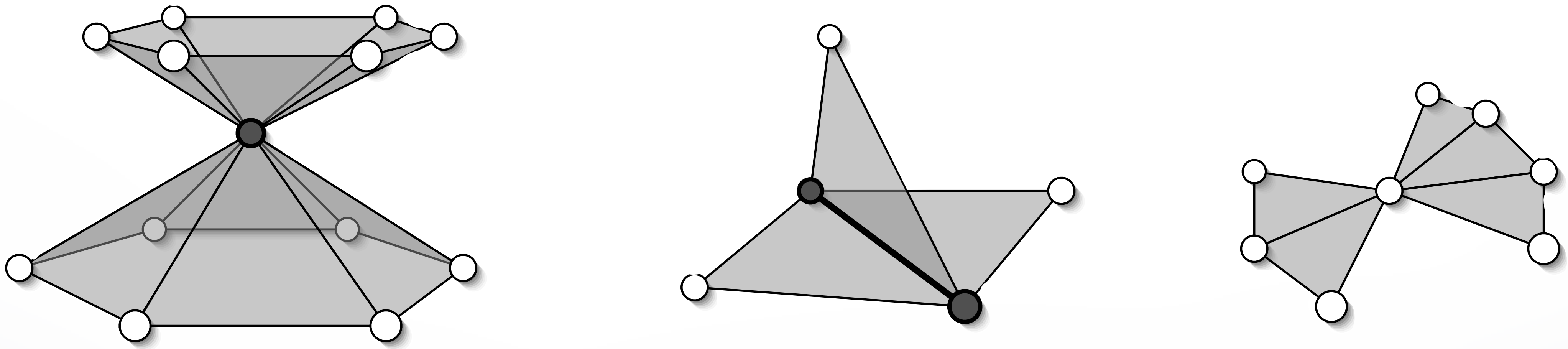
k=1 handle ≤2k edge loops

Last Time

2-Manifold Surface

- Local Neighborhood is disk-shaped $\mathbf{f}(D_\epsilon[u,v]) = D_\delta[\mathbf{f}(u,v)]$
- Guarantees meaningful neighbor enumeration
- Non-manifold

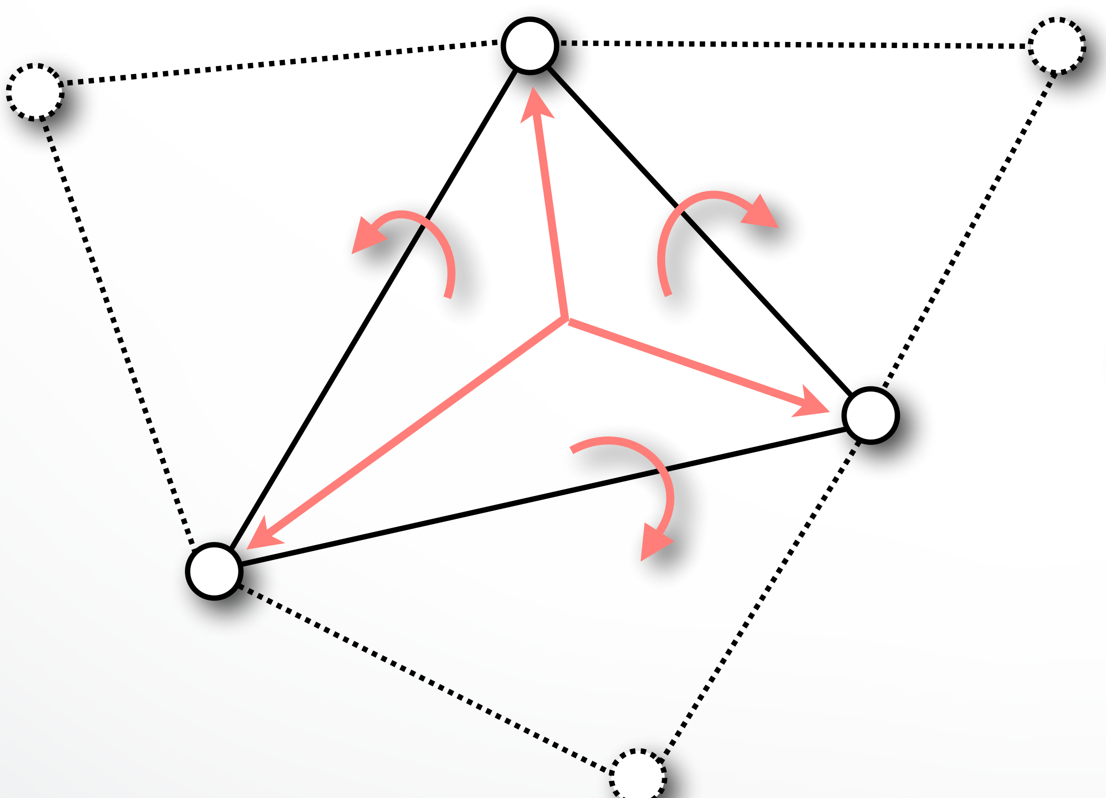


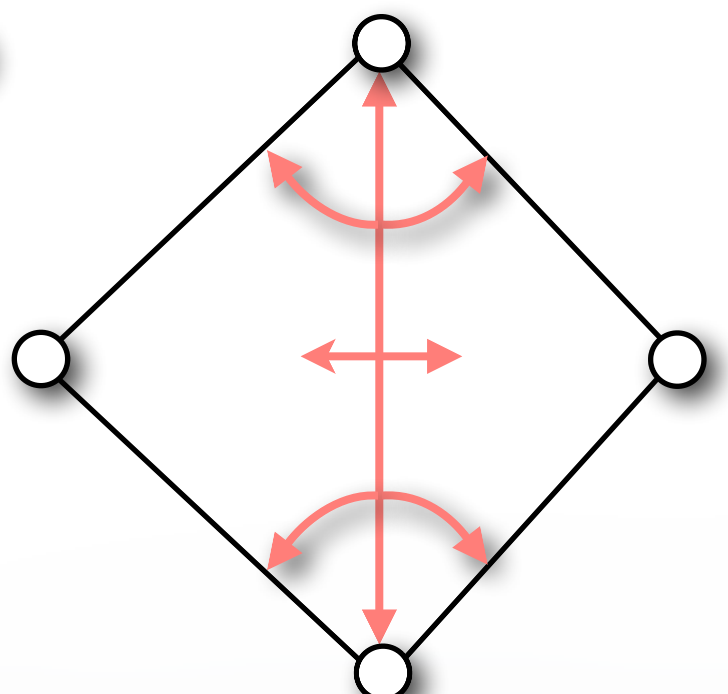


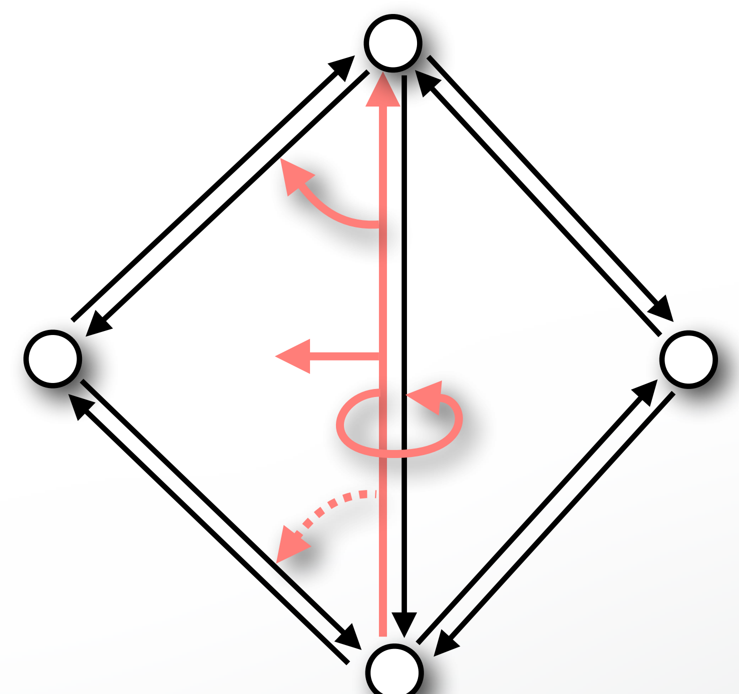
Last Time

Data Structures

- Face-Based
- Edge-Based, edges always have two faces
- Halfedge-Based

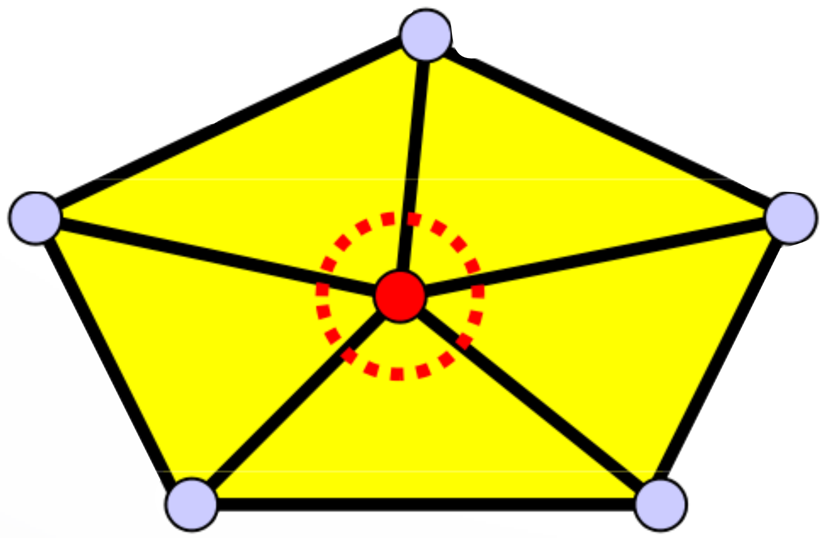




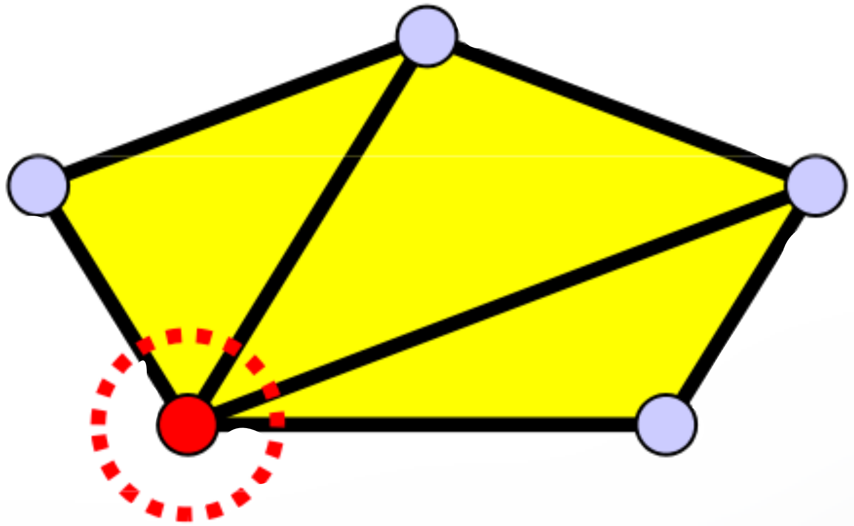


When is a Triangle Mesh a Manifold?

- Every Edge incident to 1 or 2 Triangles
- Faces incident to a vertex form closed or open fan



closed fan



open fan

Outline

- **Surface Representations**
- Explicit Surfaces
- Implicit Surfaces
- Conversion

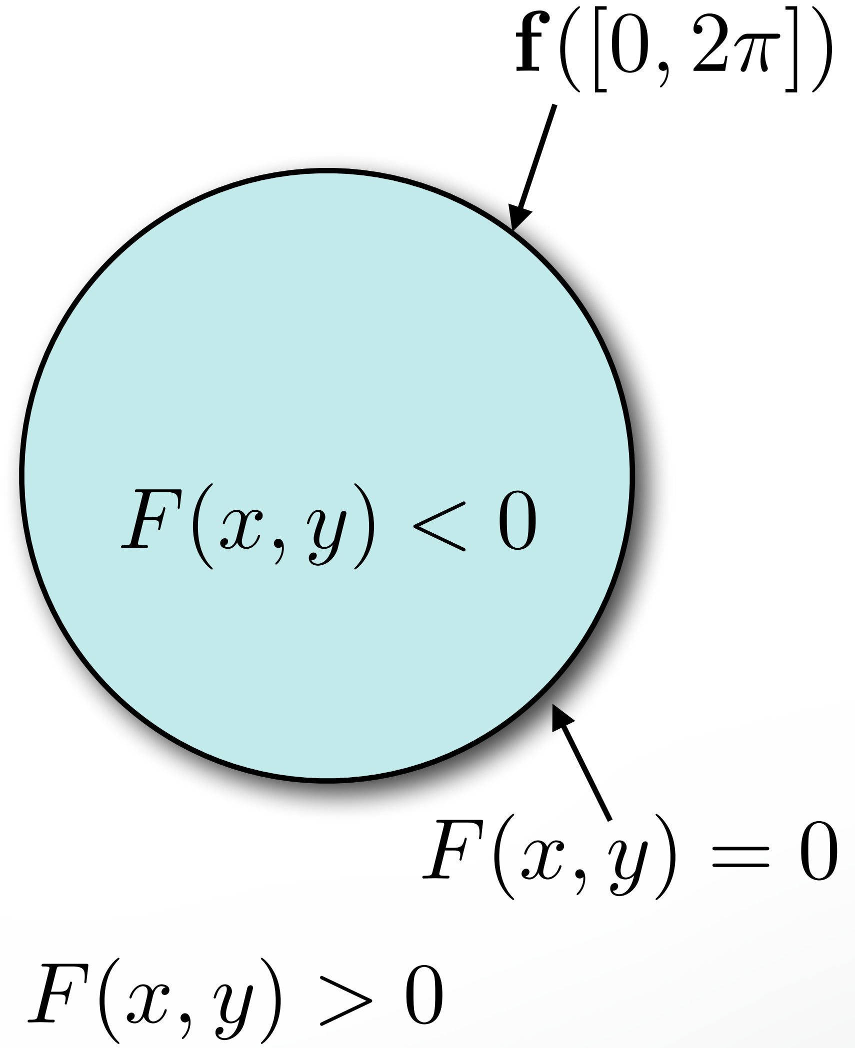
Explicit vs. Implicit

Explicit: $\mathbf{f}(x) = (r\cos(x), r\sin(x))^T$

- Range of parameterization function

Implicit: $F(x,y) = \sqrt{x^2 + y^2} - r$

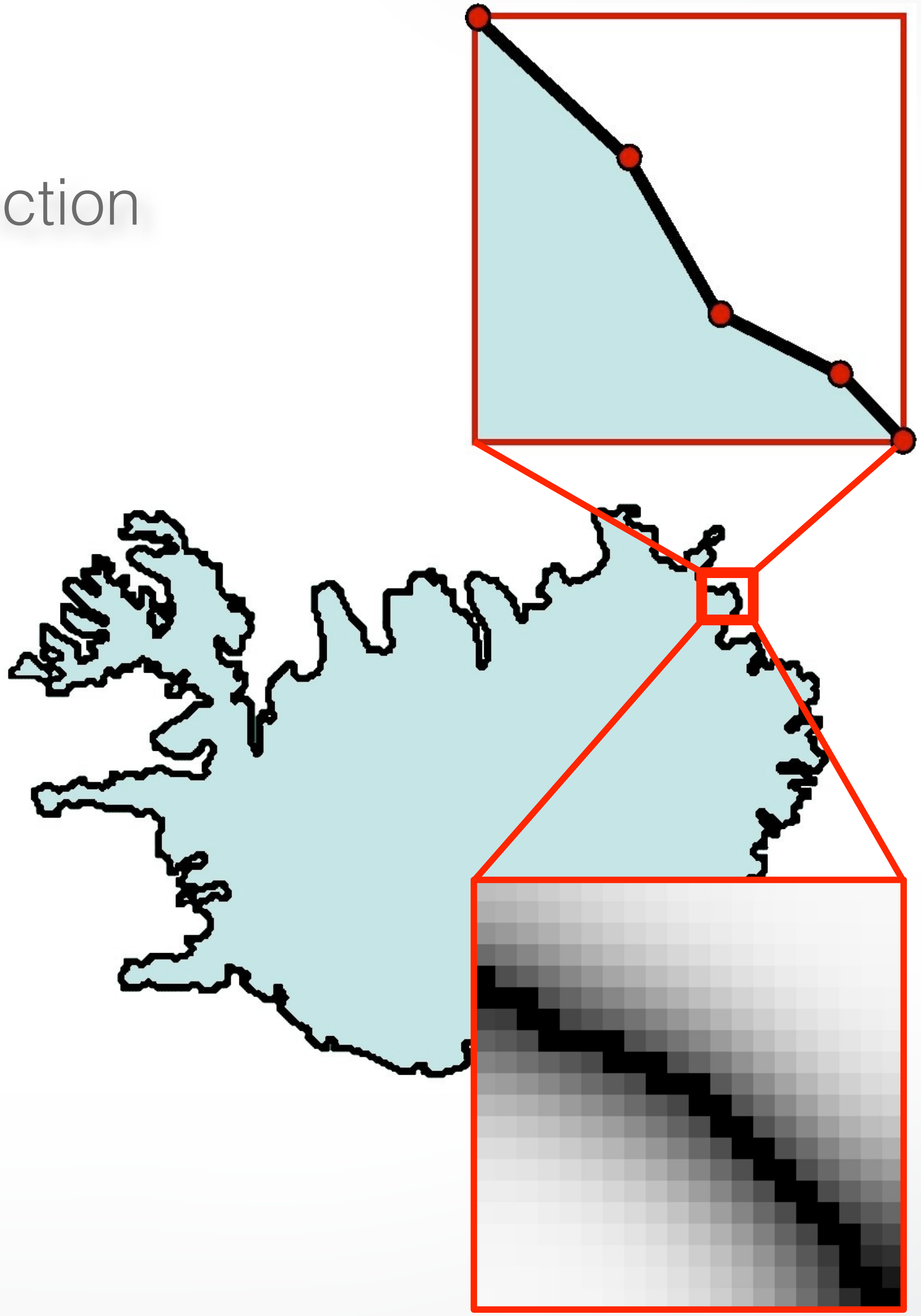
- Kernel of implicit function



Explicit vs. Implicit

Explicit: $\mathbf{f}(x) =$?

- Range of parameterization function
- Piecewise approximation



Implicit: $F(x, y) =$?

- Kernel of implicit function
- Piecewise approximation

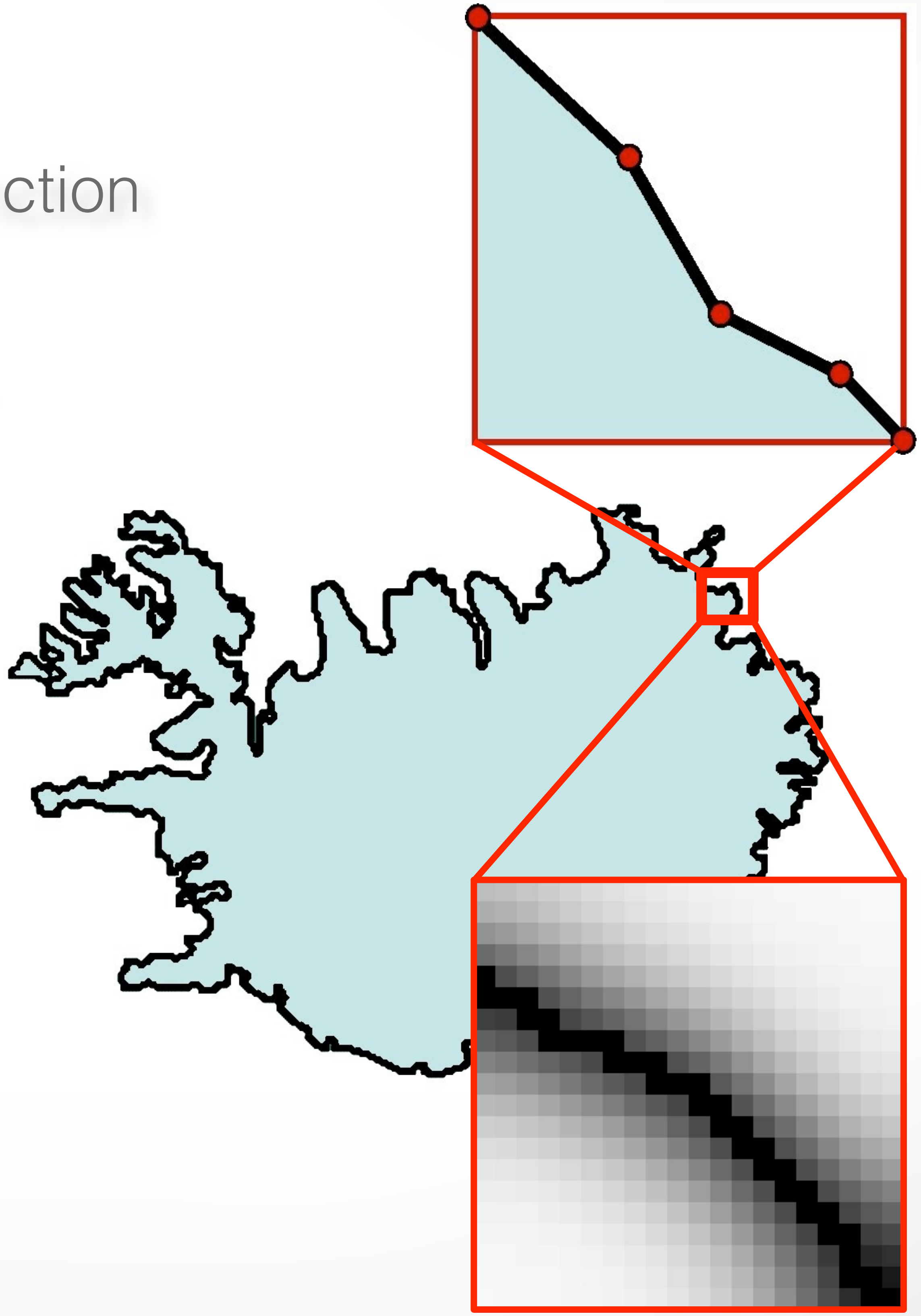
Explicit vs. Implicit

Explicit:

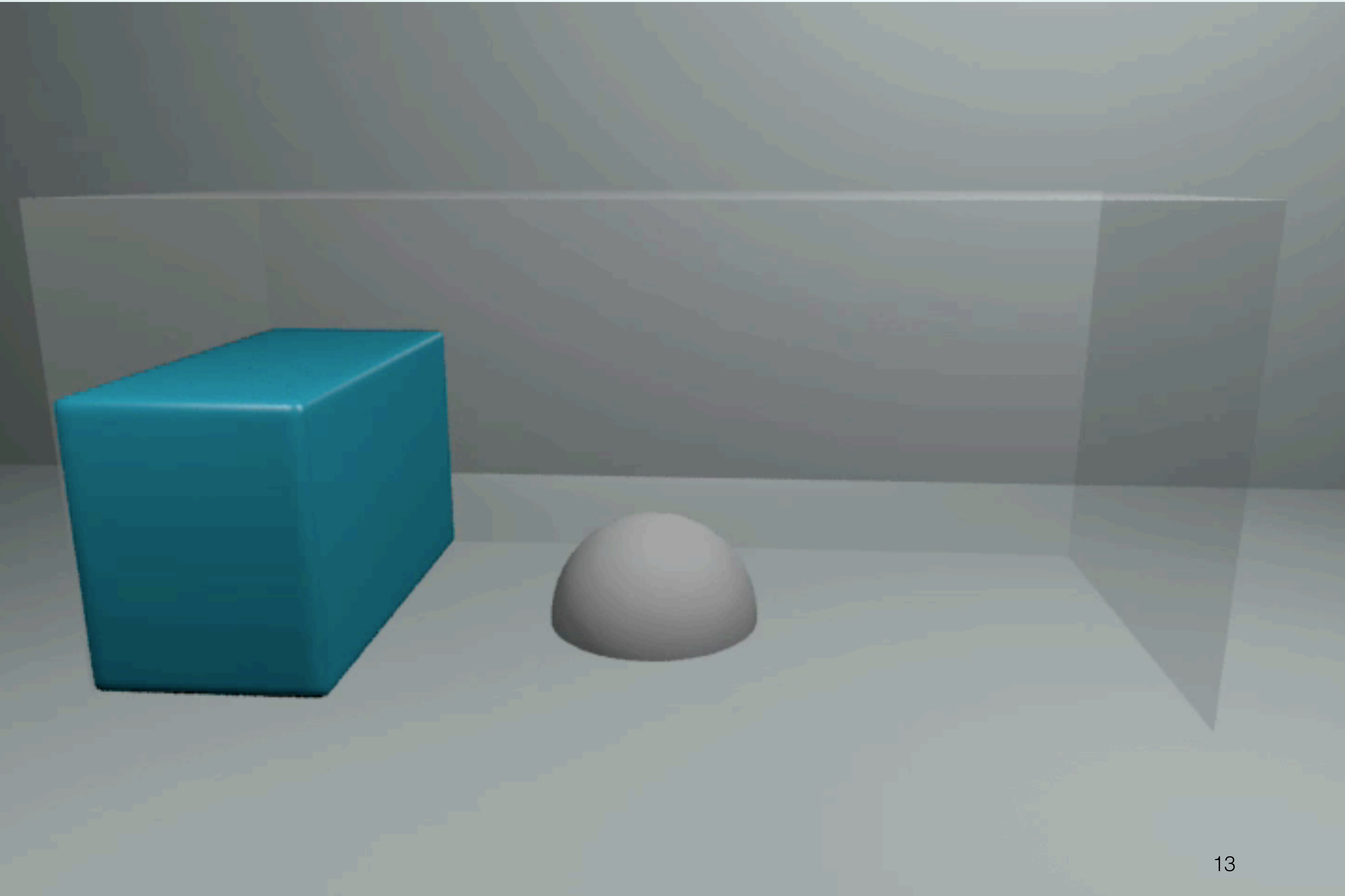
- Range of parameterization function
- Piecewise approximation
- Splines, triangle mesh, points
- Easy enumeration
- Easy geometry modification

Implicit:

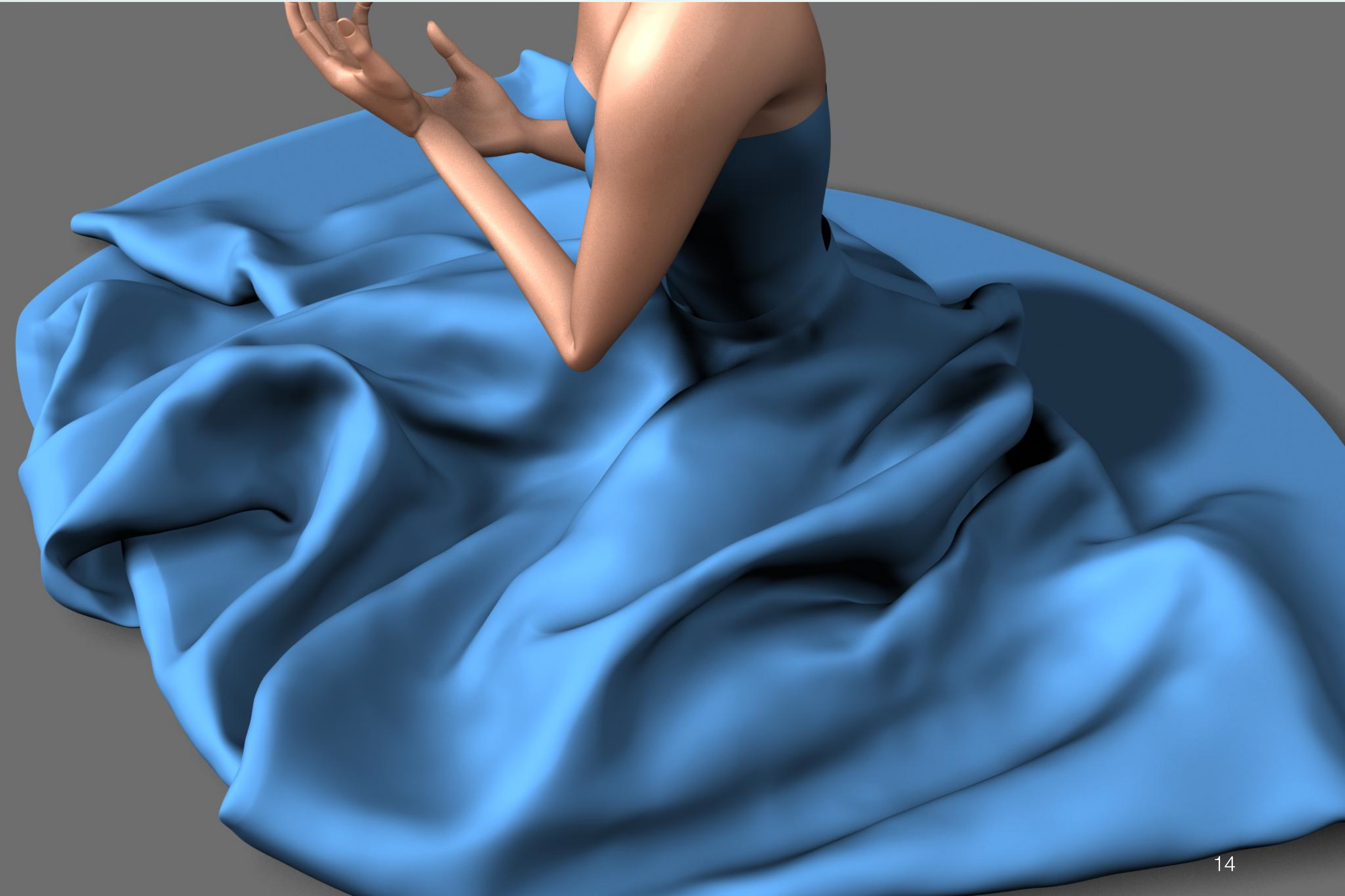
- Kernel of implicit function
- Piecewise approximation
- Scalar-valued 3D grid
- Easy in/out test
- Easy topology modification



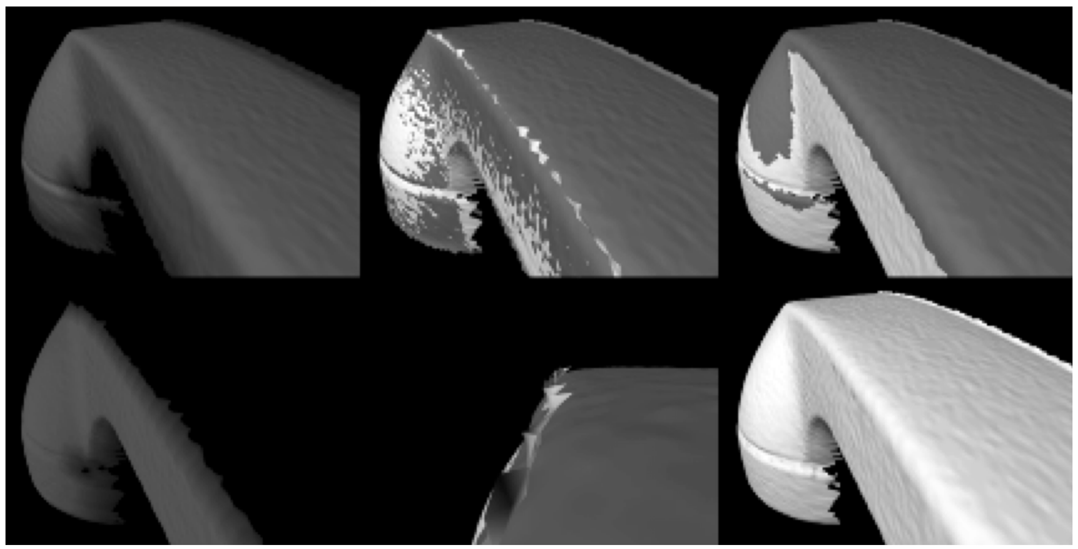
Examples: Fluid Simulation

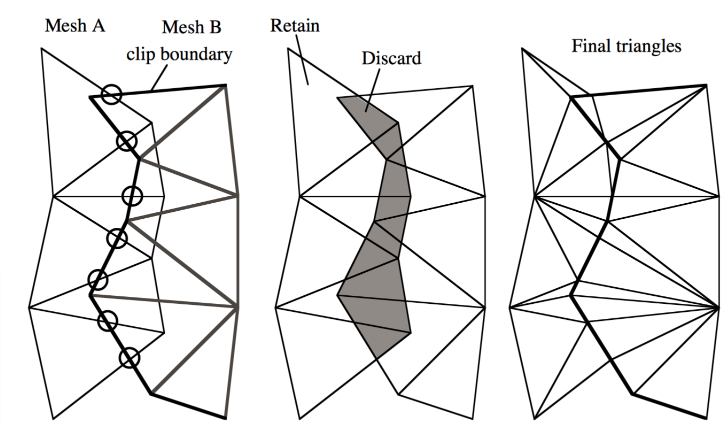


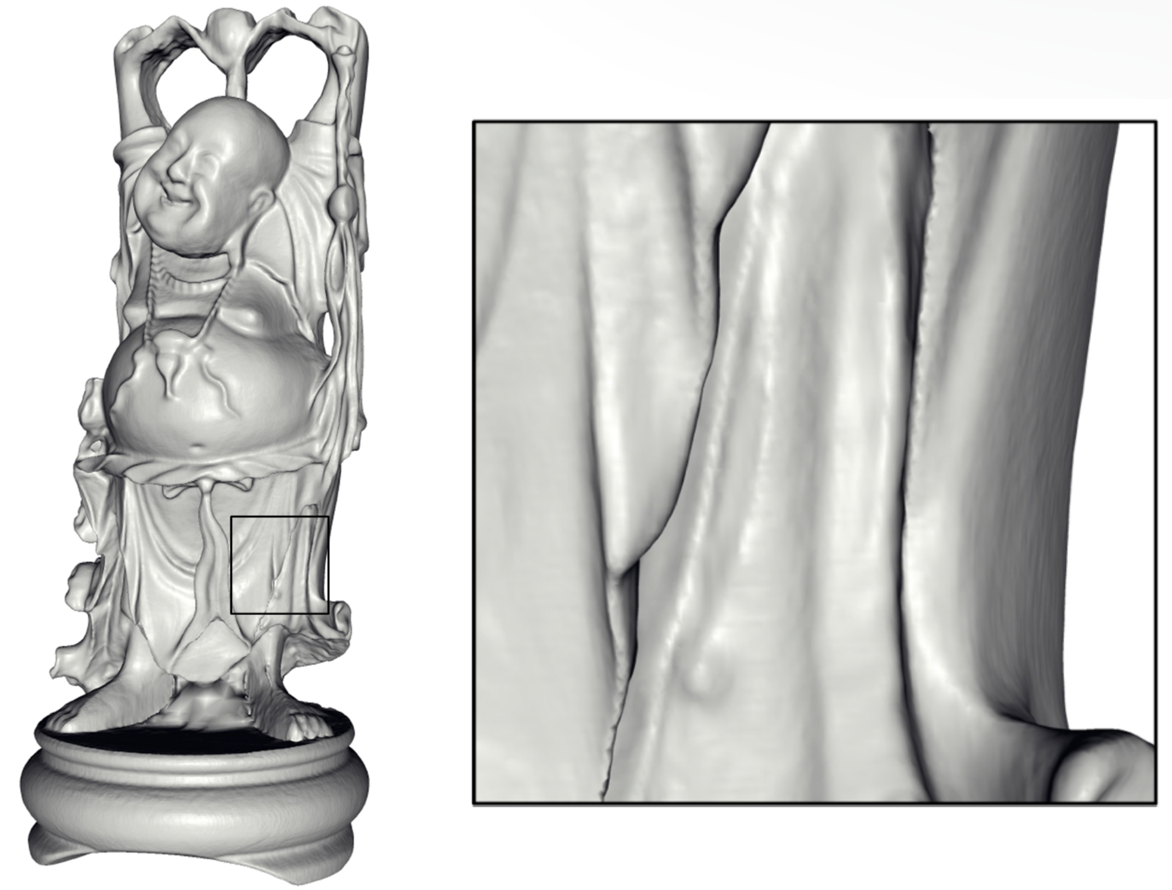
Examples: Collisions

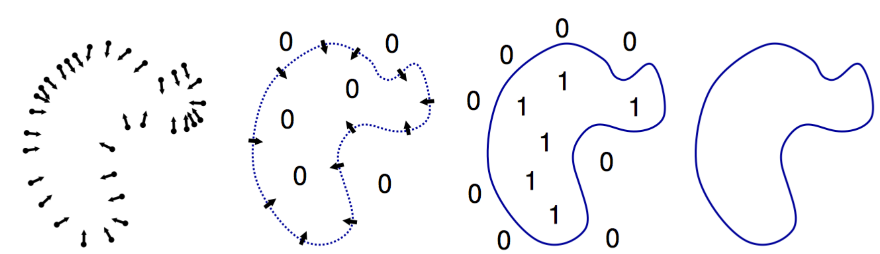


Examples: 3D Reconstruction









Zippering

Poisson Reconstruction

Examples: Kinect Fusion

1. Capture
2. Align
3. Fuse

http://msdn.microsoft.com/en-us/library/dn188670.aspx

Outline

- Surface Representations
- **Explicit Surfaces**
- Implicit Surfaces
- Conversion

Polynomial Approximation

Polynomials are computable functions

$$f(t) \;=\; \sum_{i=0}^{p} c_i\, t^i \;=\; \sum_{i=0}^{p} \tilde{c}_i\, \phi_i(t)$$

Taylor expansion up to degree p

$$g(h) \;=\; \sum_{i=0}^{p} \frac{1}{i!}\, g^{(i)}(0)\; h^i \;+\; O\big(h^{p+1}\big)$$

Error for approximation g by polynomial f

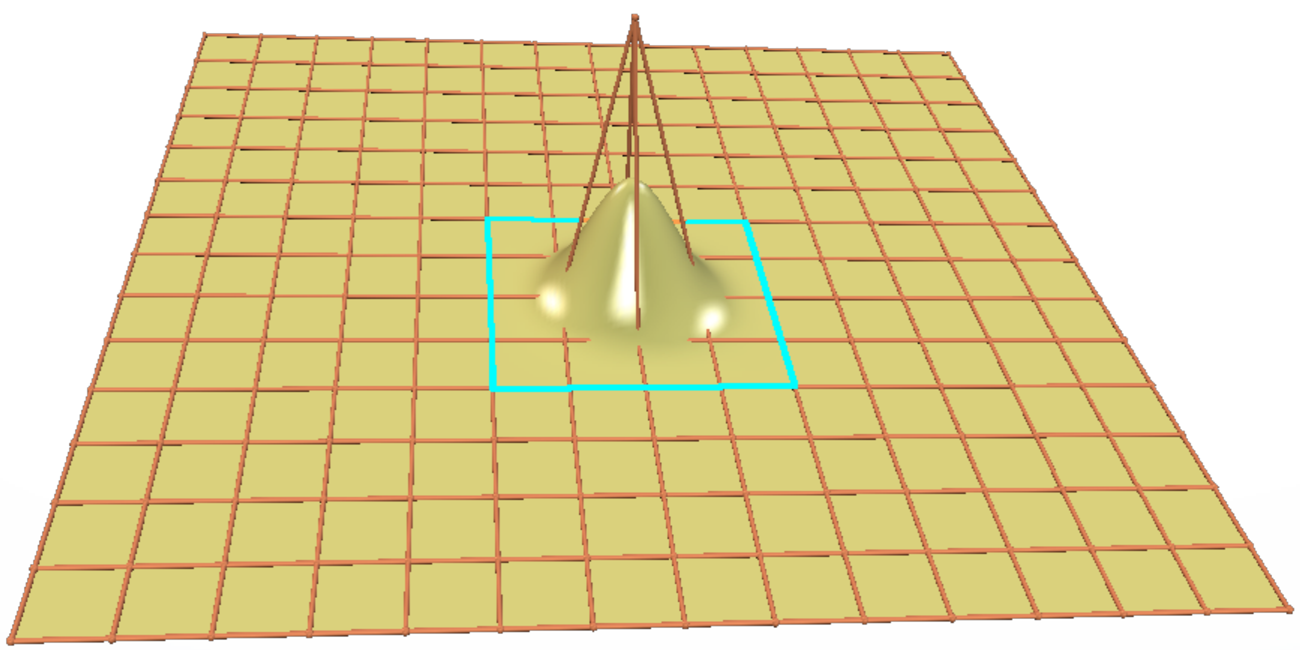
$$f(t_i) = g(t_i)\,, \quad 0 \le t_0 < \cdots < t_p \le h$$

$$|f(t) - g(t)| \;\le\; \frac{1}{(p+1)!}\; \max f^{(p+1)} \prod_{i=0}^{p} (t - t_i) \;=\; O\Big(h^{(p+1)}\Big)$$

Spline Surfaces

Piecewise polynomial approximation

$$\mathbf{f}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{c}_{ij} N_i^n(u) N_j^m(v)$$

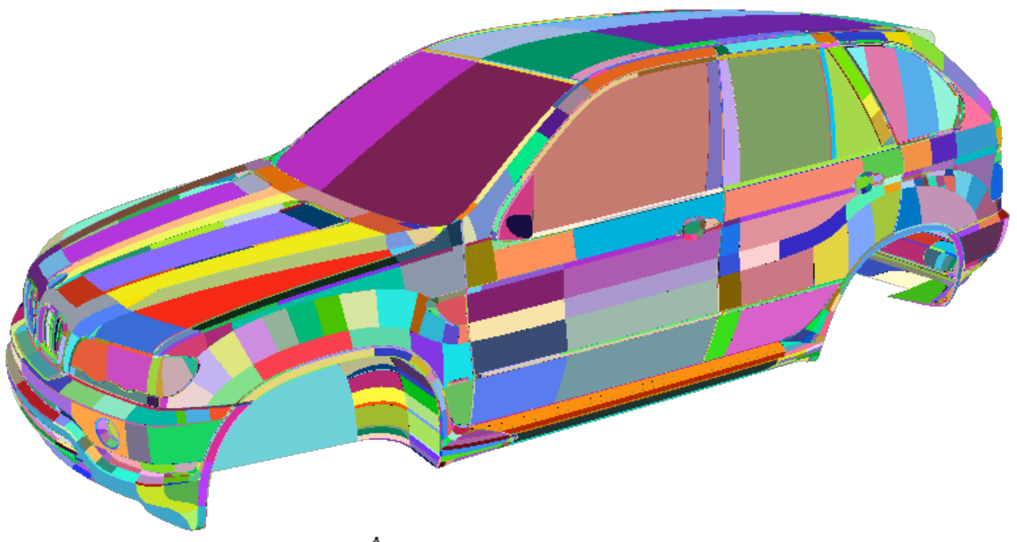


Spline Surfaces

Piecewise polynomial approximation

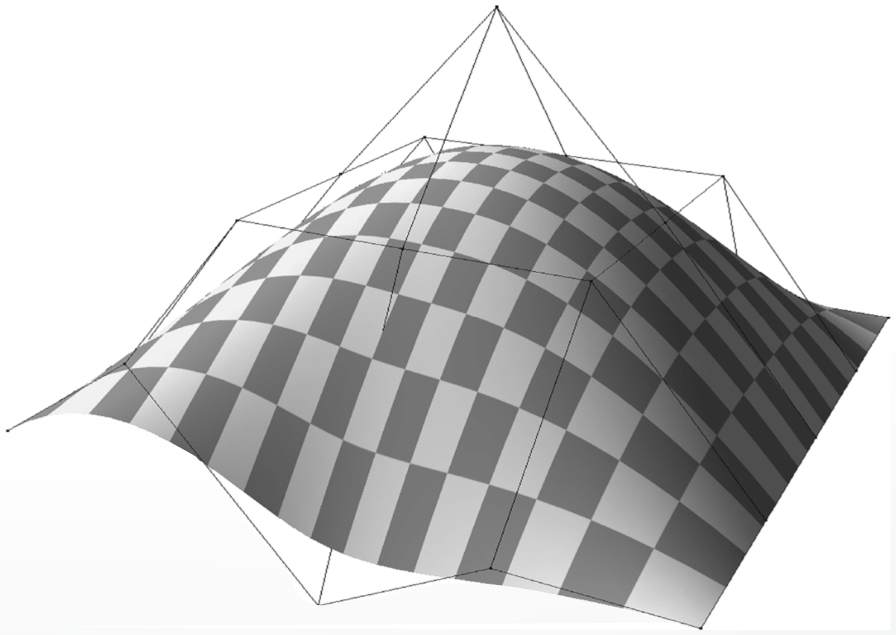
Geometric constraints

- Large number of patches
- Continuity between patches
- Trimming



Topological constraints

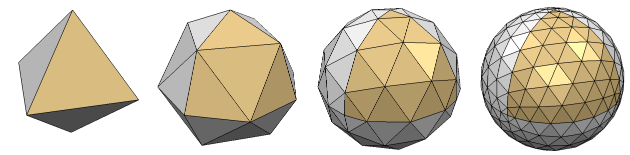
- Rectangular patches
- Regular control mesh

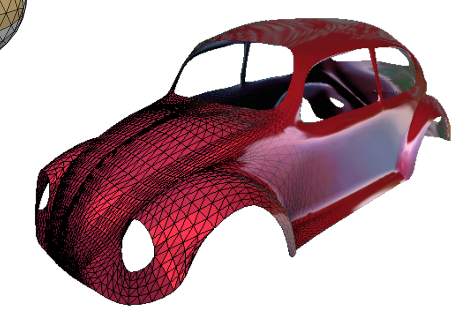


Polygon Meshes

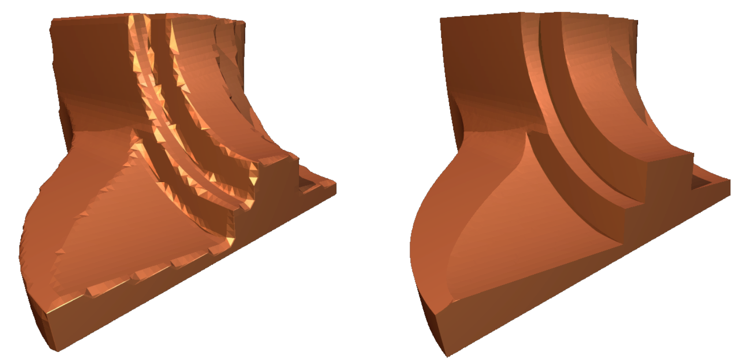
Polygonal meshes are a good compromise

- Piecewise linear approximation → error is $O(h^2)$
- Error inversely proportional to #faces

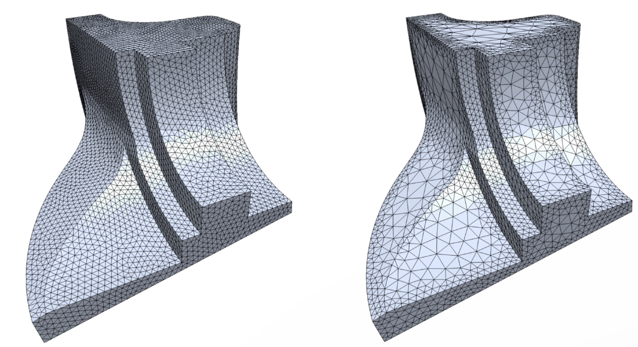




- Arbitrary topology surfaces



- Piecewise smooth surfaces



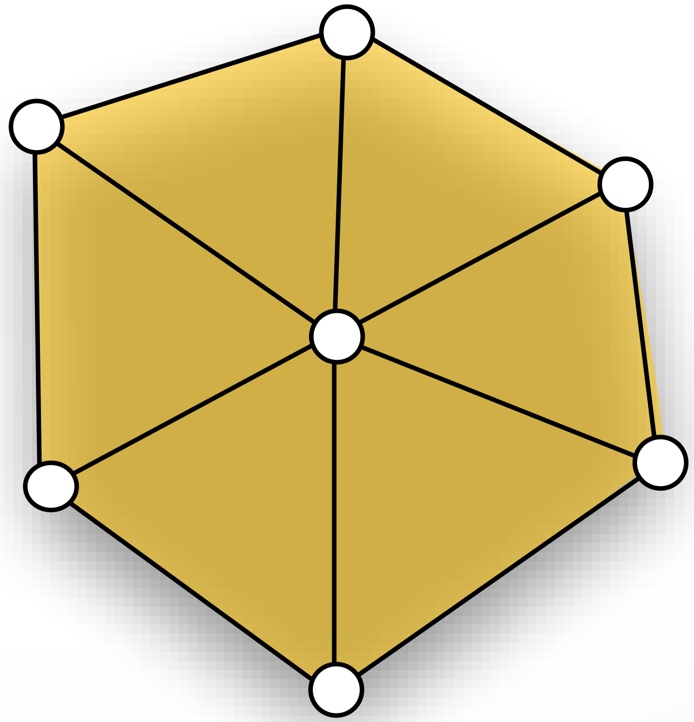
- Adaptive sampling
- Efficient GPU-based rendering/processing

Triangle Meshes

$$\mathcal{M} = (\{\mathbf{v}_i\}, \{e_j\}, \{f_k\})$$

geometry $\mathbf{v}_i \in \mathbb{R}^3$

topology $e_i, f_i \subset \mathbb{R}^3$

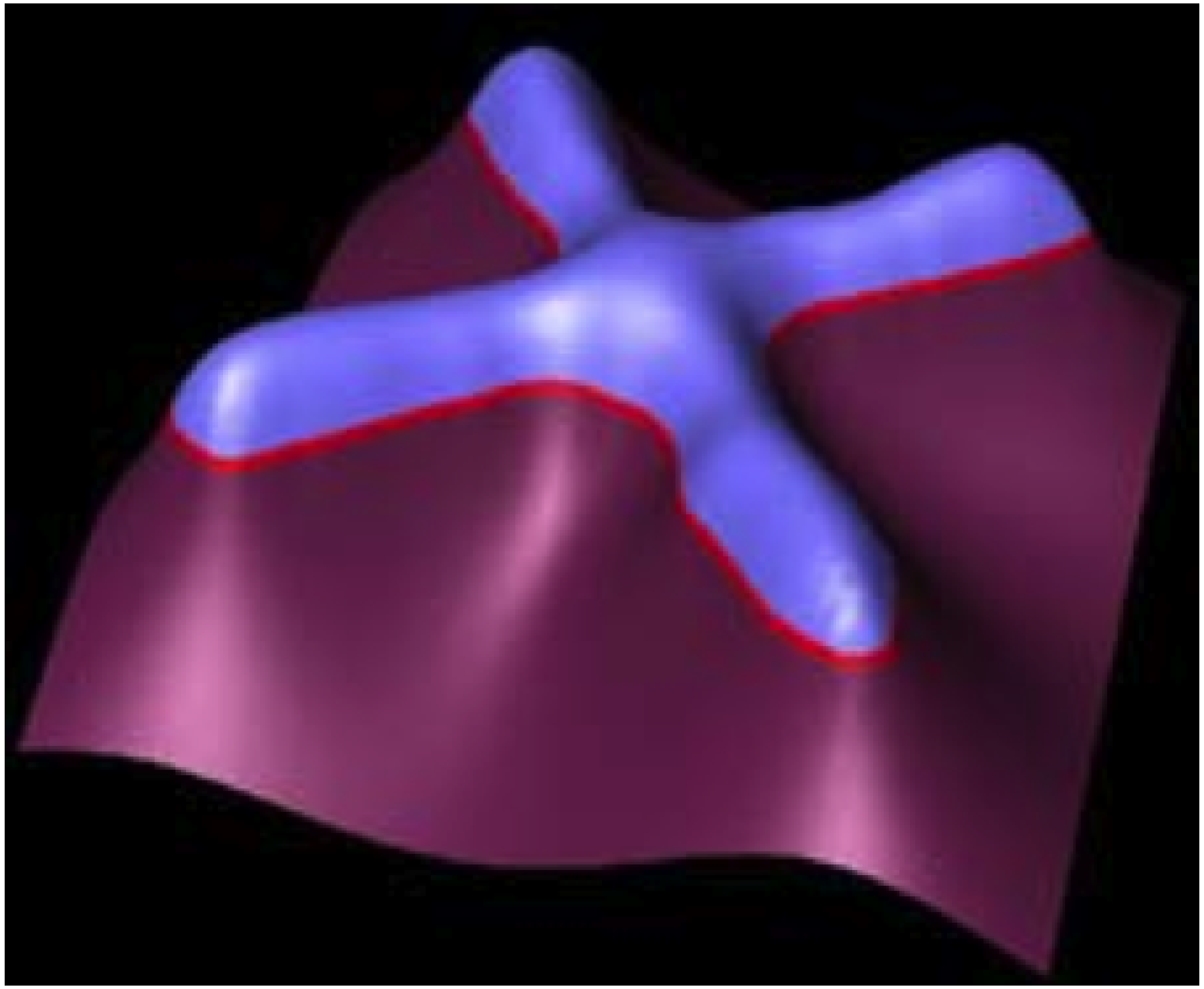


Outline

- Surface Representations
- Explicit Surfaces
- **Implicit Surfaces**
- Conversion

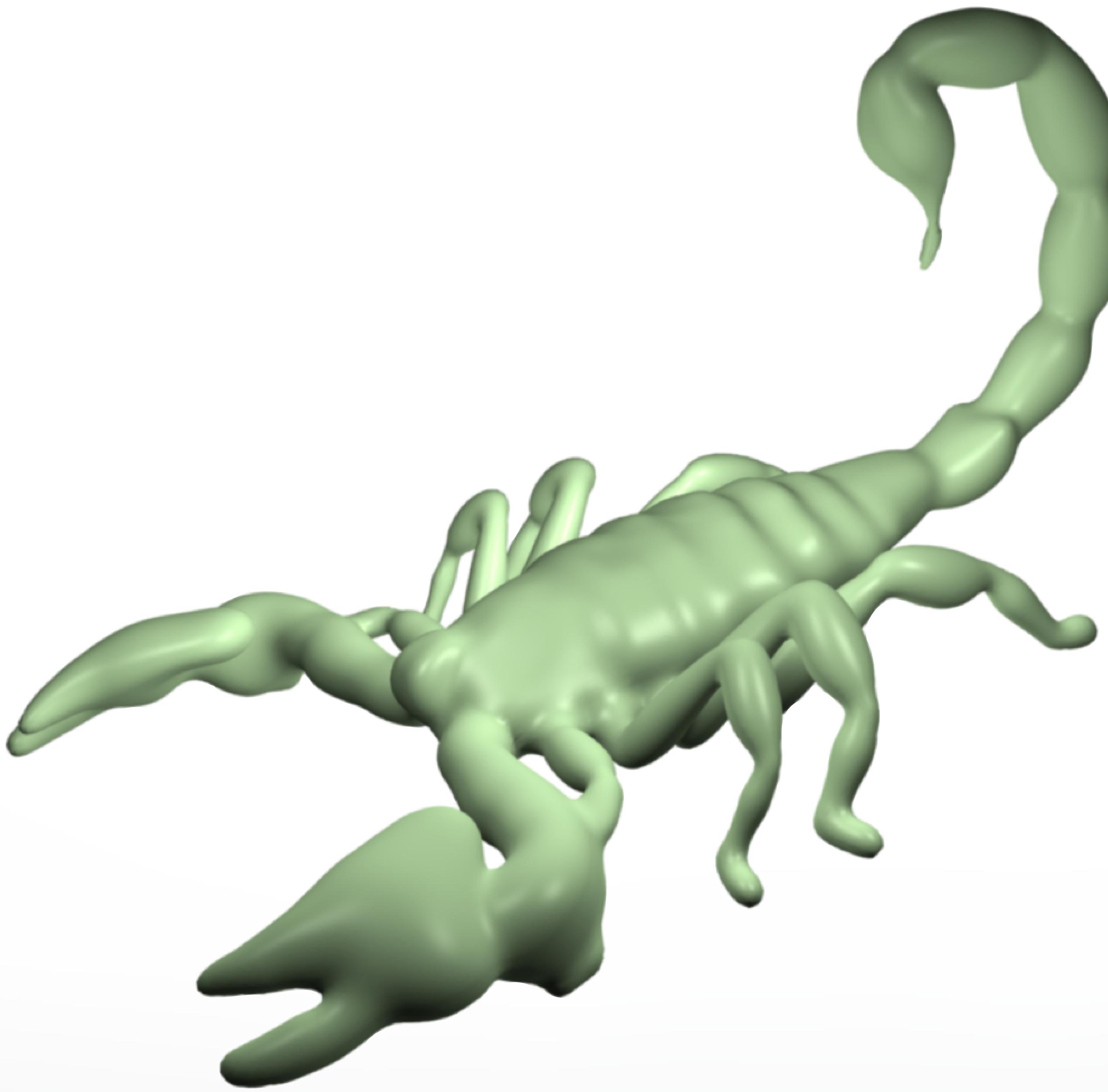
Implicit Representations

Level set of 2D function defines 1D curve



Implicit Representations

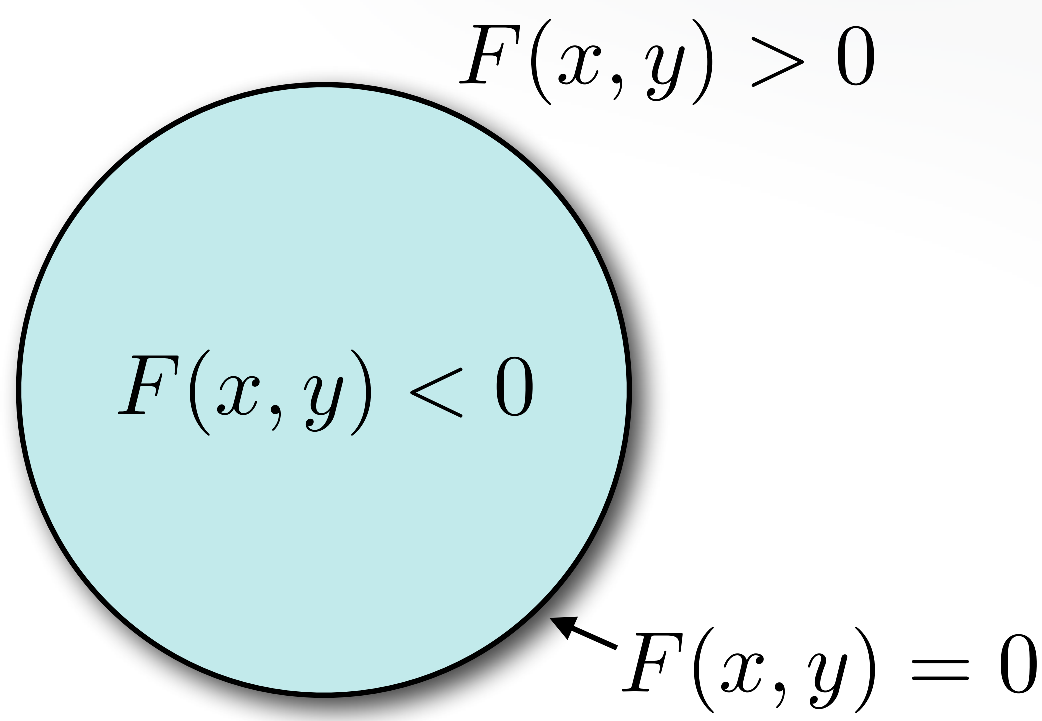
Level set of 3D function defines 2D surface



Implicit Representations

General implicit function:

- Interior: $F(x, y, z) < 0$
- Exterior: $F(x, y, z) > 0$
- Surface: $F(x, y, z) = 0$



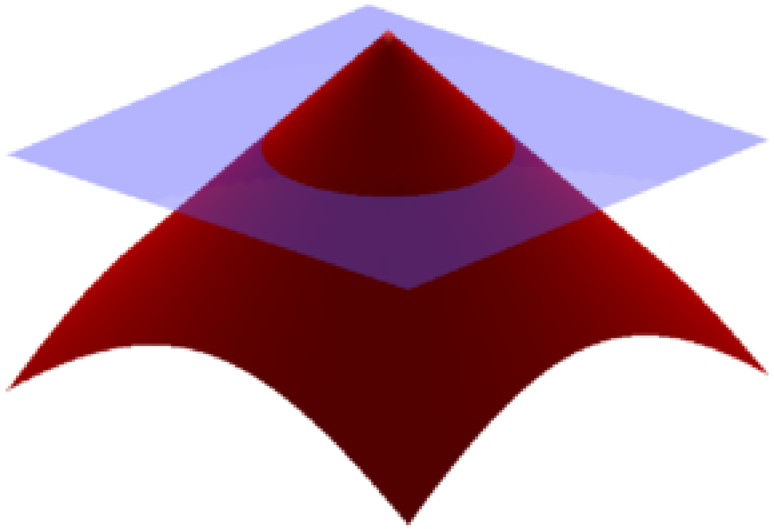
Gradient ∇F is orthogonal to level set

Special case

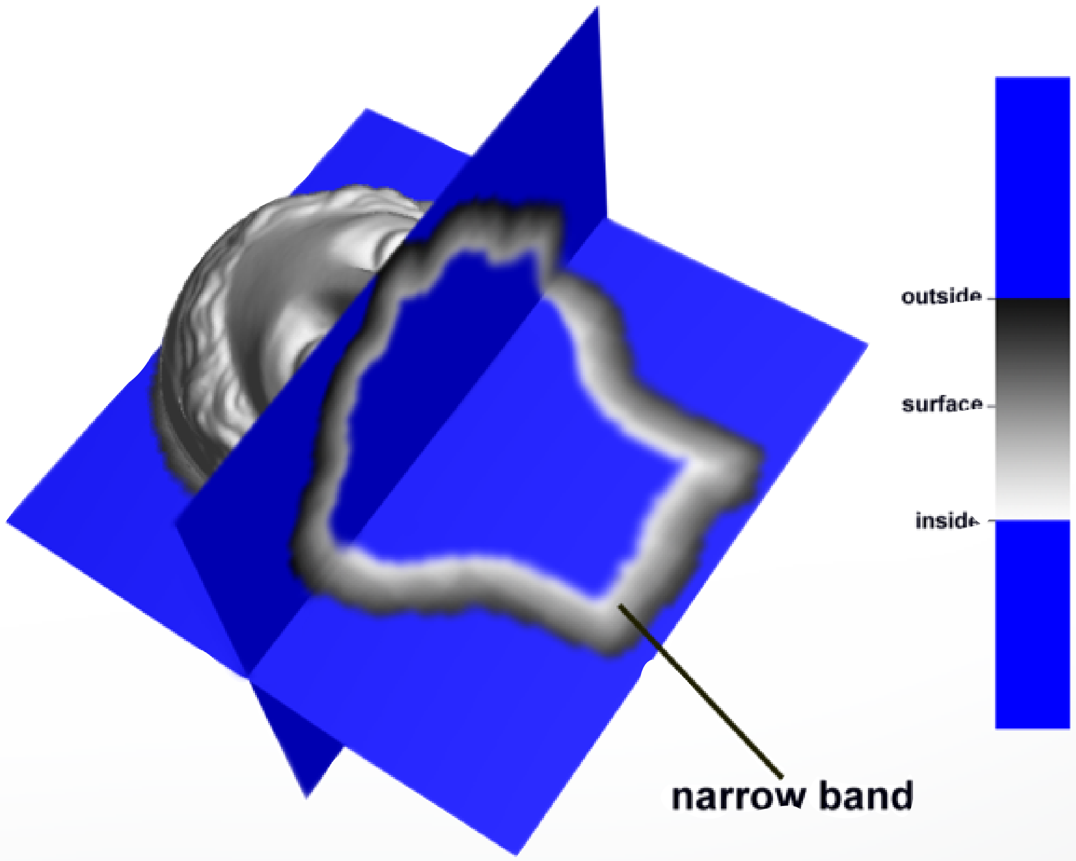
- Signed distance function (SDF)
- Gradient ∇F is unit surface normal

Signed Distance Function

SDF of a circle?



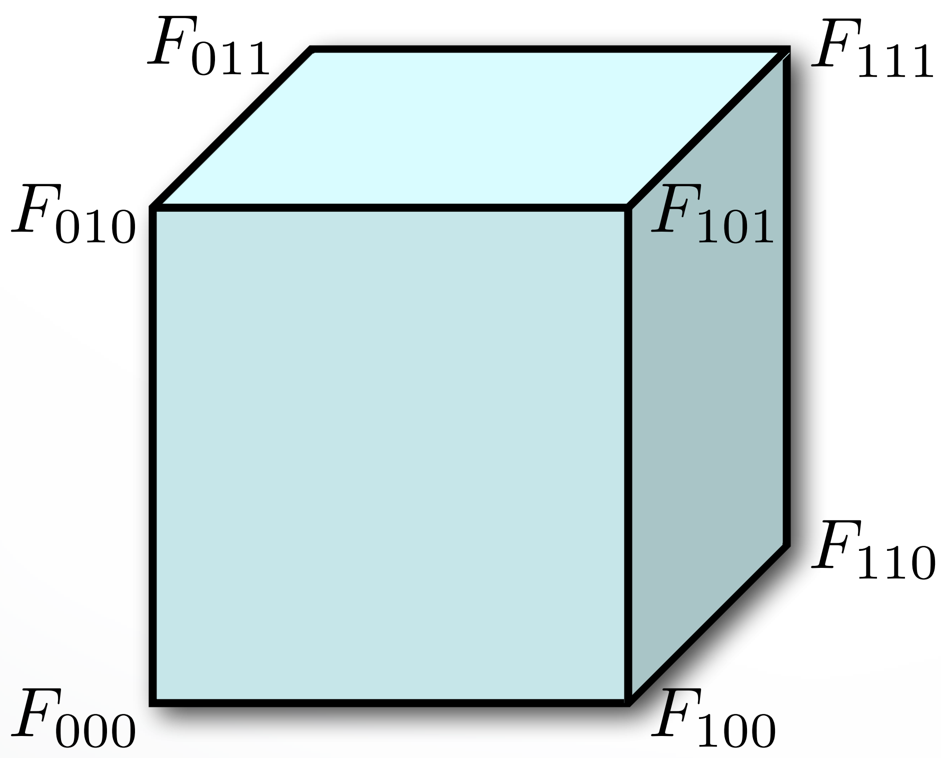
General shapes



SDF Discretization

Regular cartesian 3D grid

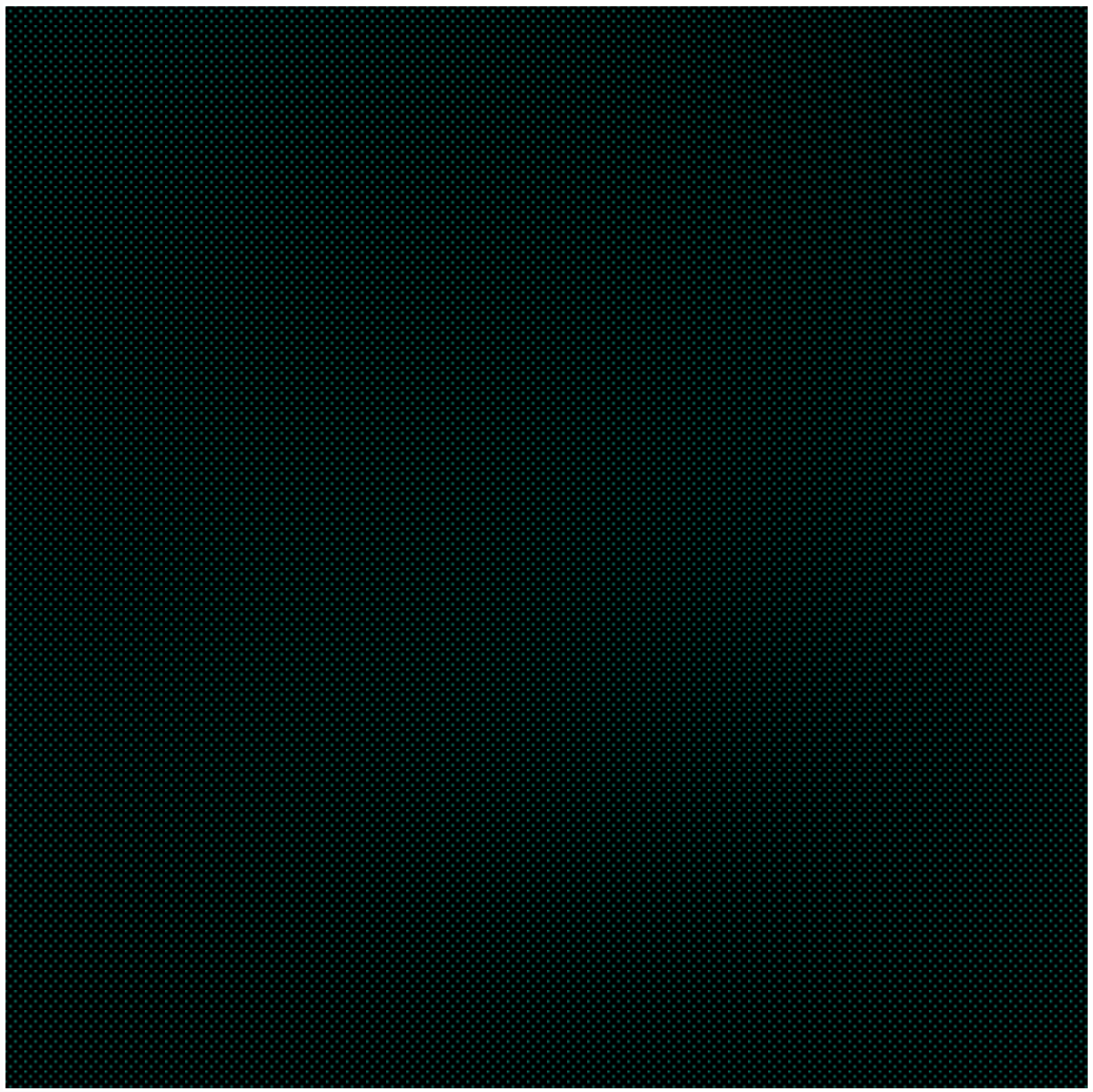
- Compute signed distance at nodes
- Tri-linear interpolation within cells



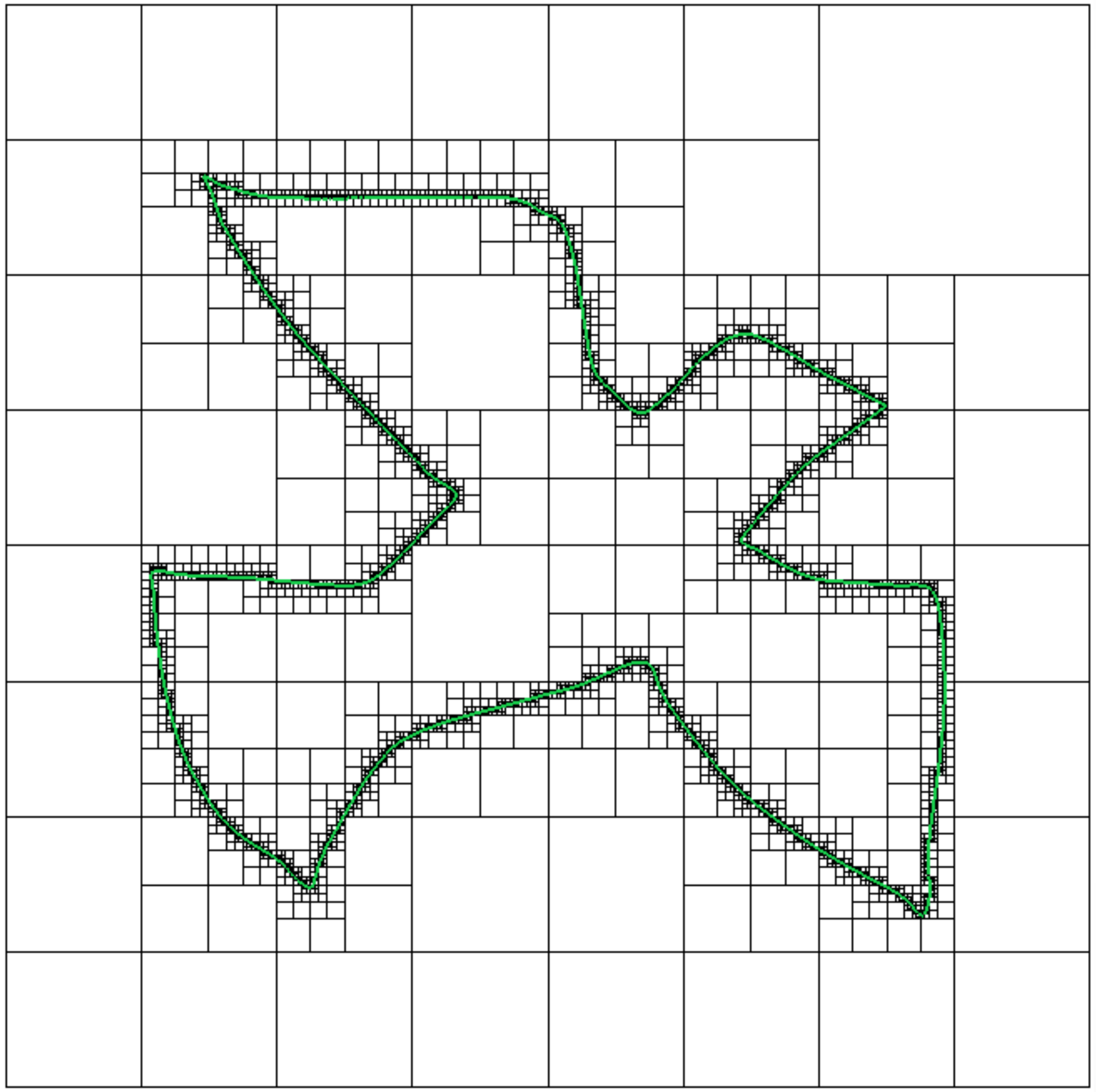
$$
\begin{array}{lcccc}
F_{000} & (1-u) & (1-v) & (1-w) & + \\
F_{100} & u & (1-v) & (1-w) & + \\
F_{010} & (1-u) & v & (1-w) & + \\
F_{001} & (1-u) & (1-v) & w & + \\
\vdots & & & & \\
F_{111} & u & v & w &
\end{array}
$$

3-Color Octree

3 Colors: **interior,exterior,boundary**

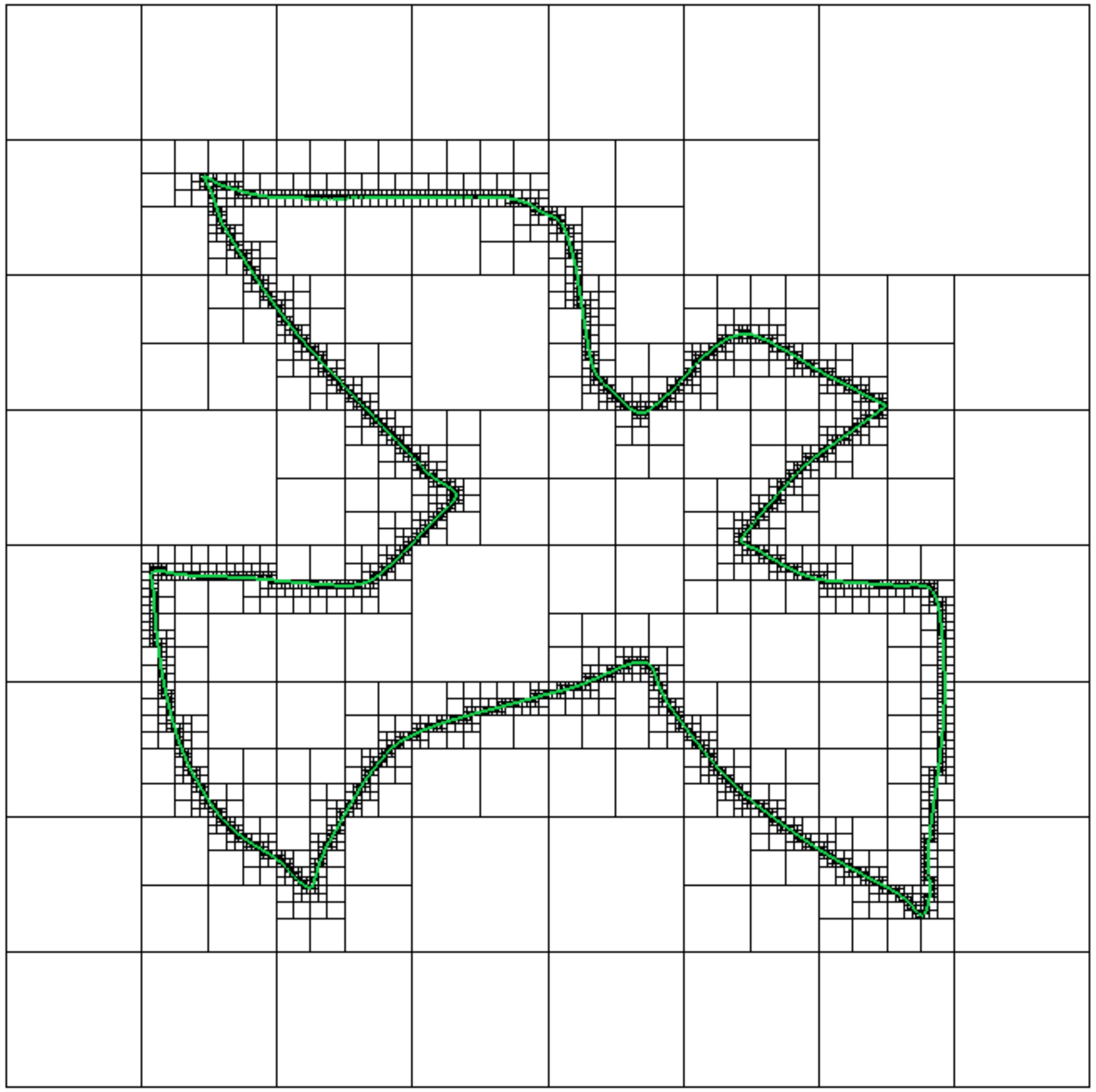


1048576 cells

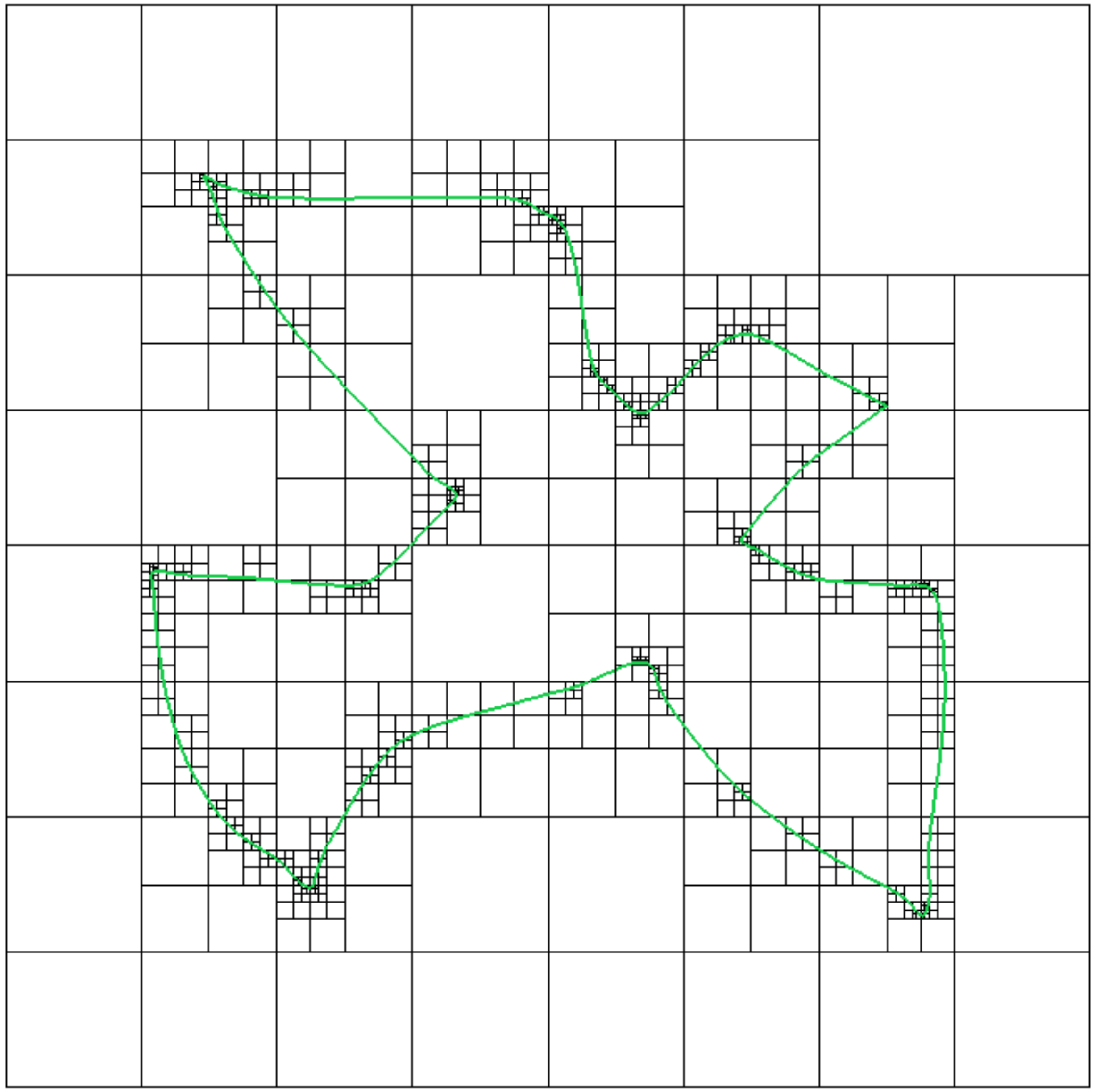


12040 cells

Adaptively Sampled Distance Fields

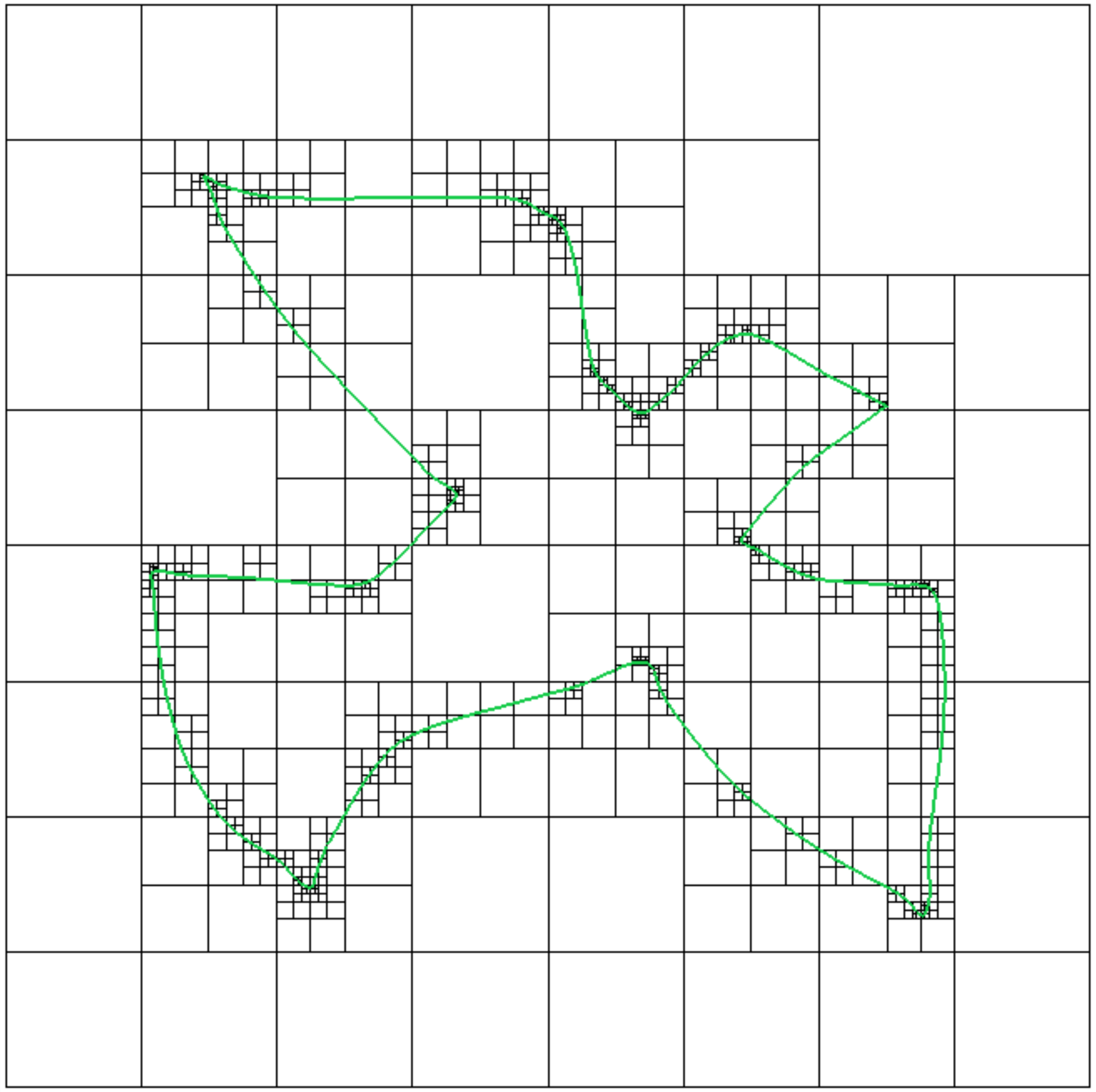


12040 cells

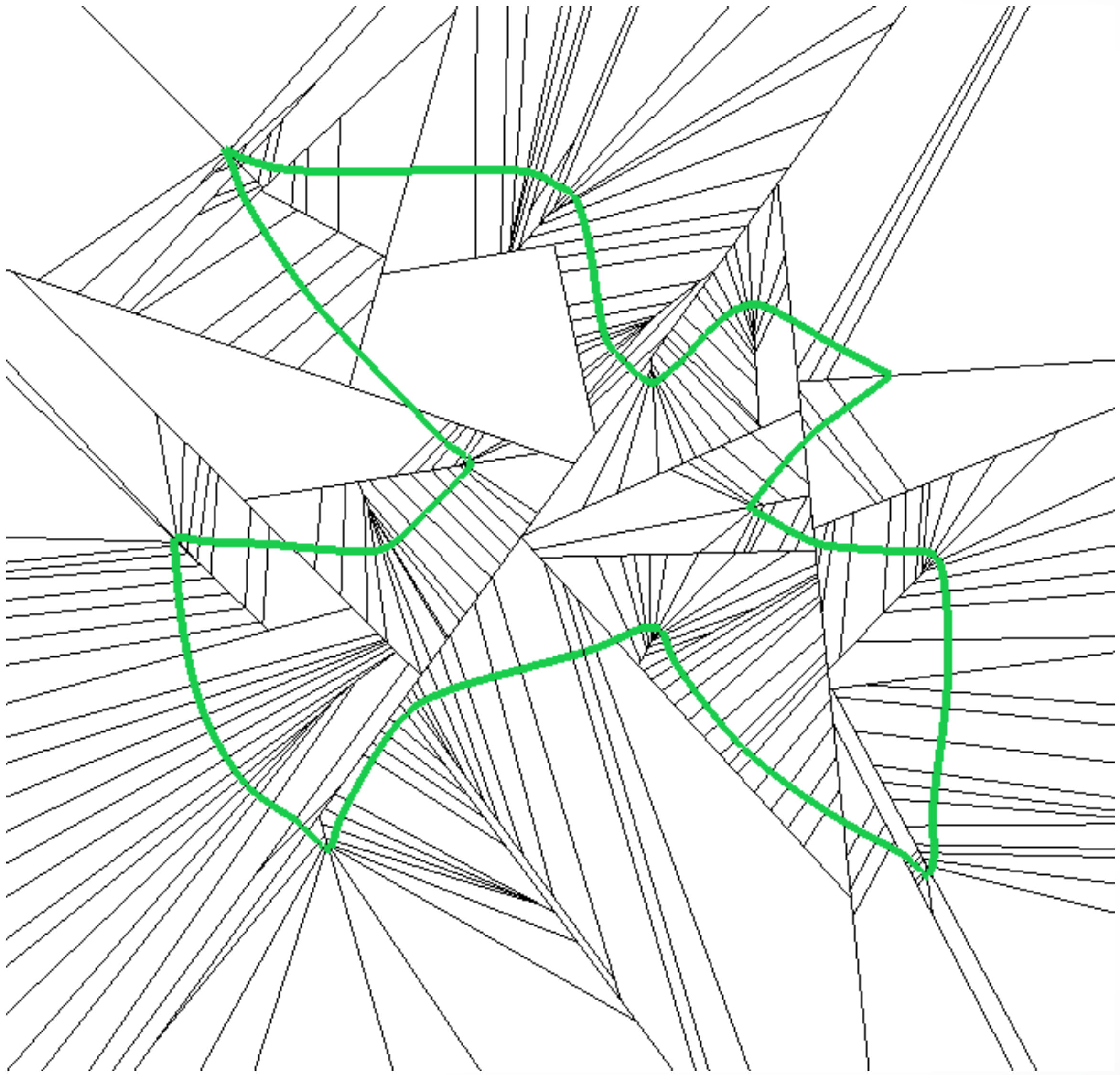


895 cells

Binary Space Partitions



895 cells



254 cells

Regularity vs. Complexity

Implicit surface discretizations

- Uniform, regular voxel grids $O(h^{-3})$
- Adaptive, 3-color octrees
  - Surface-adaptive refinement $O(h^{-2})$
  - Feature-adaptive refinement $O(h^{-1})$
- Irregular hierarchies
  - Binary space partition (BSP) $O(h^{-1})$

Literature

- Frisken et al., “Adaptively Sampled Distance Fields: A general representation of shape for computer graphics”, SIGGRAPH 2000

- Wu & Kobbel, “Piecewise Linear Approximation of Signed Distance Fields”, VMV 2003

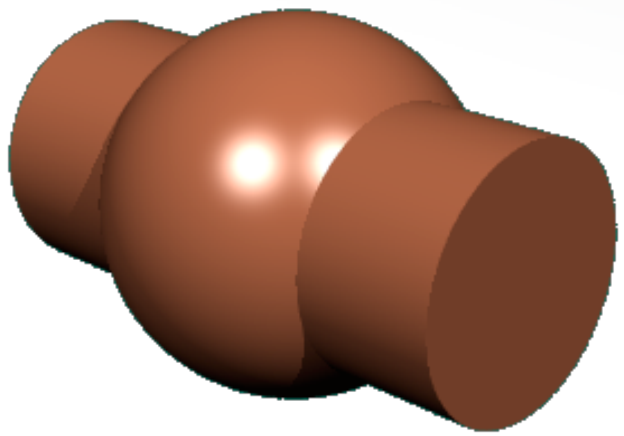
Implicit Representations

- Natural representation for **volumetric data**: CT scans, density fields, etc.
- Advantageous when modeling shapes with **complex and/or changing topology** (e.g., fluids)
- Very suitable representation for **Constructive Solid Geometry** *(CSG)*

CSG Example

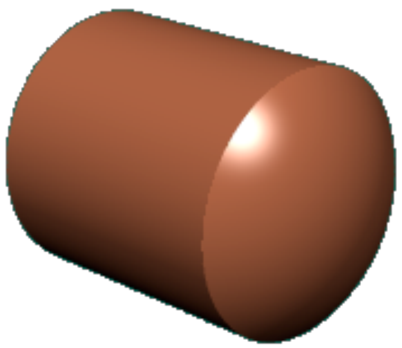
Union

$$F_{C\cup S}(\cdot) = \min\left\{F_C(\cdot)\,, F_S(\cdot)\right\}$$



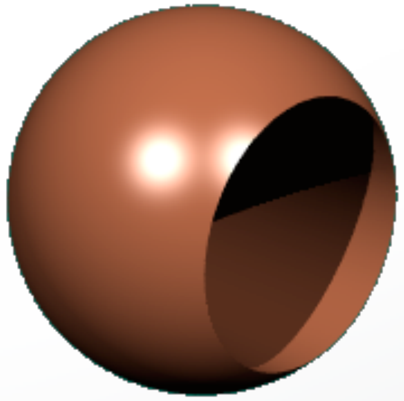
Intersection

$$F_{C\cap S}(\cdot) = \max\left\{F_C(\cdot)\,, F_S(\cdot)\right\}$$

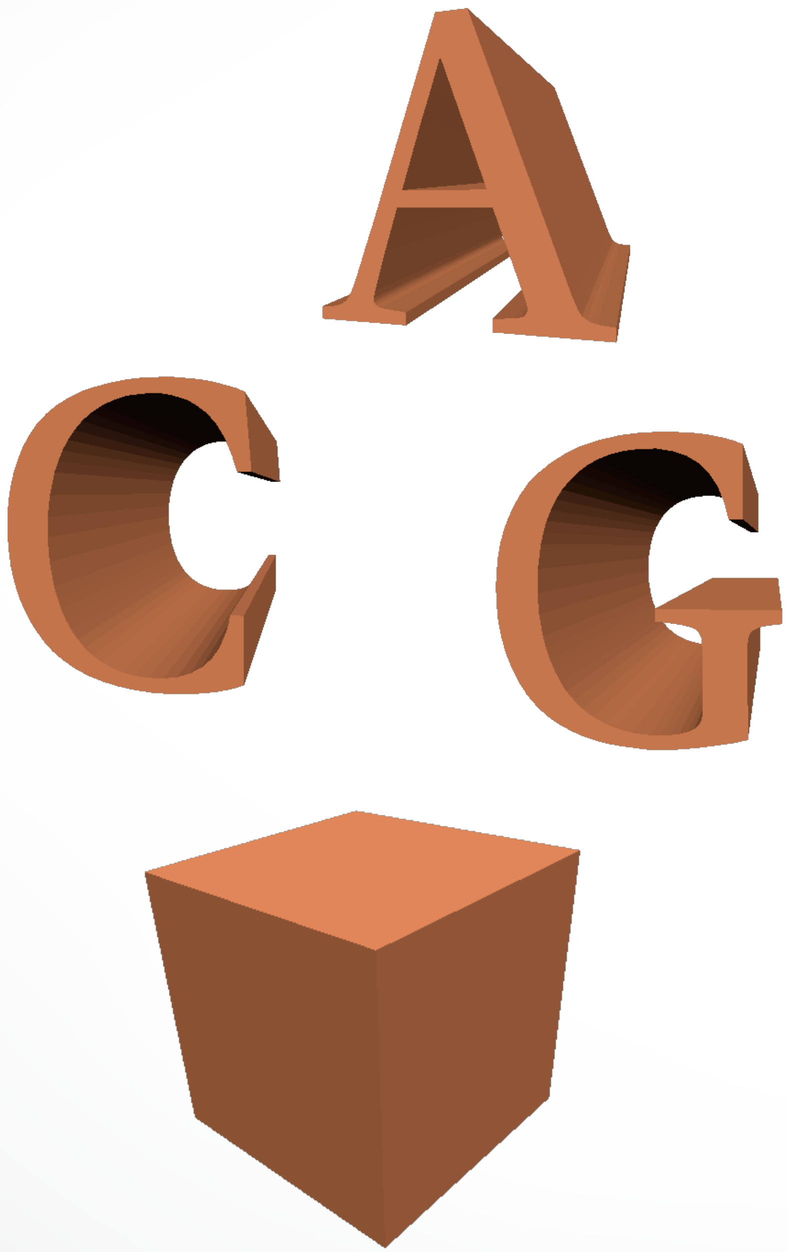


Difference

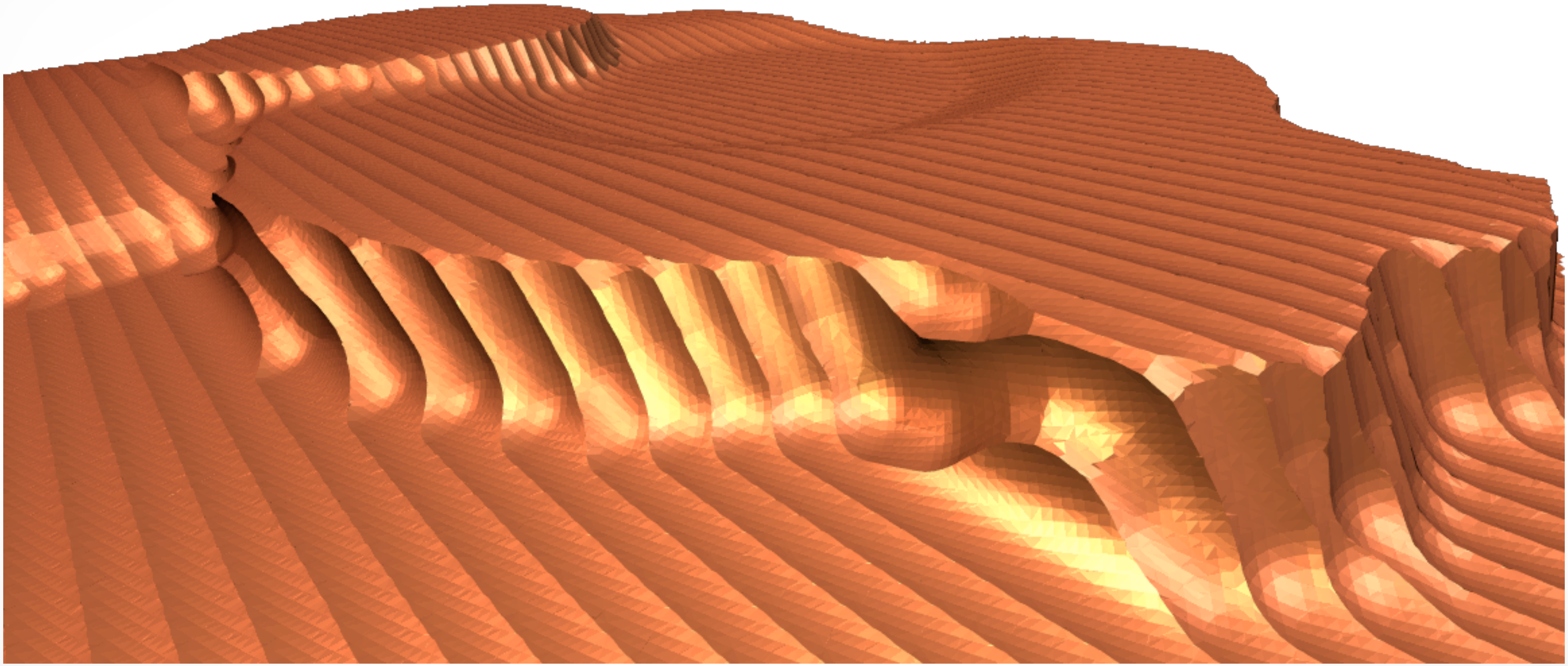
$$F_{S\setminus C}(\cdot) = \max\left\{-F_C(\cdot)\,, F_S(\cdot)\right\}$$



CSG Example



CSG Example: Milling



Outline

- Surface Representations
- Explicit Surfaces
- Implicit Surfaces
- **Conversion**

Conversion

Explicit to Implicit

- Compute signed distance at grid points
- Compute distance point-mesh
- Fast marching

Implicit to Explicit

- Extract zero-level iso-surface $F(x, y, z) = 0$
- Other iso-surfaces $F(x, y, z) = C$
- Medical imaging, simulations, measurements, …

Signed Distance Computation

Find closest mesh triangle

- Use spatial hierarchies (octree, BSP tree)

Distance point-triangle

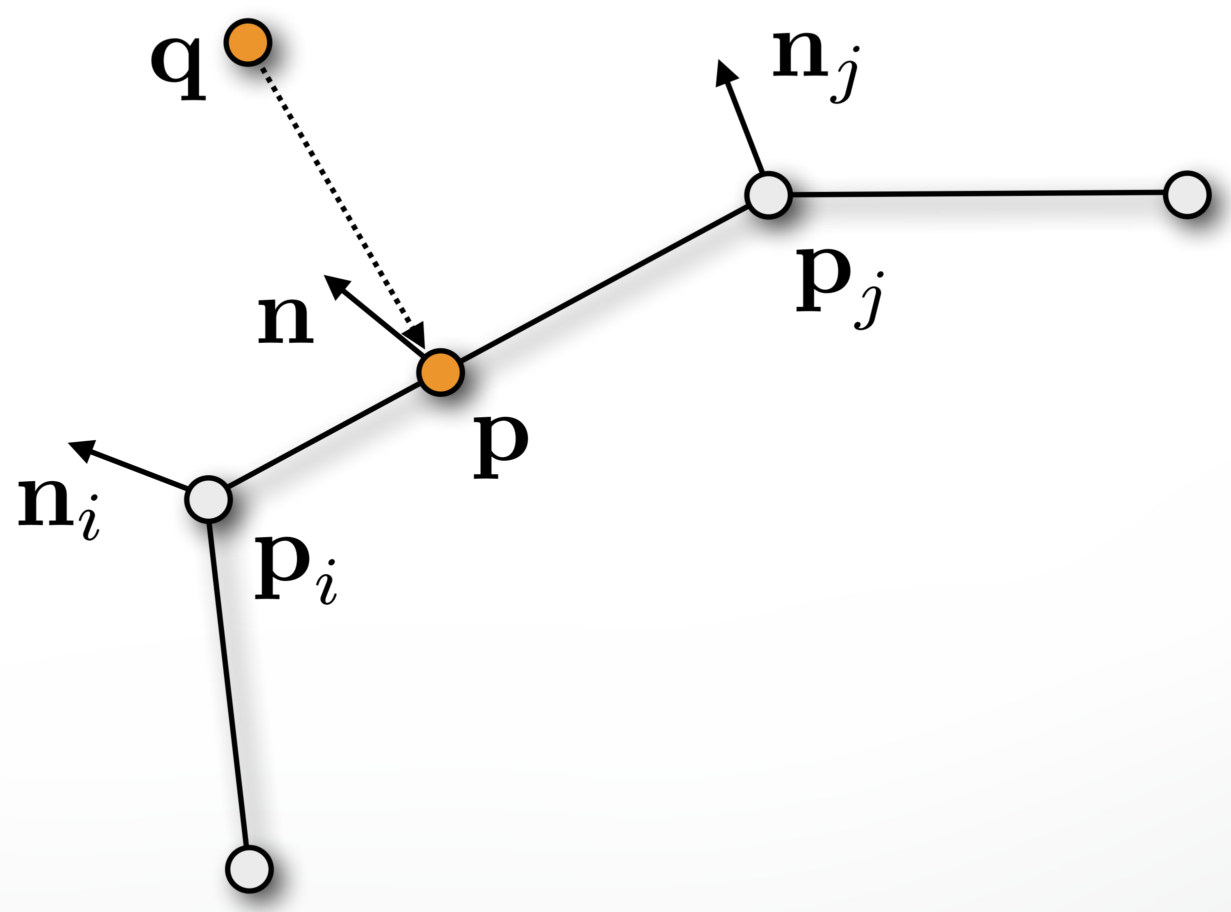
- Distance to plane, edge, or vertex
- http://www.geometrictools.com

Inside or outside?

- Based on interpolated surface normals

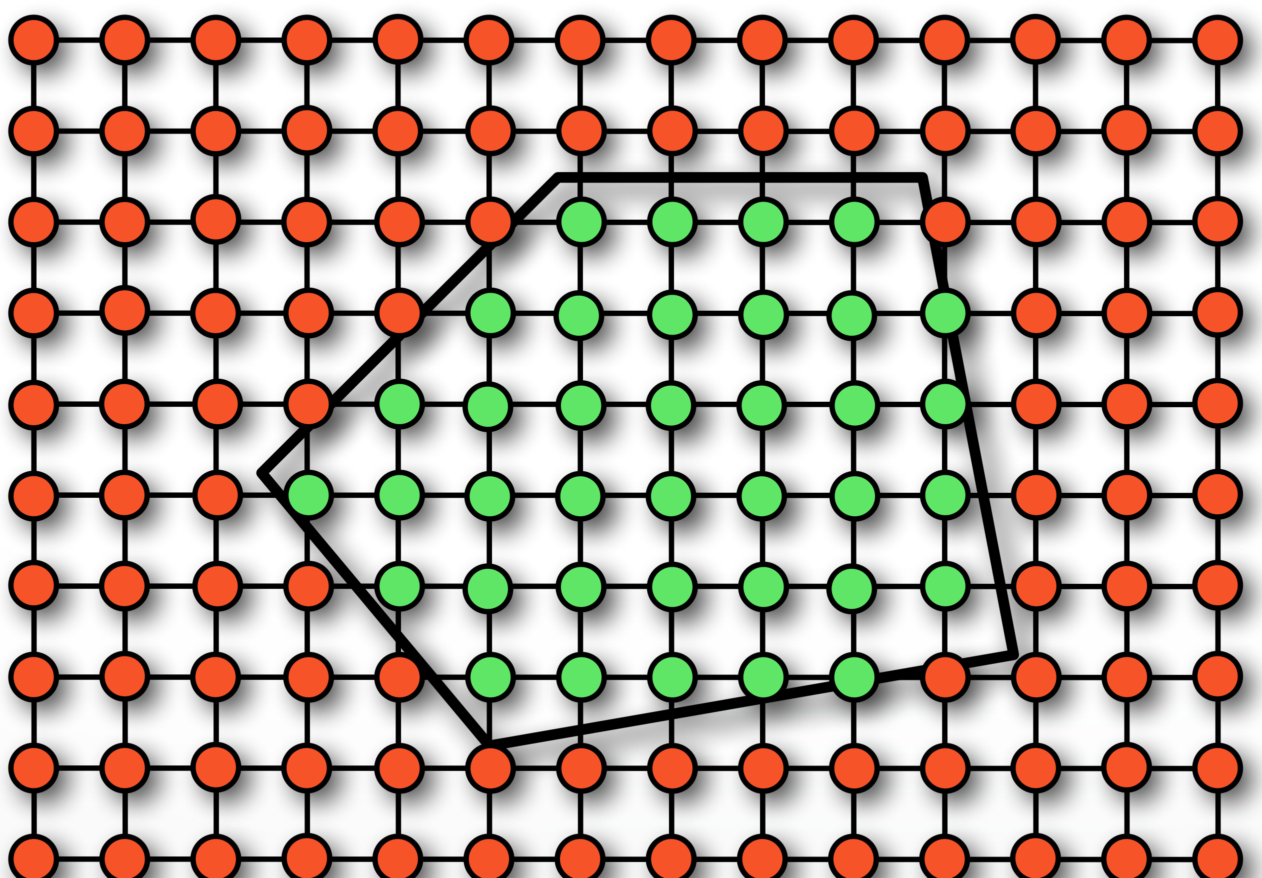
Signed Distance Computation

- Closest point $\mathbf{p} = \alpha \mathbf{p}_i + (1-\alpha)\mathbf{p}_j$
- Interpolated normal $\mathbf{n} = \alpha \mathbf{n}_i + (1-\alpha)\mathbf{n}_j$
- Inside if $(\mathbf{q}-\mathbf{p})^\top \mathbf{n} < 0$



Fast Marching Techniques

- Initialize with exact distance in mesh's vicinity
- Fast-march outwards
- Fast-march inwards



Literature

- Schneider, Eberly, "Geometric Tools for Computer Graphics", Morgan Kaufmann, 2002
- Sethian, "Level Set and Fast Marching Methods", Cambridge University Press, 1999

Conversion

Explicit to Implicit

- Compute signed distance at grid points
- Compute distance point-mesh
- Fast marching

Implicit to Explicit

- Extract zero-level iso-surface $F(x, y, z) = 0$
- Other iso-surfaces $F(x, y, z) = C$
- Medical imaging, simulations, measurements, …

2D: Marching Square

1. Classify grid nodes as inside/outside

- Is $F(\mathbf{x}_{i,j}) > 0$ or < 0 ?

2. Classify cell: 2^4 configurations

- In/out for each corner

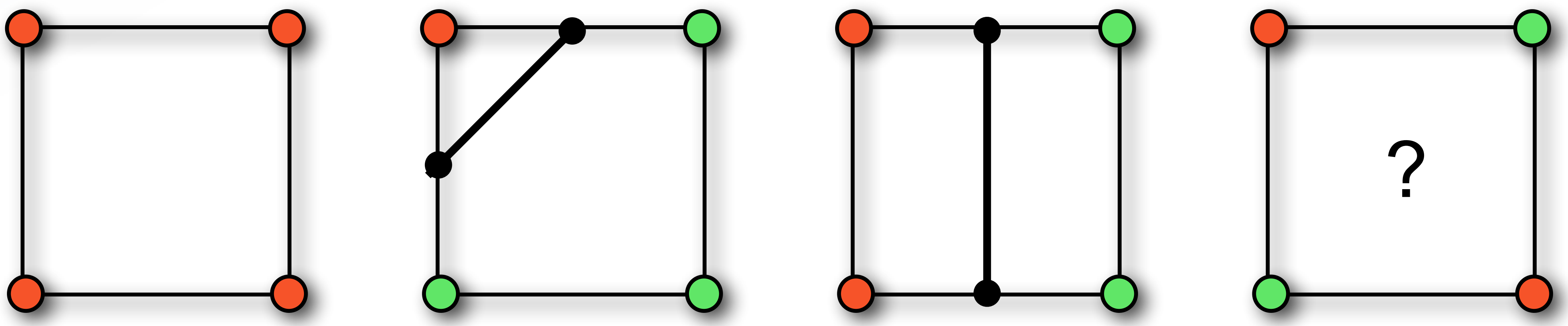
3. Compute intersection points

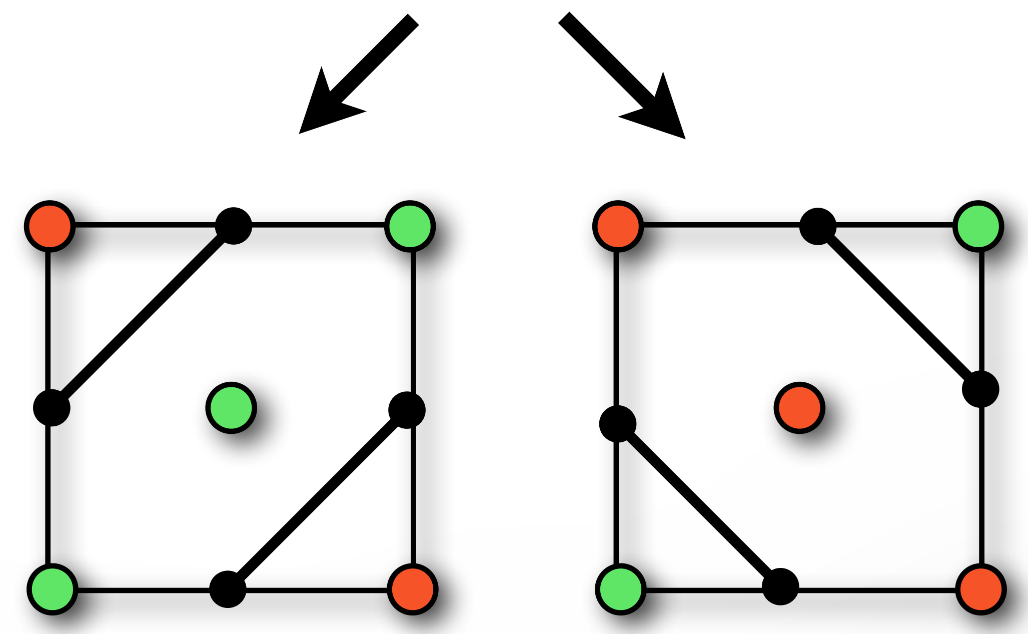
- Linear interpolation along edges

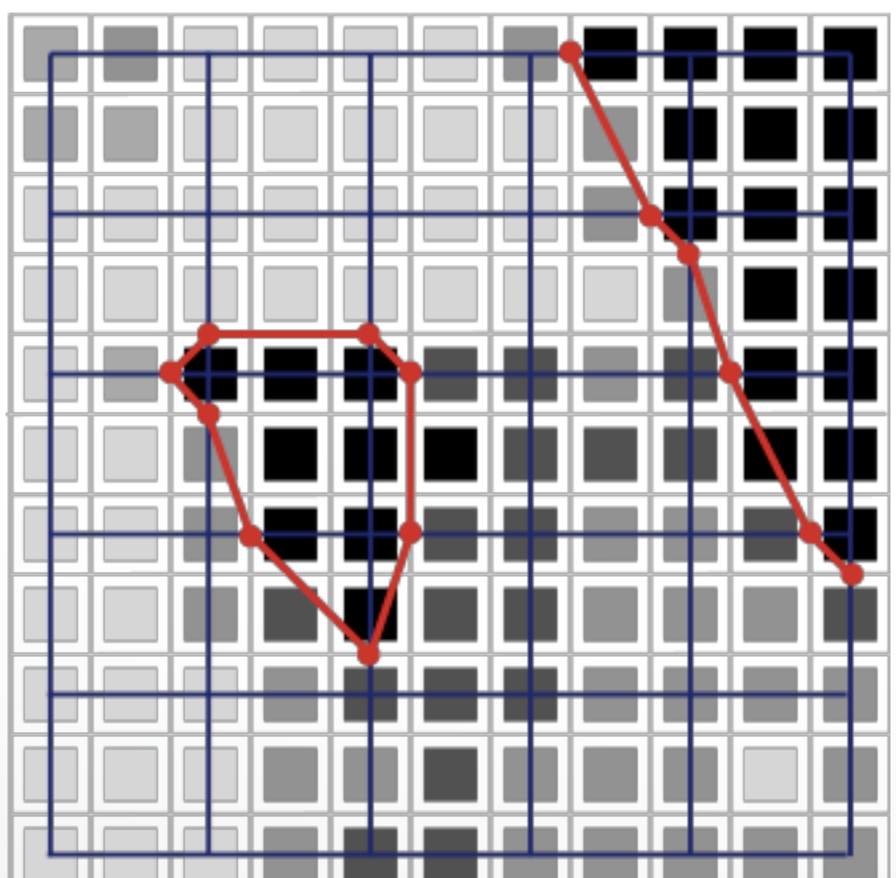
4. Connect them by edges

- Look-up table for edge configuration

2D: Marching Square







3D: Marching Cubes

1. **Classify grid nodes as inside/outside**
   - Is $F(\mathbf{x}_{i,j,k}) > 0$ or < 0
2. **Classify cell: 2^8 configurations**
   - In/out for each corner
3. **Compute intersection points**
   - Linear interpolation along edges
4. **Connect them by edges**
   - Look-up table for path configuration
   - Disambiguation by modified table [Montani '94]

3D: Marching Cubes

Classify grid nodes $\mathbf{x}_{i,j,k}$ based on $F_{i,j,k} = F(\mathbf{x}_{i,j,k})$

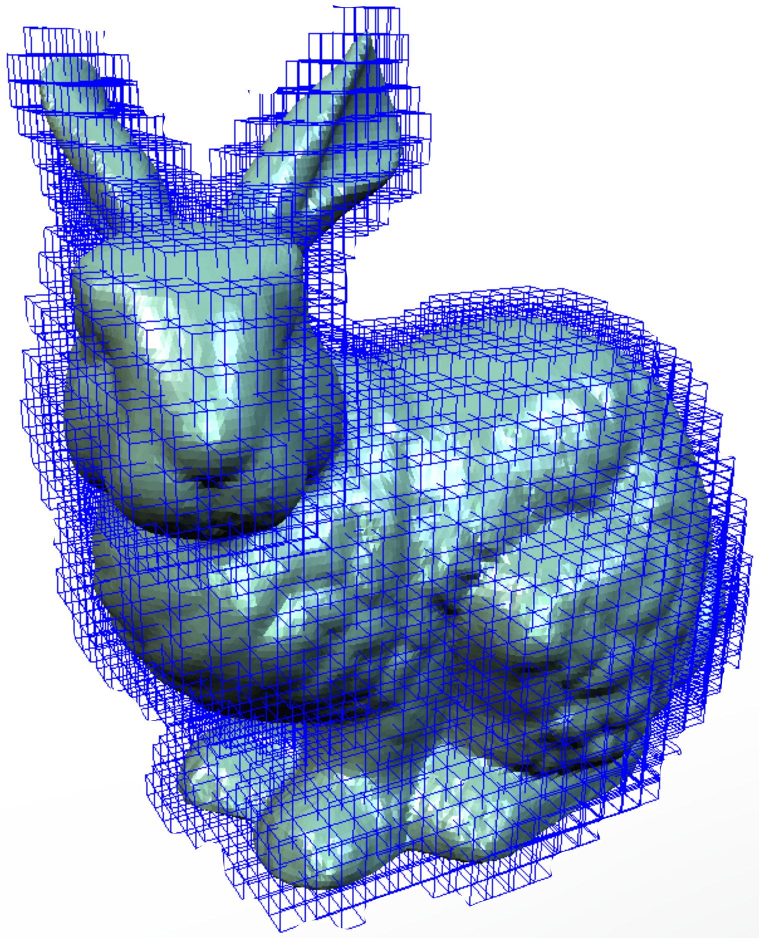
- Inside or outside

Classify all cubes based on $F_{i,j,k}$

- Inside, outside, or intersecting

Refined only intersected cells

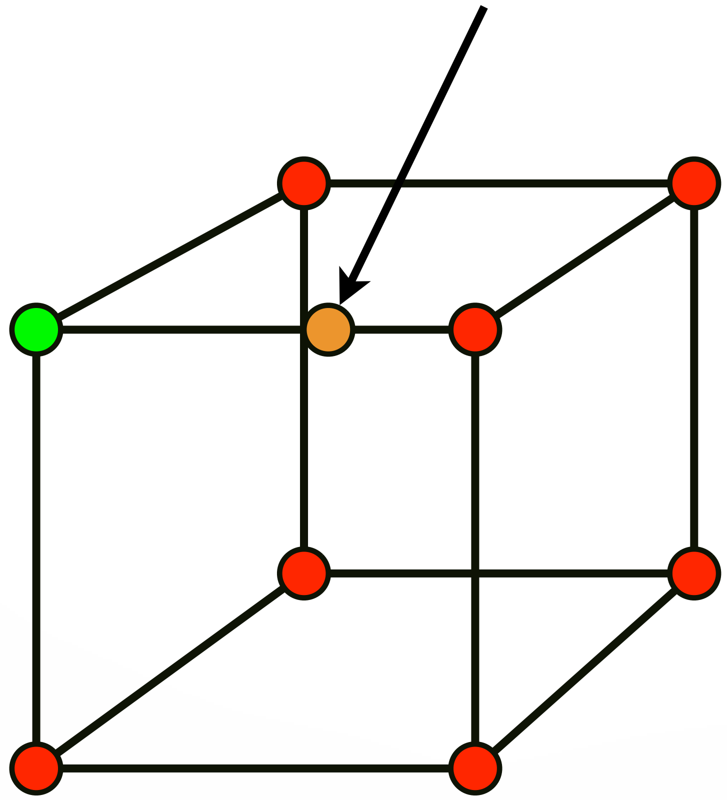
- 3-color adaptive octree
- $O(h^{-2})$ complexity



Intersection Points

Linear interpolation along edges

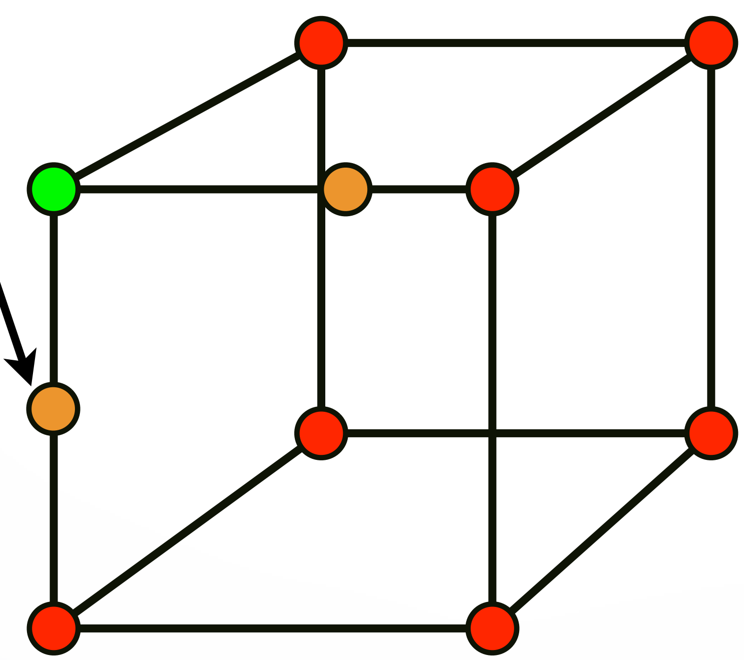
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i+1,j,k}| \; + \; \mathbf{x}_{i+1,j,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i+1,j,k}|}$$



Intersection Points

Linear interpolation along edges

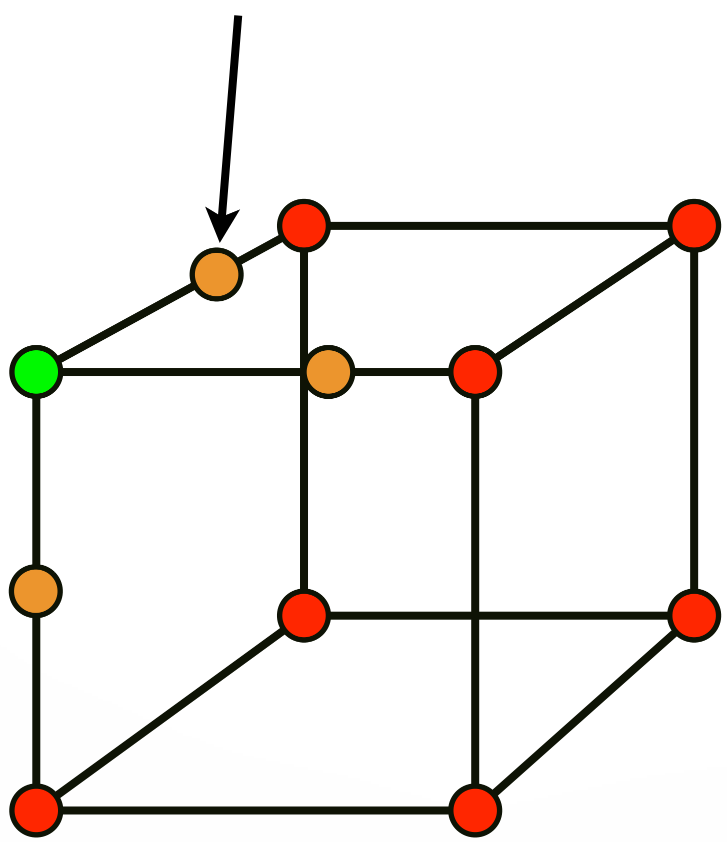
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j+1,k}| \;+\; \mathbf{x}_{i,j+1,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j+1,k}|}$$



Intersection Points

Linear interpolation along edges

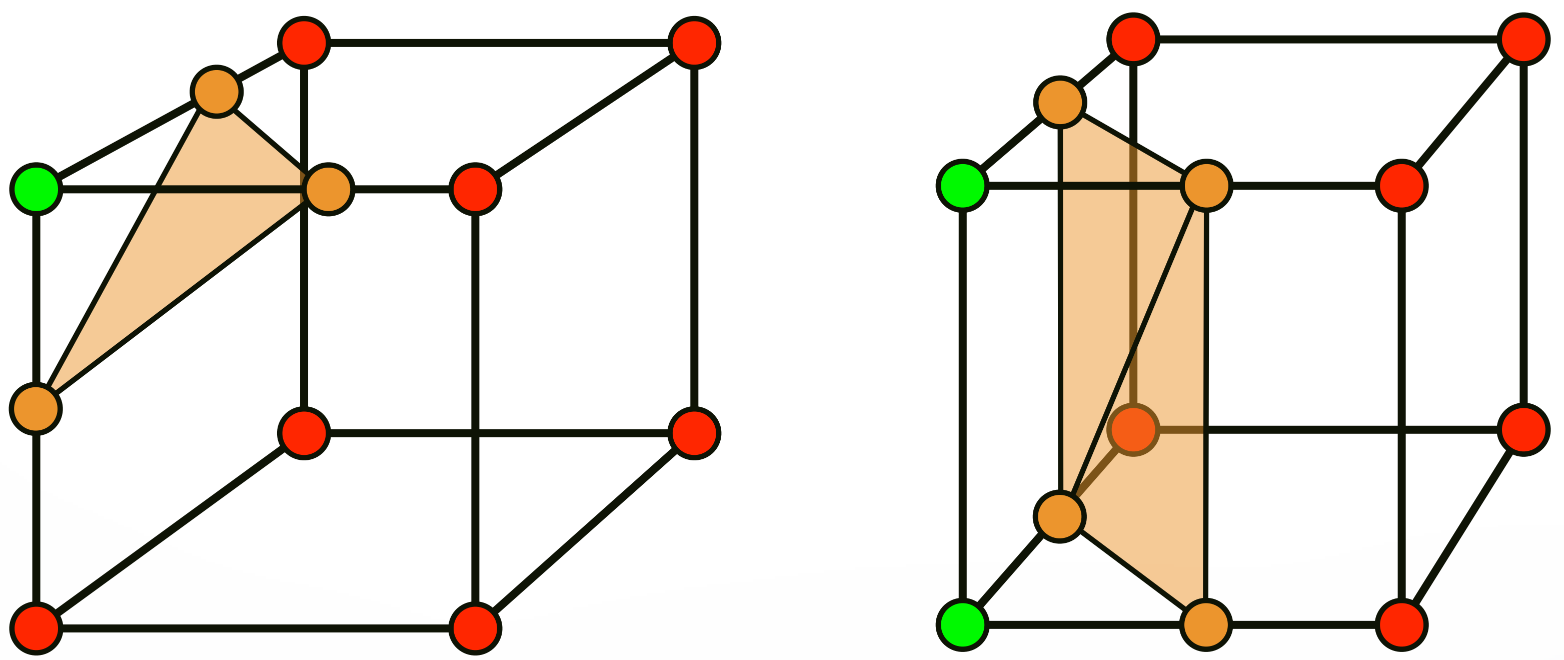
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j,k+1}| \; + \; \mathbf{x}_{i,j,k+1} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j,k+1}|}$$



Intersection Points

Linear interpolation along edges

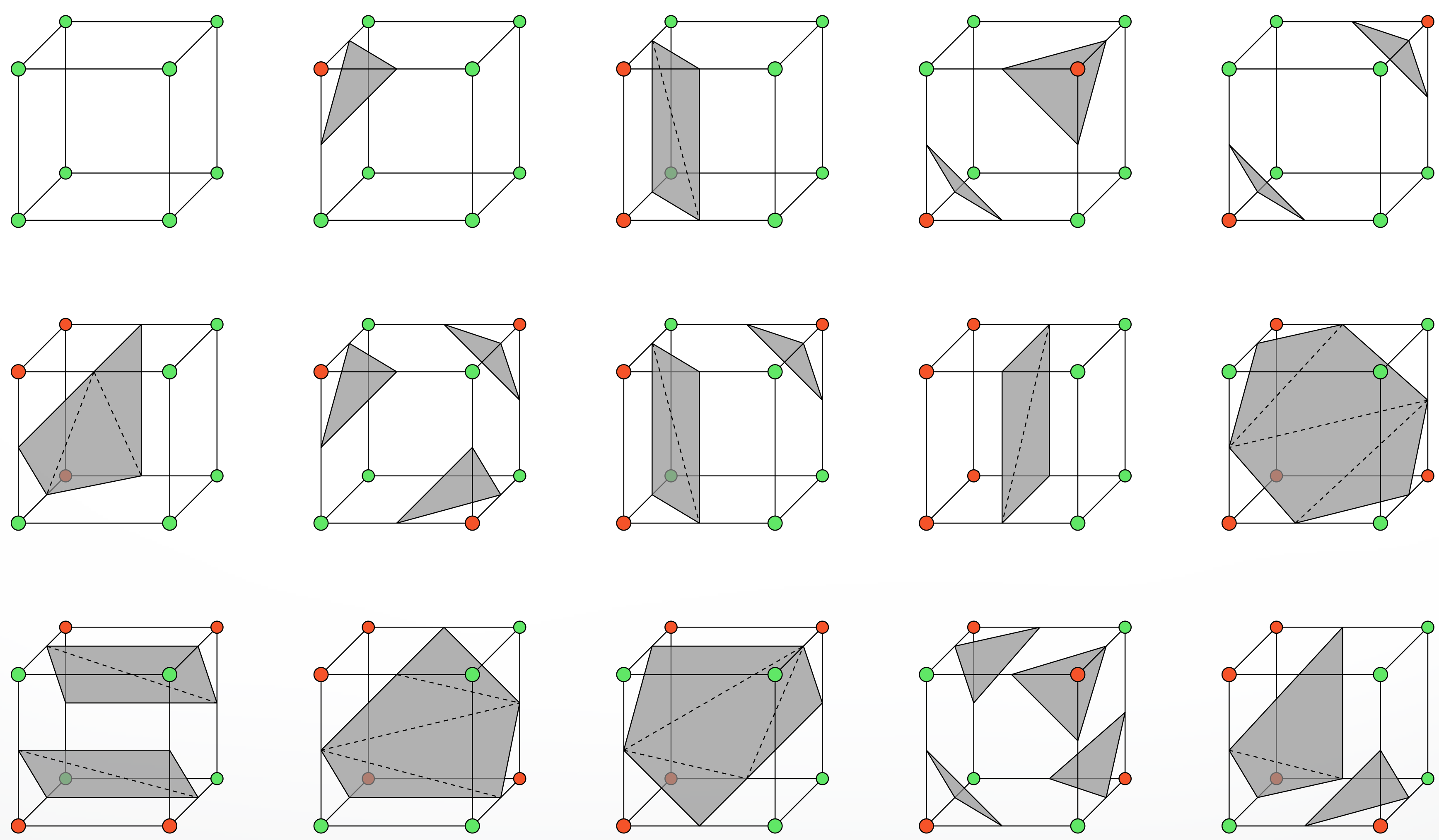
Lookup table for patch configuration



Marching Cubes

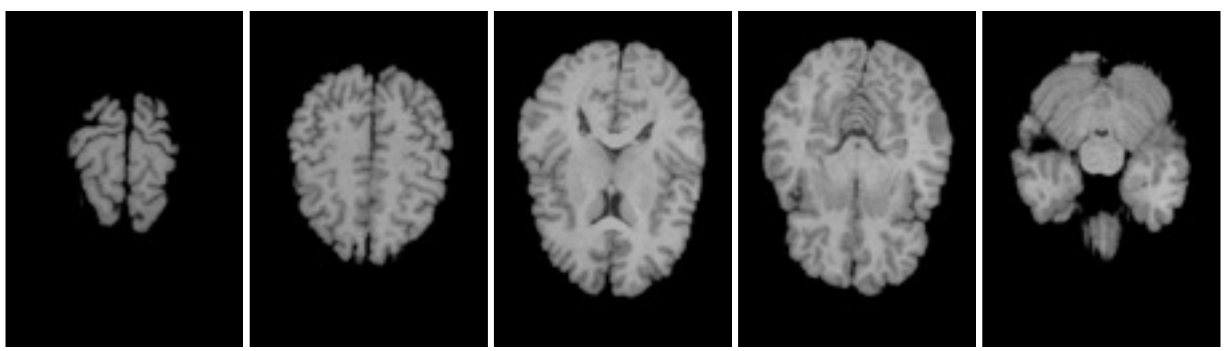
Look-up table with 2^8 entries

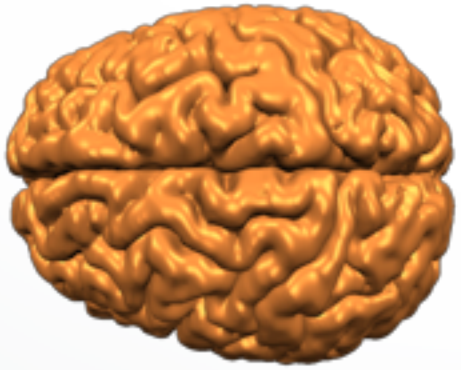
- 15 representative cases shown
- Others follow by symmetry

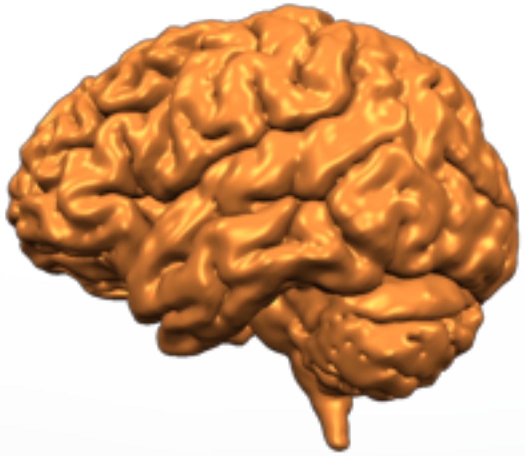


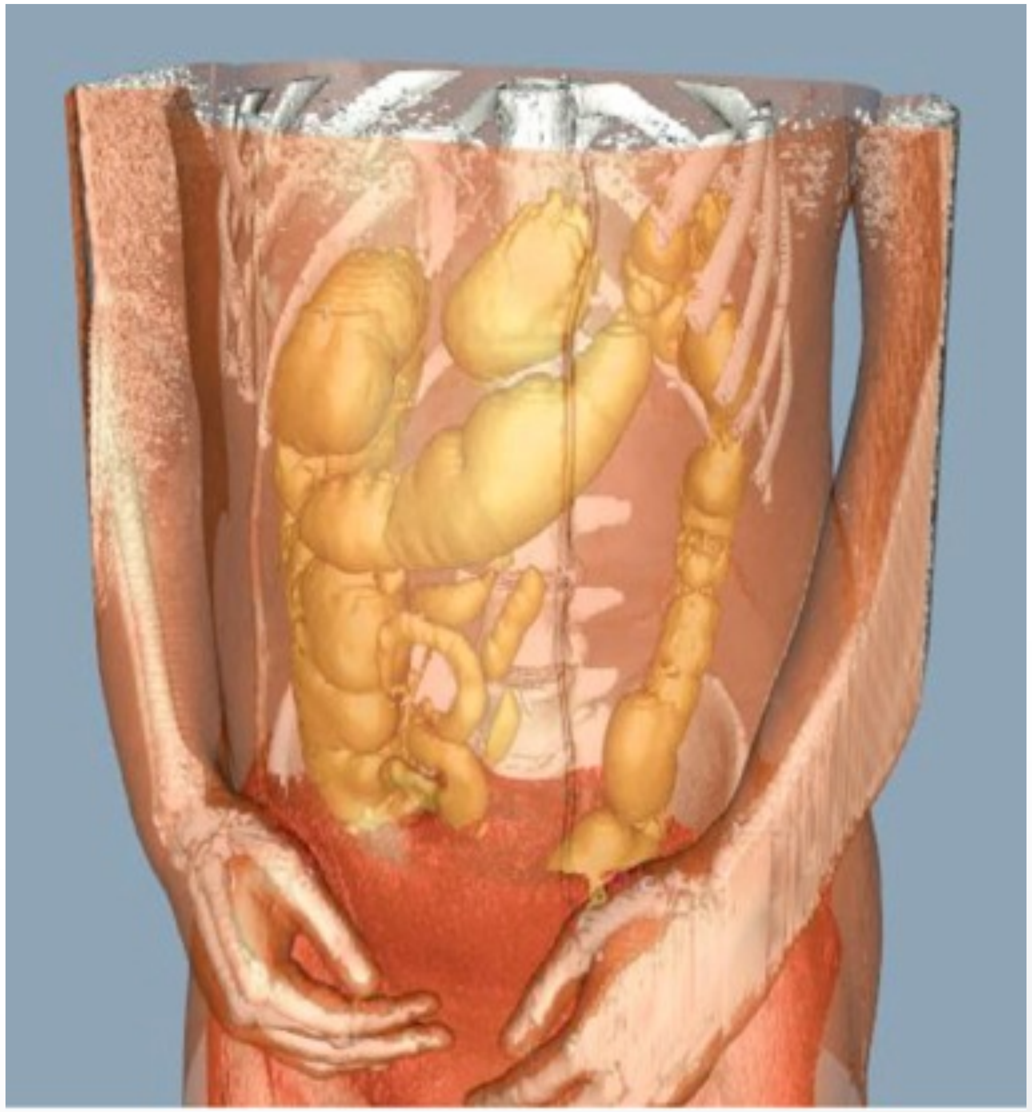
Marching Cubes

Algorithm for isosurface extraction from medical scans (CT, MRI)



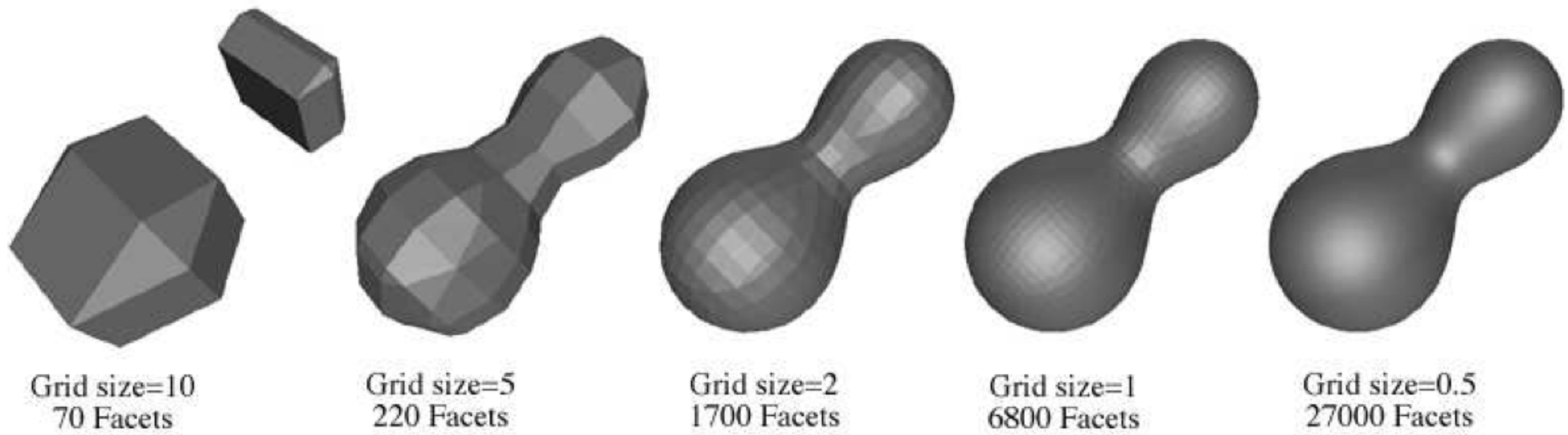






Marching Cubes

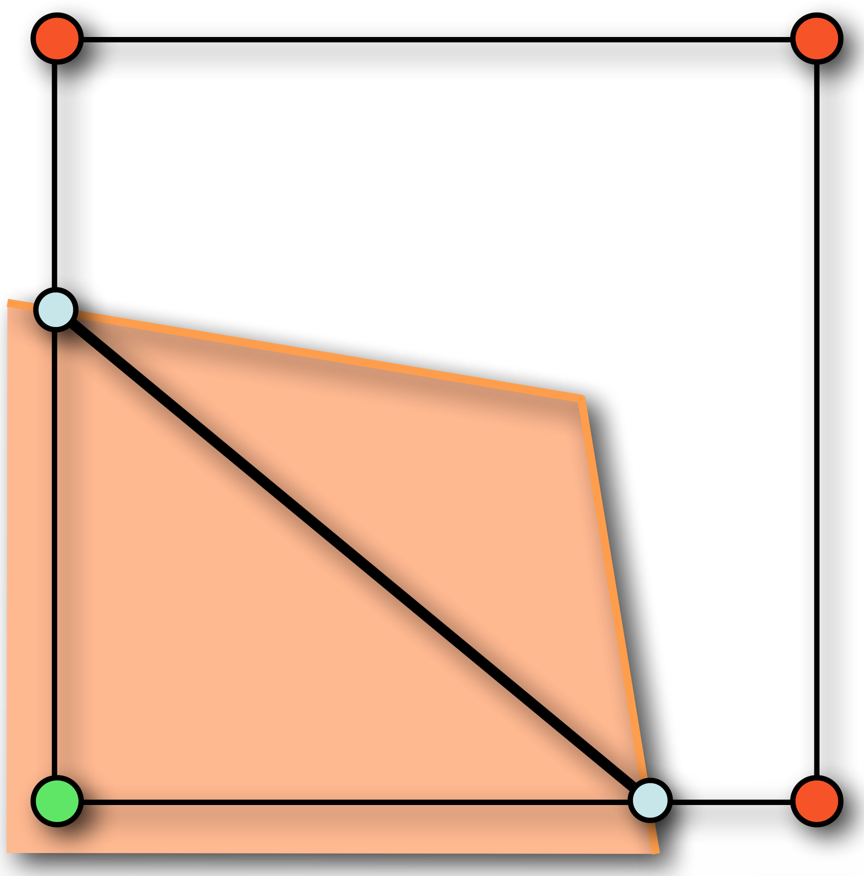
Effect of grid size

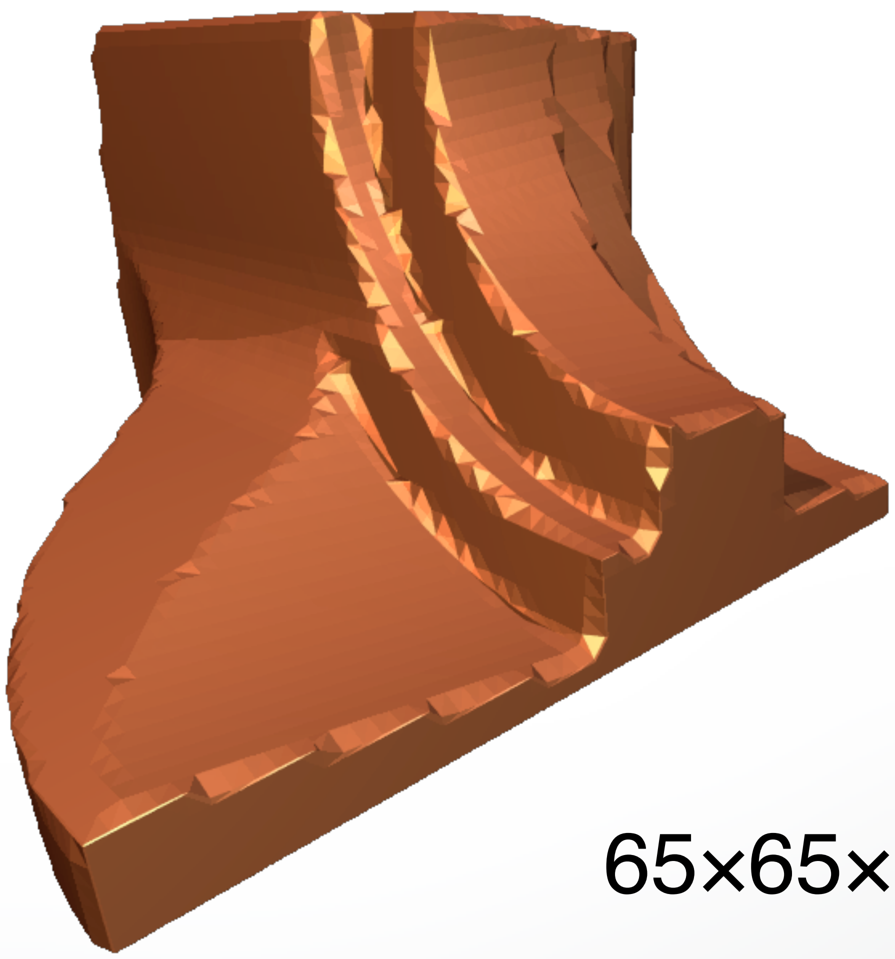


Marching Cubes

Sample points restricted to edges of regular grid

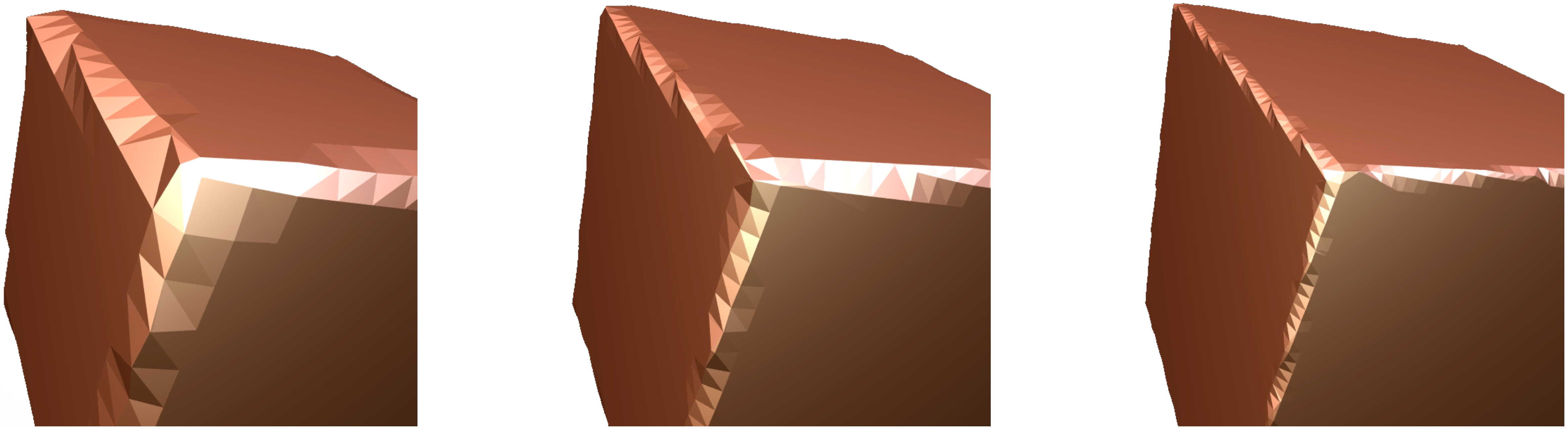
Alias artifacts at sharp features





65×65×65

Increasing Resolution

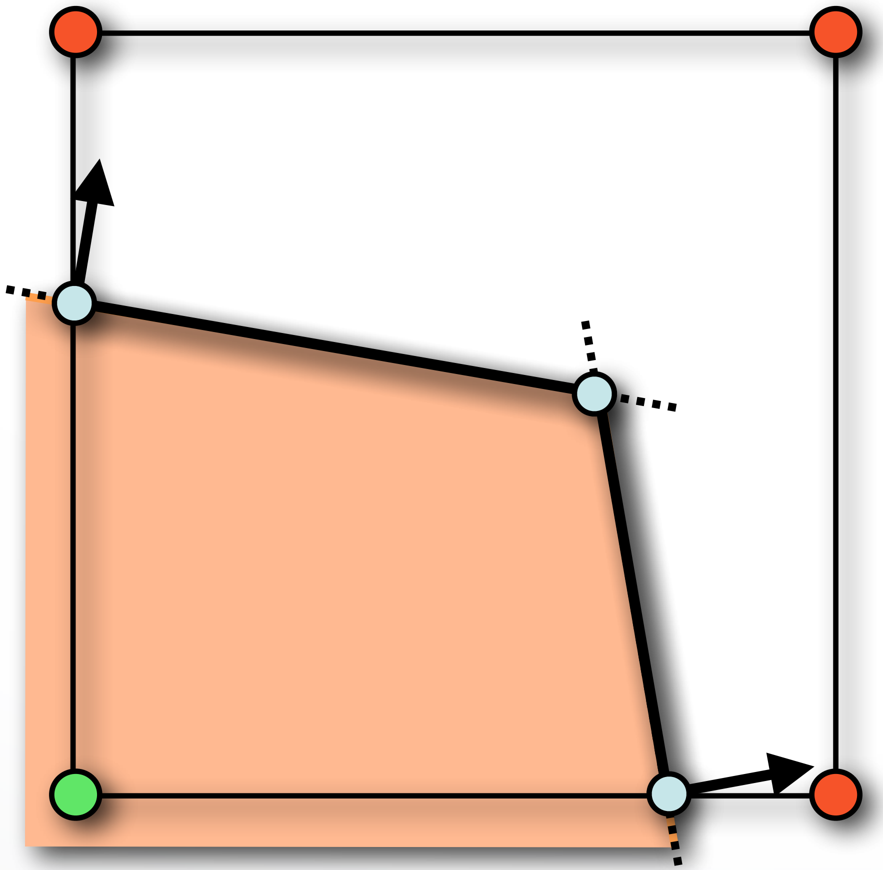


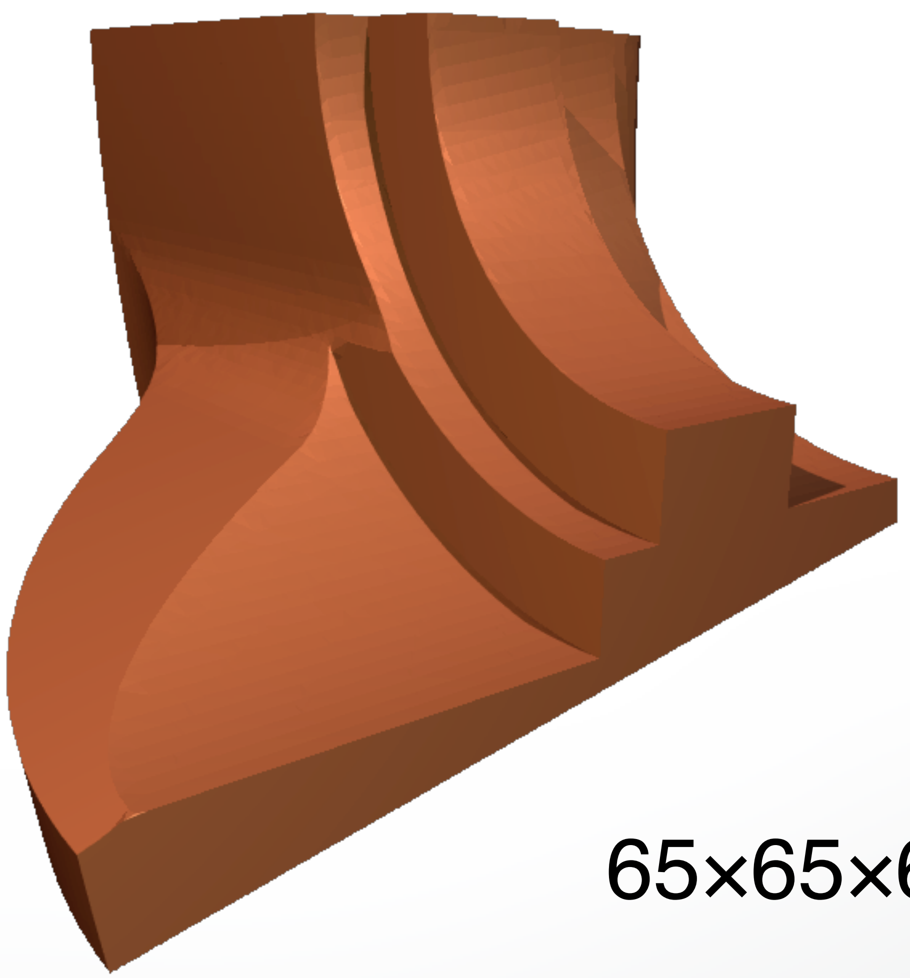
Does not remove alias problems!

Extended Marching Cubes

Locally extrapolate distance gradient

Place samples on estimated features



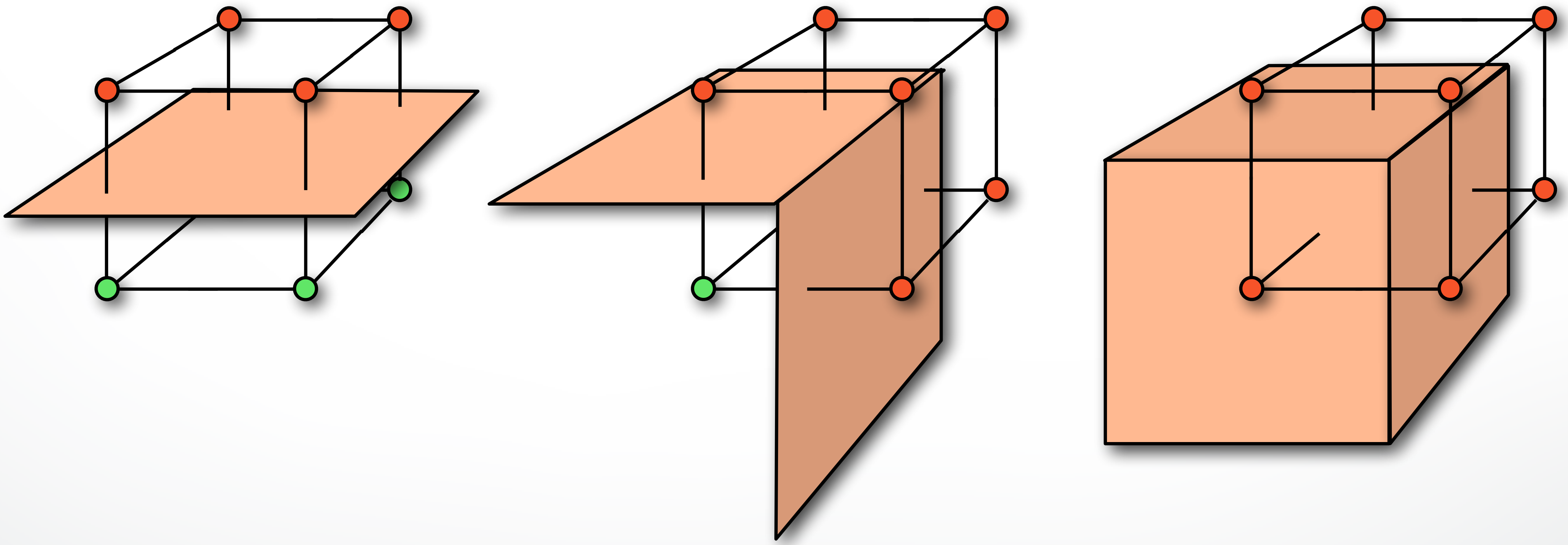


65×65×65

Extended Marching Cubes

Feature detection

- Based on angle between normals $\mathbf{n}_i$
- Classify into edges / corners



Extended Marching Cubes

Feature sampling

- Intersect tangent planes $(\mathbf{s}_i, \mathbf{n}_i)$

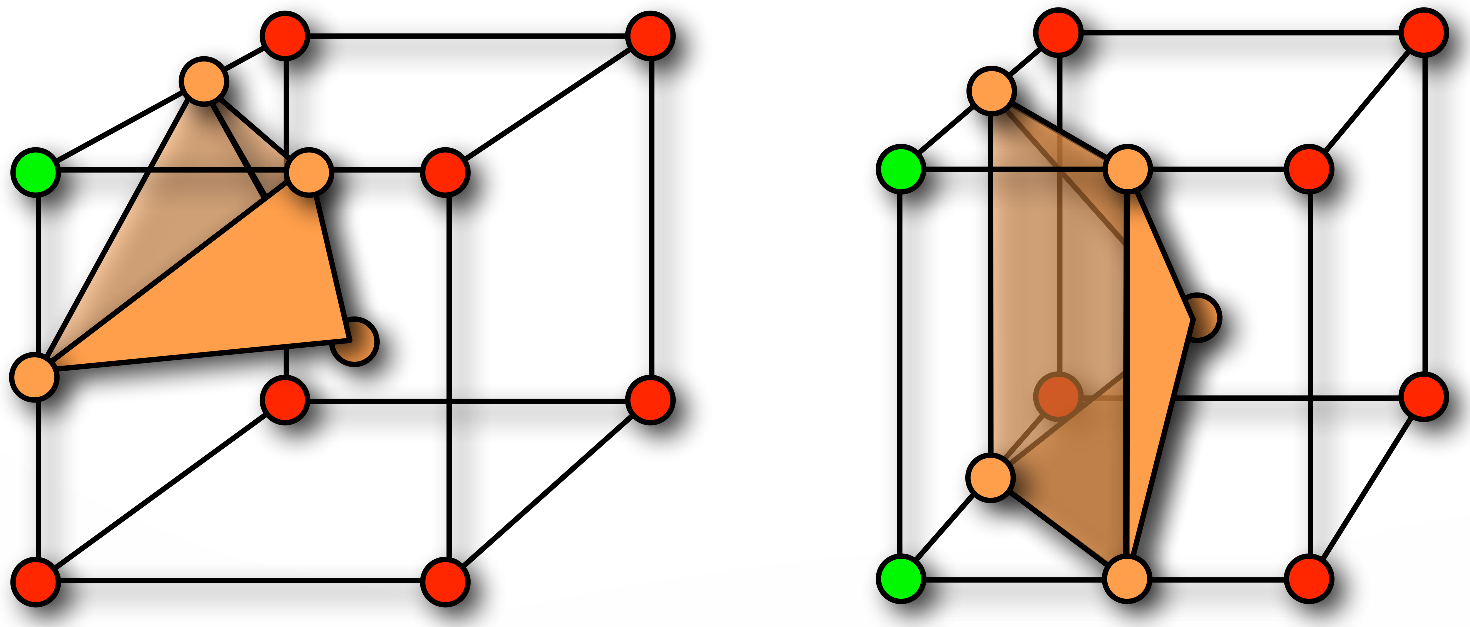
$$\begin{pmatrix} \vdots \\ \mathbf{n}_i \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{n}_i^T \mathbf{s}_i \\ \vdots \end{pmatrix}$$

- Over- or under-determined system
- Solve by SVD pseudo-inverse

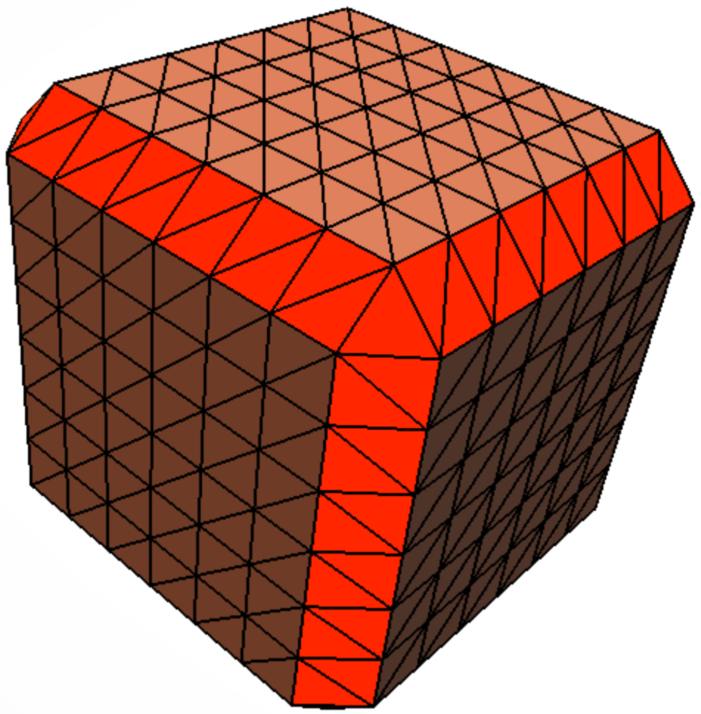
Extended Marching Cubes

Feature sampling

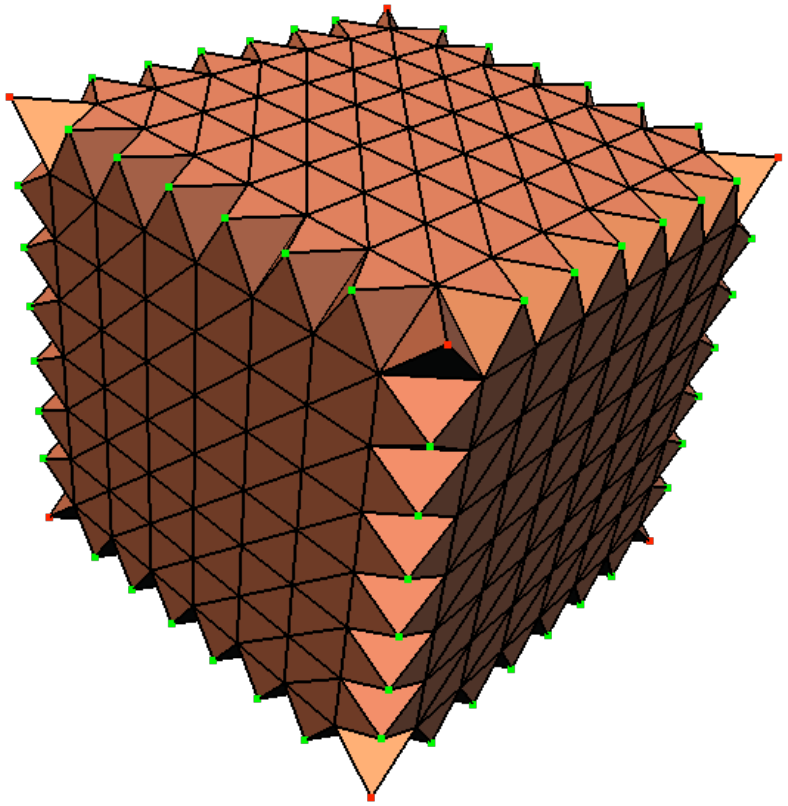
- Intersect tangent planes $(\mathbf{s}_i, \mathbf{n}_i)$
- Triangle fans centered at feature point



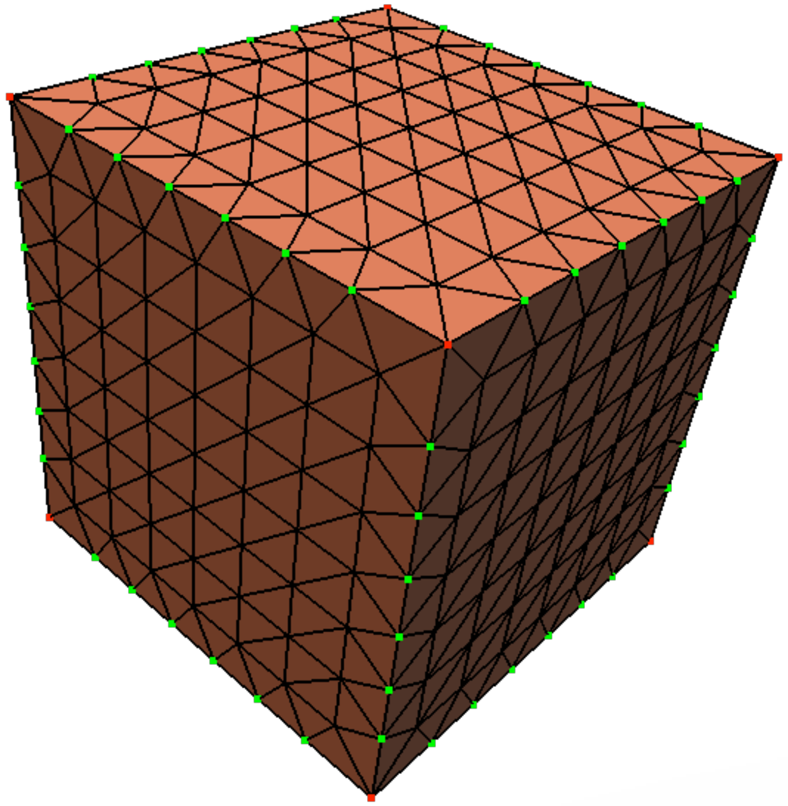
Extended Marching Cubes



Feature
Detection

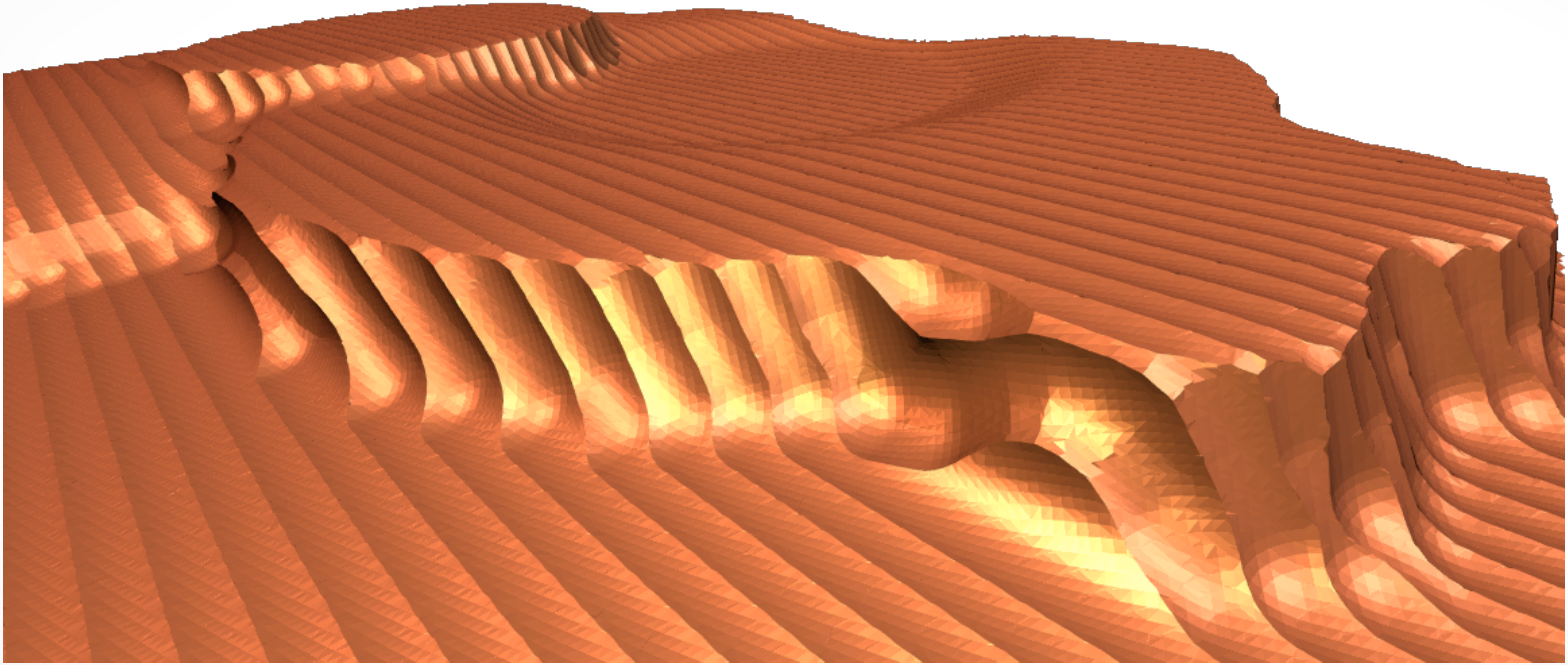


Feature
Sampling



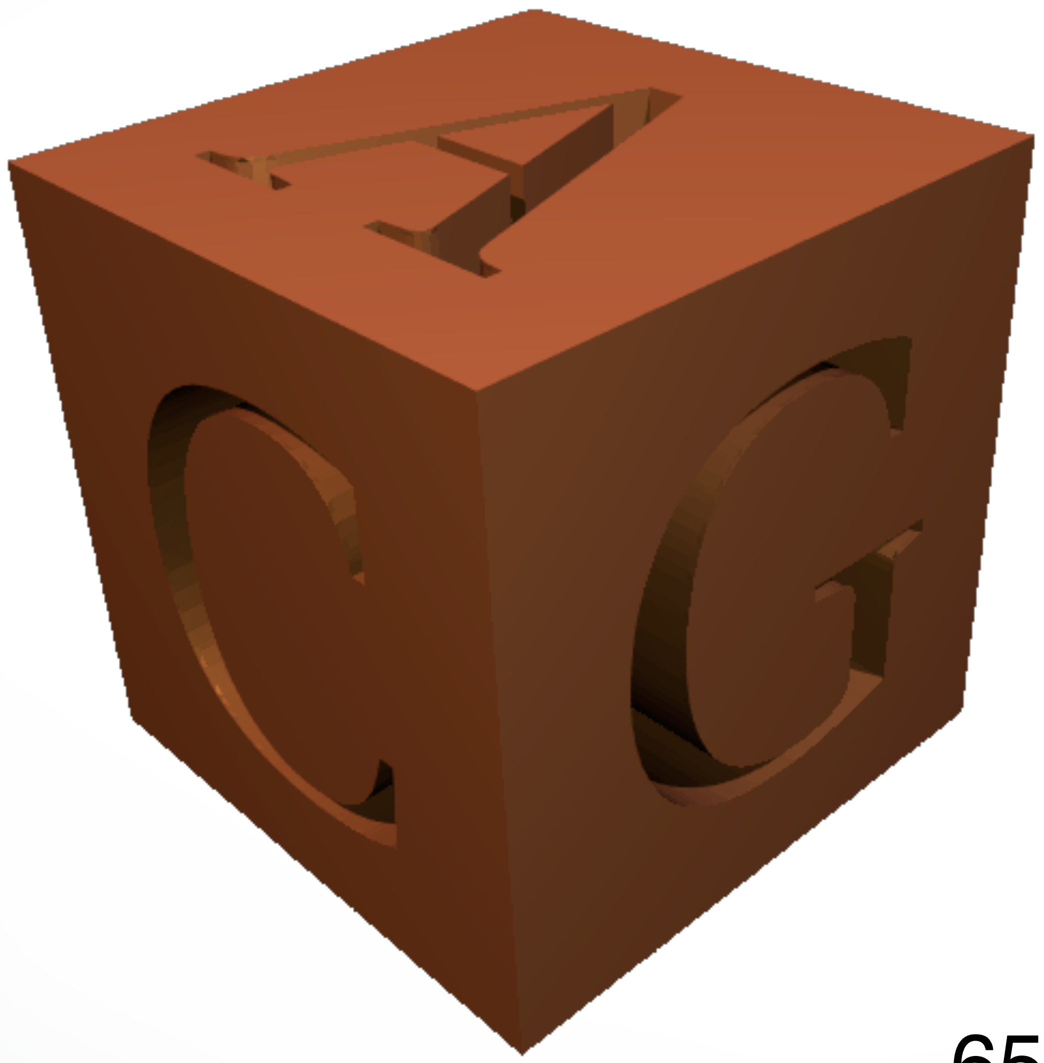
Edge
Flipping

Milling Simulation



257×257×257

CSG Modeling



65×65×65

Marching Cubes

+ Result is watertight, closed 2-manifold surface!

+ Easy to parallelize

- Uniform (over-) sampling (→ mesh decimation)

- Degenerate triangles (→ remeshing)

- MC does not preserve features

+ EMC preserves features, but…

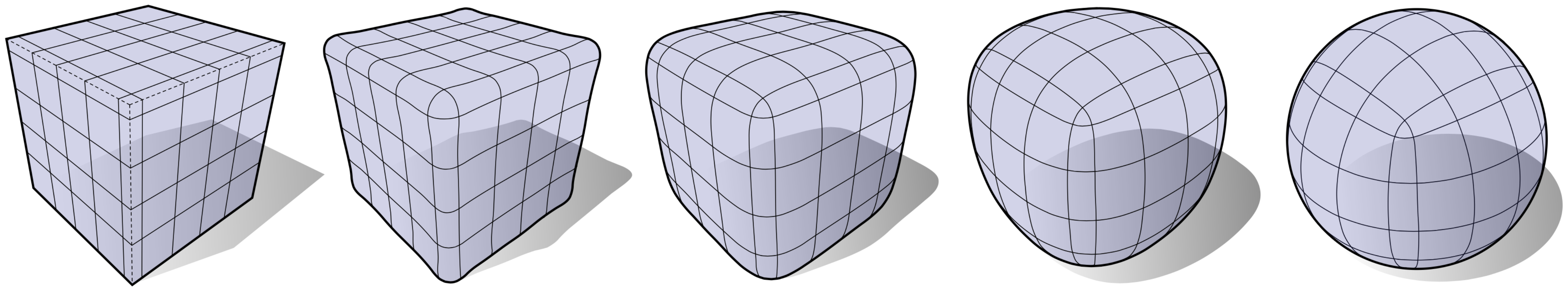
  about 10% more triangles

  20-40% computational overhead

Literature

- Lorensen & Cline, "Marching Cubes: A High Resolution 3D Surface Construction Algorithm", SIGGRAPH 1987
- Montani et al., "A modified look-up table for implicit disambiguation of Marching Cubes", Visual Computer 1994
- Kobbelt et al., "Feature Sensitive Surface Extraction from Volume Data", SIGGRAPH 2001

Next Time



Discrete Differential Geometry

http://cs621.hao-li.com

Thanks!