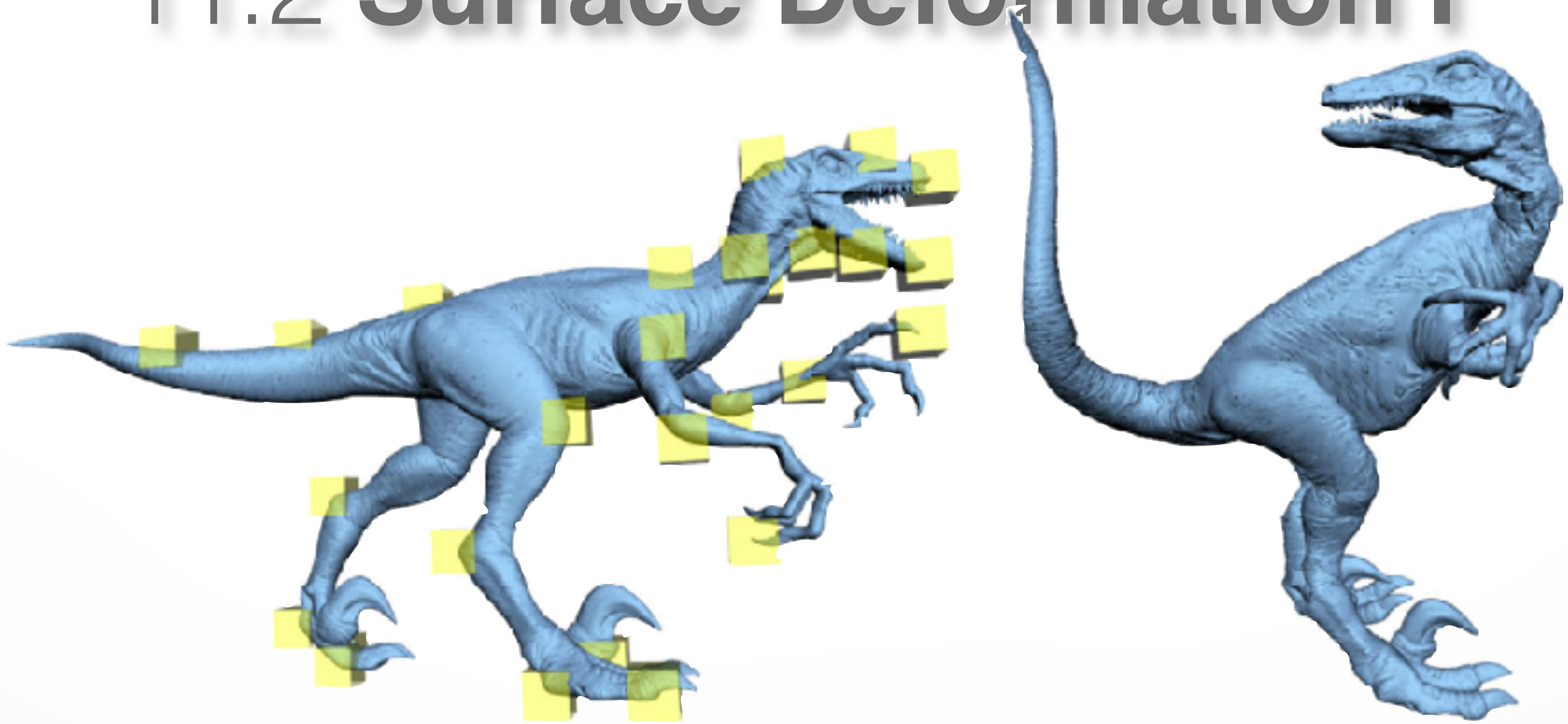
CSCI 621: **Digital Geometry Processing**

11.2 **Surface Deformation I**



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Acknowledgement

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- Prof. Olga Sorkine, ETH Zurich

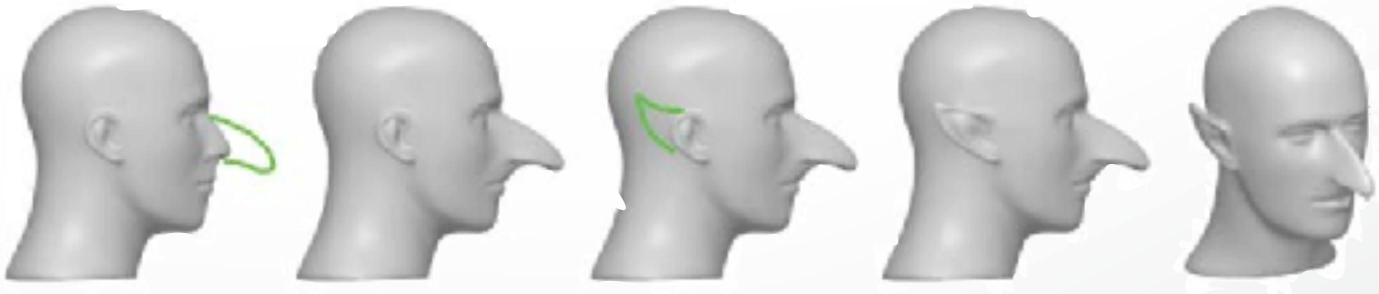
Shapes & Deformation

Why deformations?

- Sculpting, customization
- Character posing, animation

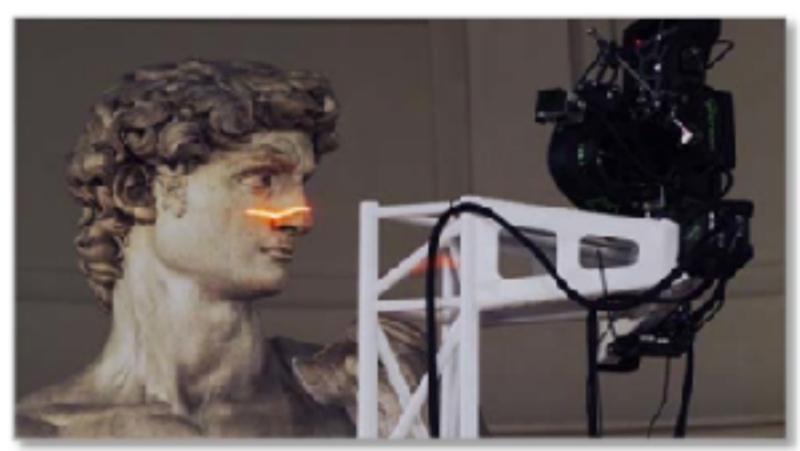
Criteria?

- Intuitive behavior and interface
- semantics
- Interactivity



Shapes & Deformation

- Manually modeled and scanned shape data
- Continuous and discrete shape representations



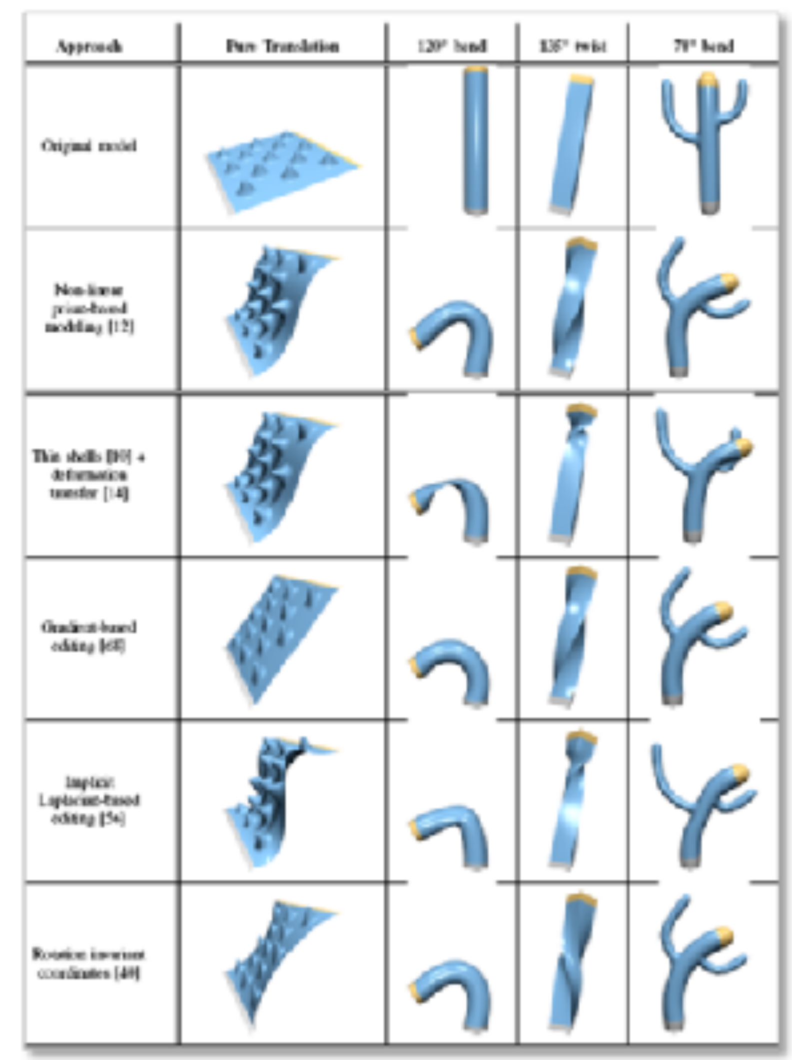
Goals

State of research in shape editing

Discuss practical considerations

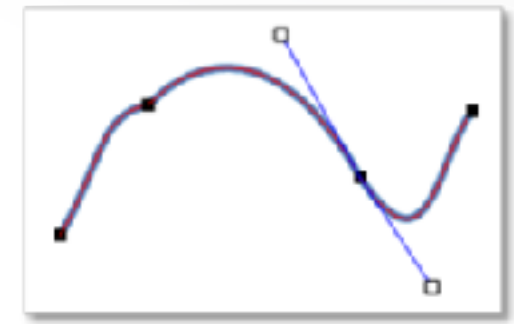
- Flexibility
- Numerical issues
- Admissible interfaces

Comparison, tradeoffs



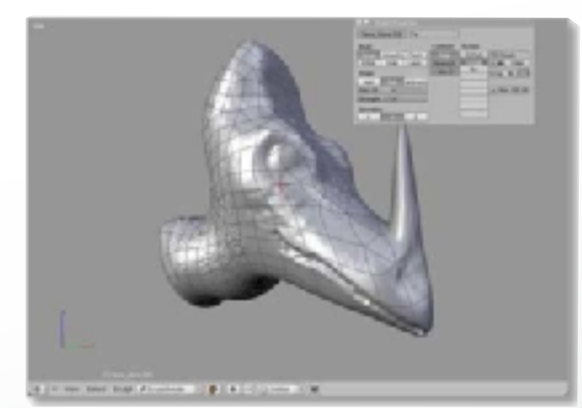
Continuous/Analytical Surfaces

- Tensor product surfaces (e.g. Bézier, B-Spline, NURBS)



- Subdivision Surfaces

- Editability is inherent to the representation

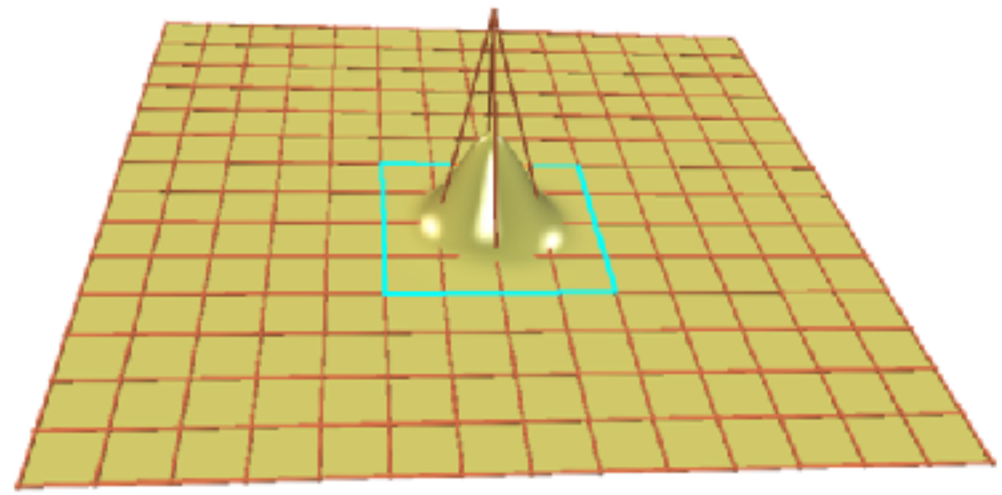


Spline Surfaces

Tensor product surfaces ("curves of curves")

- Rectangular grid of control points

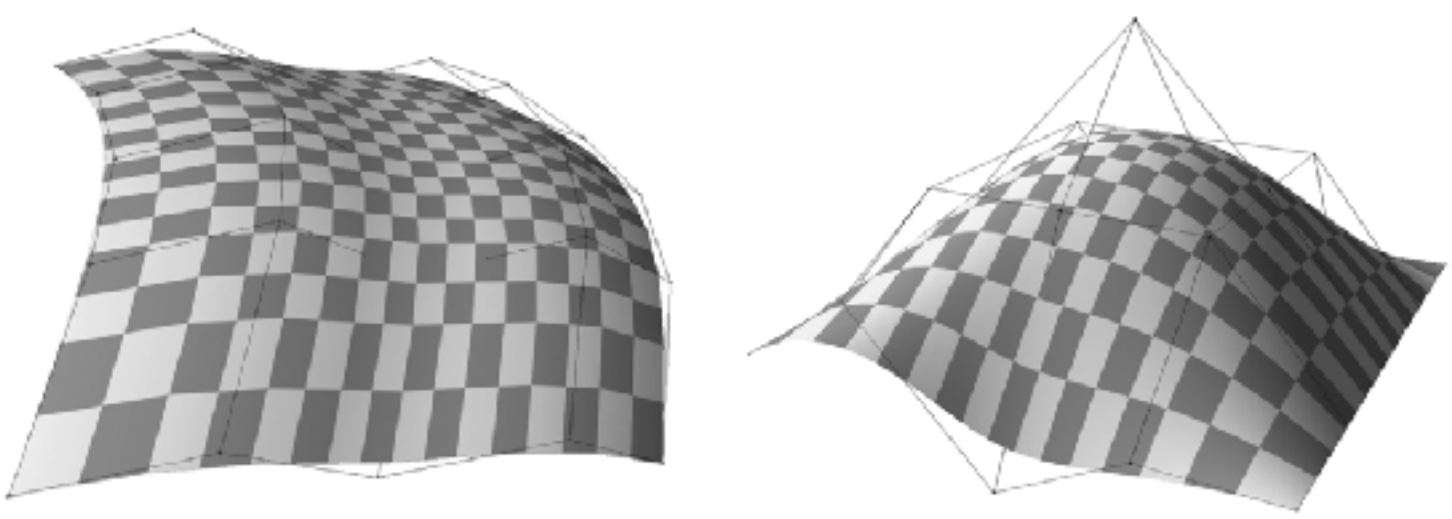
$$\mathbf{p}(u,v) = \sum_{i=0}^{k} \sum_{j=0}^{l} \mathbf{p}_{ij} \, N_i^n(u) N_j^n(v)$$



Spline Surfaces

Tensor product surfaces ("curves of curves")

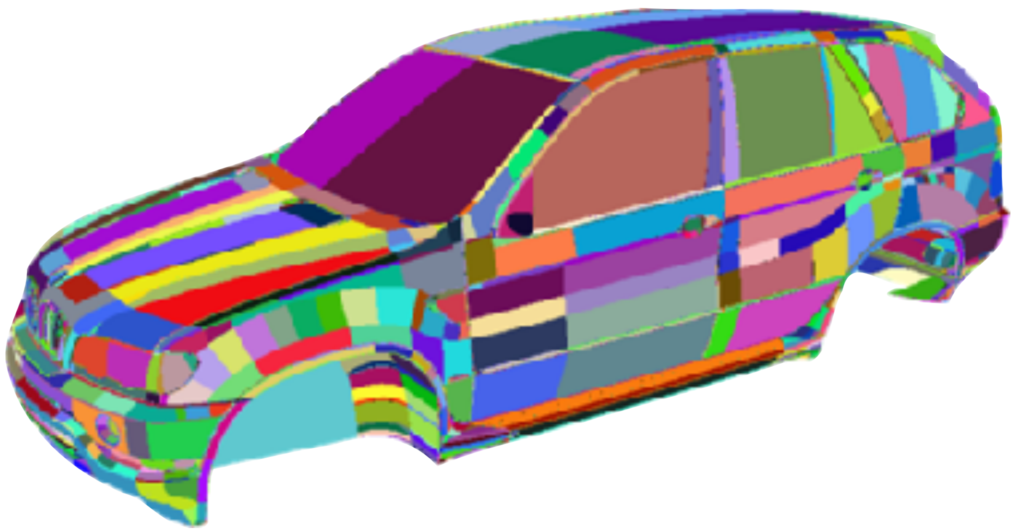
- Rectangular grid of control points
- Rectangular surface patch



Spline Surfaces

Tensor product surfaces ("curves of curves")

- Rectangular grid of control points
- Rectangular surface patch



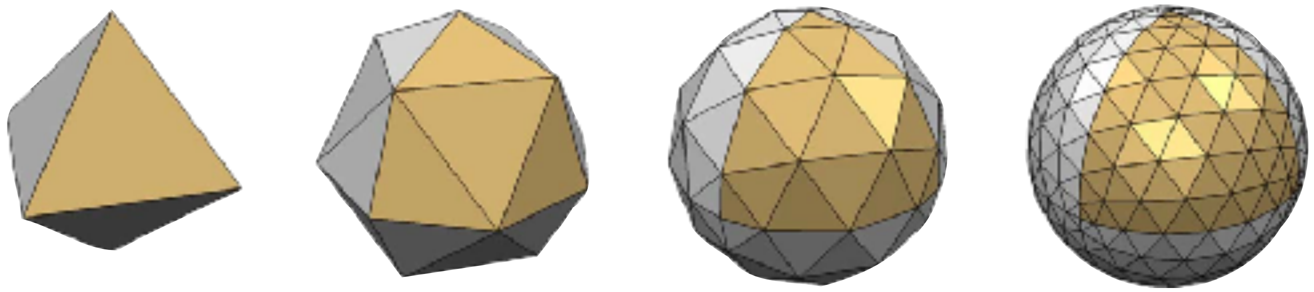
Problems:

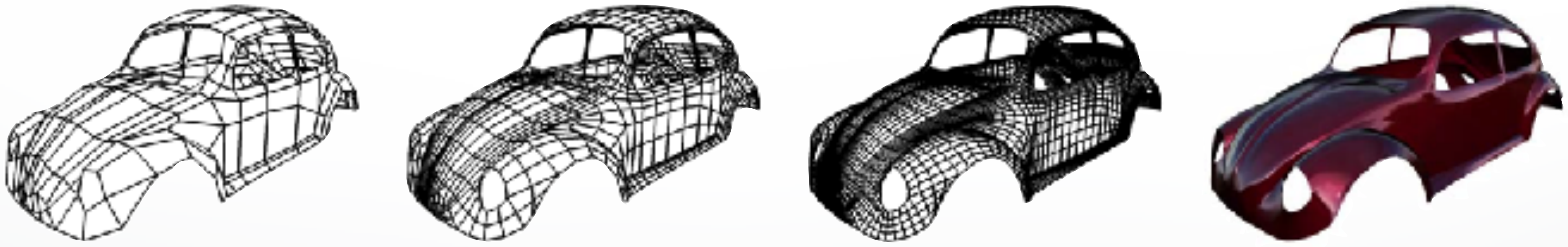
- Many patches for complex models
- Smoothness across patch boundaries
- Trimming for non-rectangular patches

Subdivision Surfaces

Generalization of spline curves/surfaces

- Arbitrary control meshes
- Successive refinement (subdivision)
- Converges to smooth limit surface
- Connection between splines and meshes

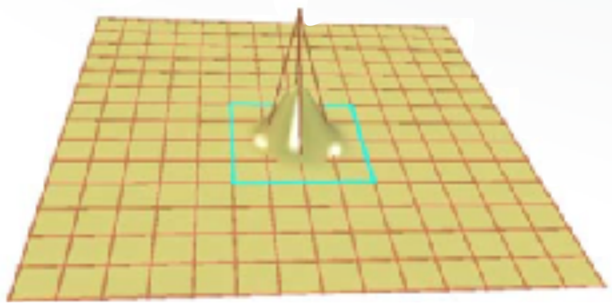


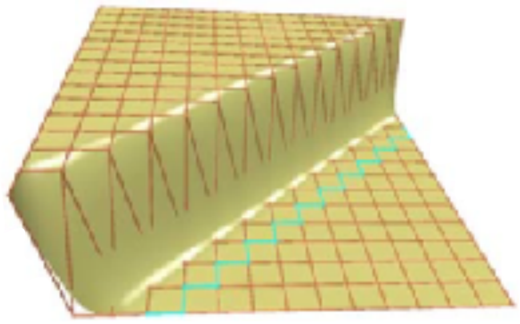


Spline & Subdivision Surfaces

Basis functions are smooth bumps

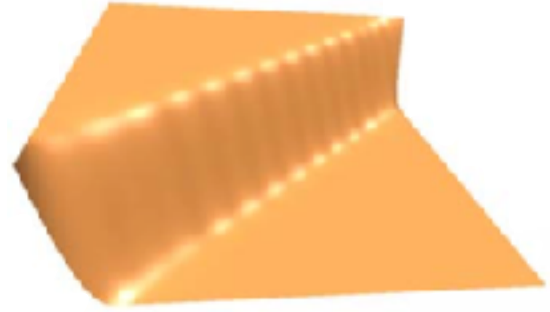
- Fixed support
- Fixed control grid





Bound to control points

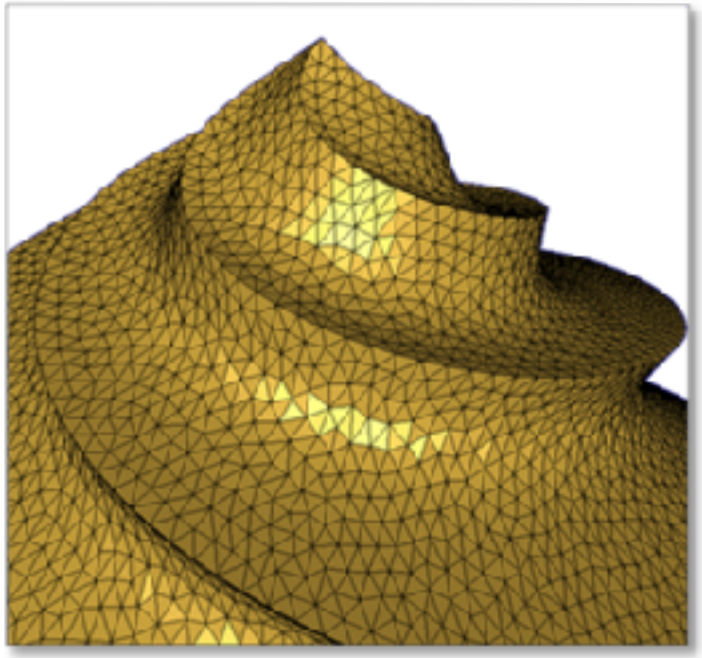
- Initial patch layout is crucial
- Requires experts!

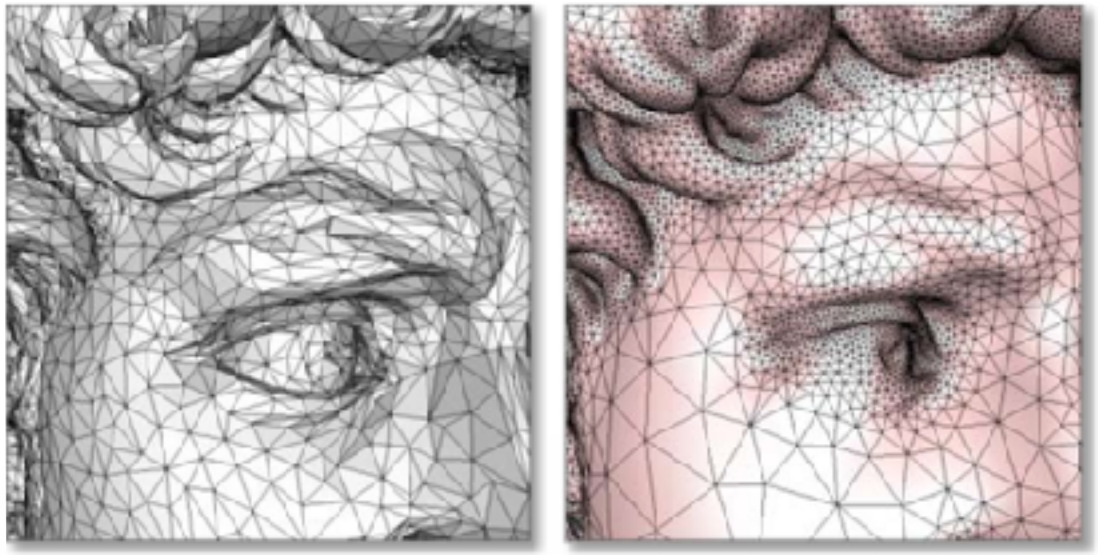


De-couple deformation from surface representation!

Discrete Surfaces: Point Sets, Meshes

- Flexible
- Suitable for highly detailed scanned data
- No analytic surface
- No inherent “editability”



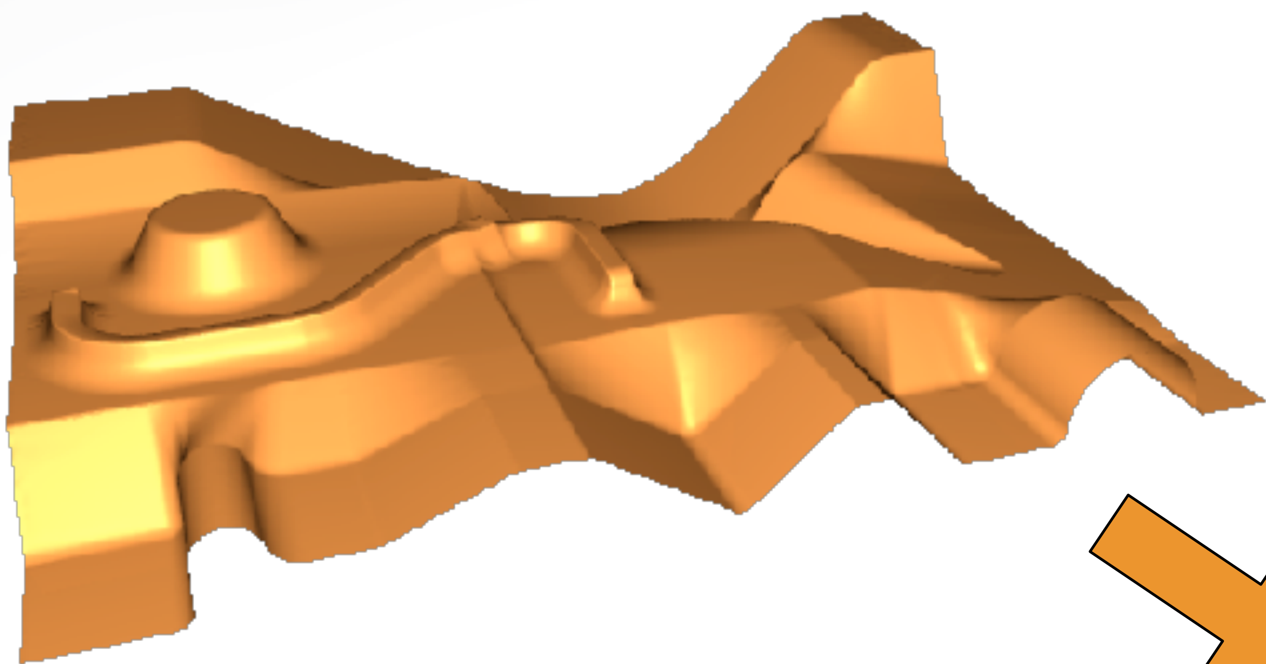


Mesh Editing

Outline

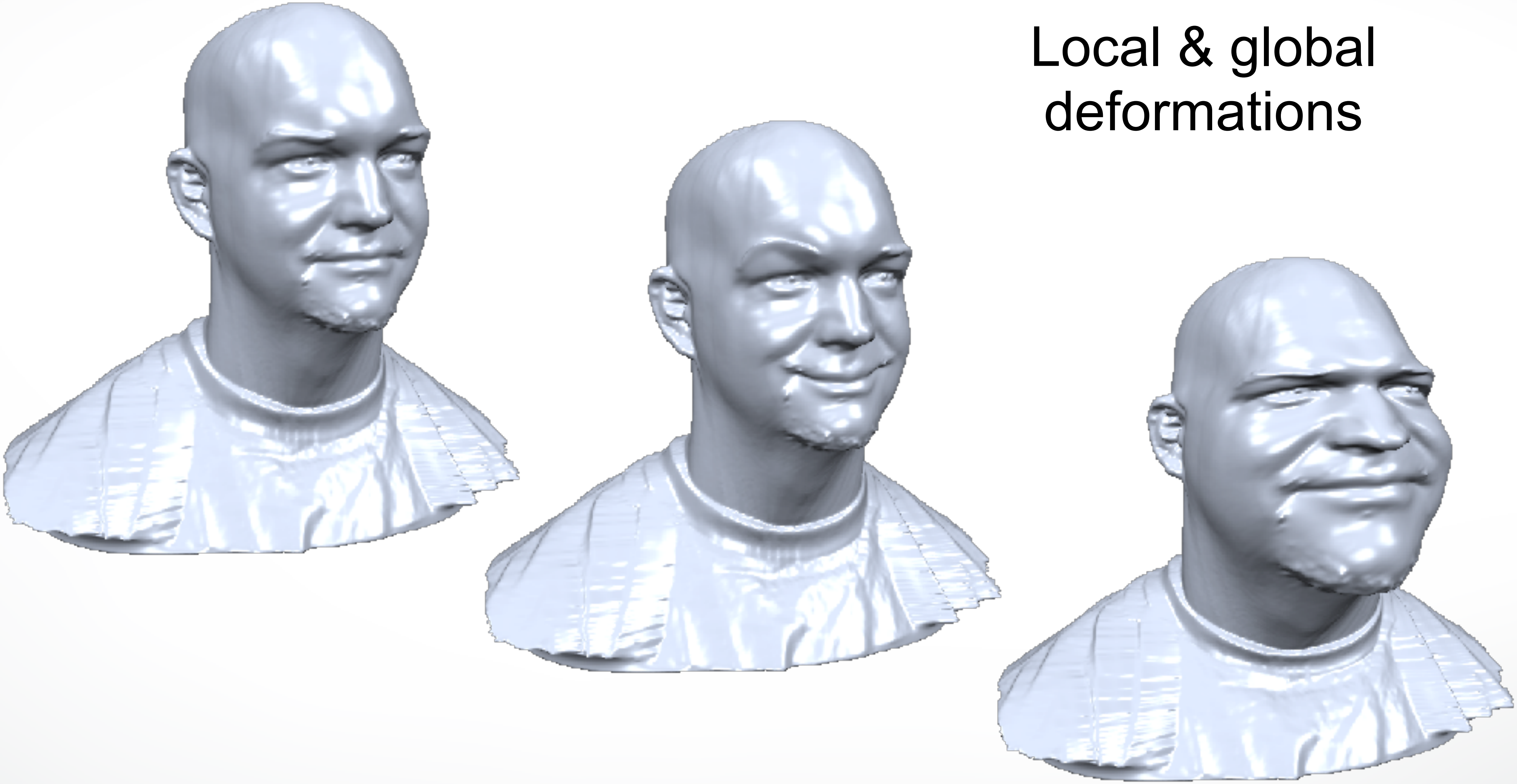
- **Surface-Based Deformation**
  - **Linear Methods**
  - Non-Linear Methods
- Spatial Deformation

Mesh Deformation



Global deformation
with intuitive
detail preservation

Mesh Deformation

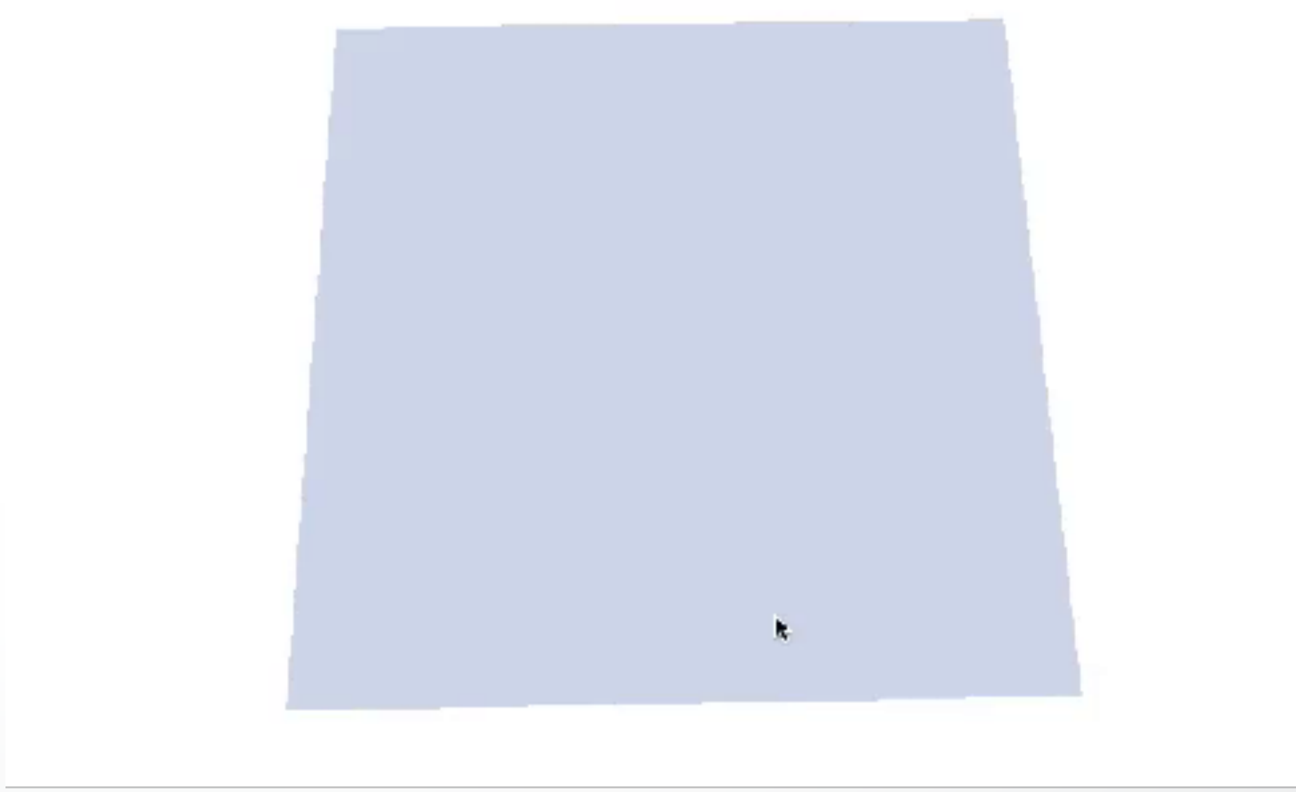


Local & global deformations

Linear Surface-Based Deformation

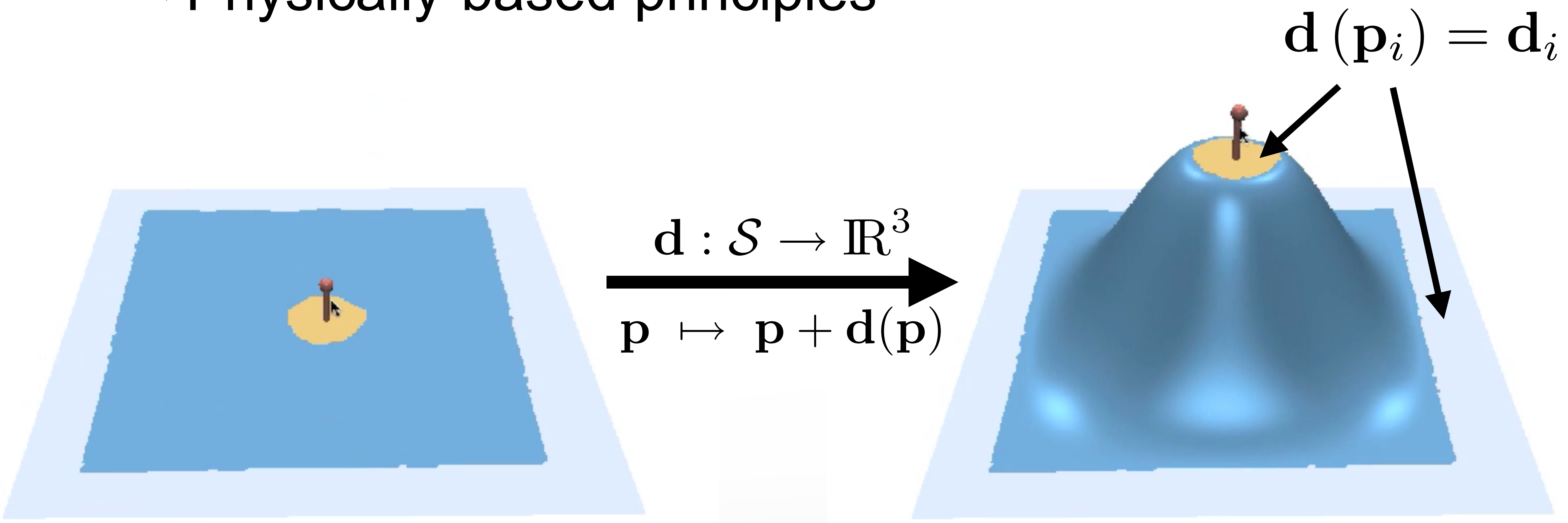
- **Shell-Based Deformation**
- Multiresolution Deformation
- Differential Coordinates

Modeling Metaphor



Modeling Metaphor

- Mesh deformation by displacement function **d**
  - Interpolate prescribed constraints
  - Smooth, intuitive deformation
  - ➡Physically-based principles



Shell Deformation Energy

- **Stretching**
  - Change of local distances
  - Captured by 1st fundamental form

$$\int_\Omega k_s \left\| \mathbf{I} - \bar{\mathbf{I}} \right\|^2$$

$$\mathbf{I} = \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

- **Bending**
  - Change of local curvature
  - Captured by 2nd fundamental form

$$\int_\Omega k_b \left\| \mathbf{II} - \bar{\mathbf{II}} \right\|^2$$

$$\mathbf{II} = \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{vu}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

- Stretching & bending is sufficient
  - Differential geometry: "1st and 2nd fundamental forms determine a surface up to rigid motion."

Physically-Based Deformation

- Nonlinear stretching & bending energies

$$\int_{\Omega} k_s \underbrace{\left\| \mathbf{I} - \mathbf{I}' \right\|^2}_{\text{stretching}} + k_b \underbrace{\left\| \mathbf{II} - \mathbf{II}' \right\|^2}_{\text{bending}} \; \mathrm{d}u\mathrm{d}v$$

- Linearize terms → Quadratic energy

$$\int_{\Omega} k_s \underbrace{\left(\|\mathbf{d}_u\|^2 + \|\mathbf{d}_v\|^2 \right)}_{\text{stretching}} + k_b \underbrace{\left(\|\mathbf{d}_{uu}\|^2 + 2\,\|\mathbf{d}_{uv}\|^2 + \|\mathbf{d}_{vv}\|^2 \right)}_{\text{bending}} \mathrm{d}u\mathrm{d}v$$

Physically-Based Deformation

- Minimize linearized bending energy

$$E(\mathbf{d}) \;=\; \int_{\mathcal{S}} \left\|\mathbf{d}_{uu}\right\|^2 + 2\left\|\mathbf{d}_{uv}\right\|^2 + \left\|\mathbf{d}_{vv}\right\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

$$f(x) \rightarrow \min$$

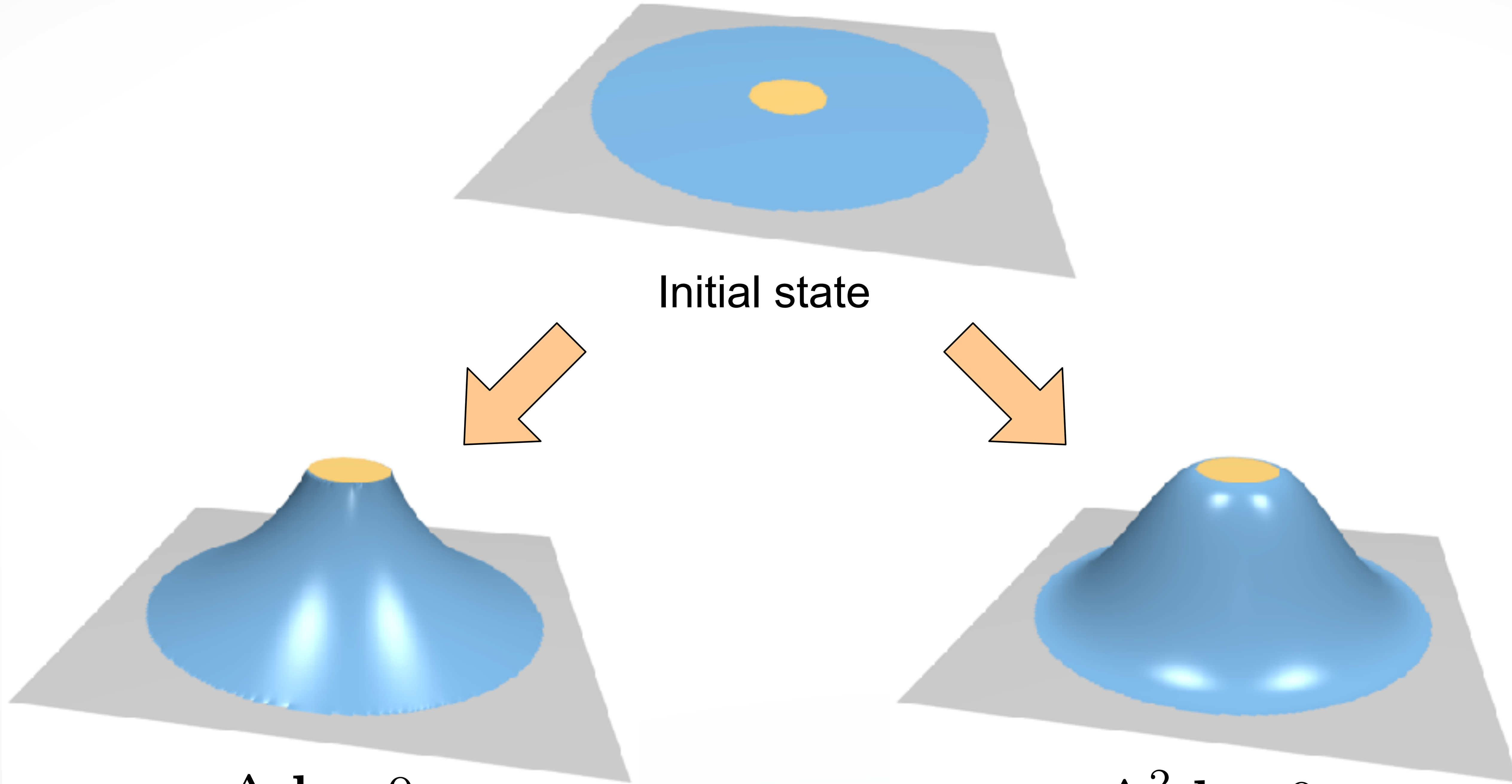
- Variational calculus → Euler-Lagrange PDE

$$\Delta^2\mathbf{d} \;:=\; \mathbf{d}_{uuuu} + 2\mathbf{d}_{uuvv} + \mathbf{d}_{vvvv} \;=\; 0$$

$$f'(x) = 0$$

➡ "Best" deformation that satisfies constraints

Deformation Energies



$$\Delta \mathbf{d} = 0$$

(Membrane)

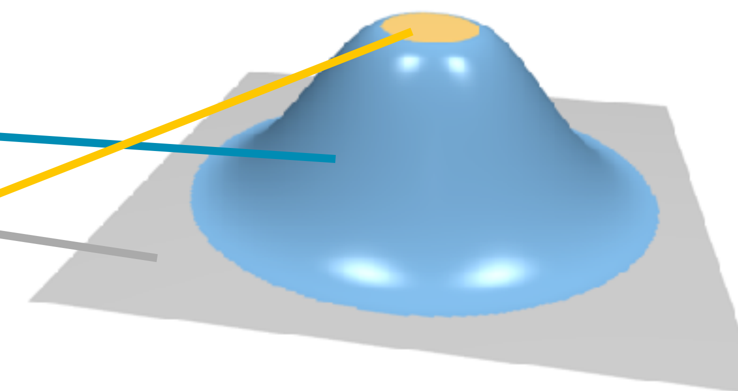
$$\Delta^2 \mathbf{d} = 0$$

(Thin plate)

PDE Discretization

- Euler-Lagrange PDE

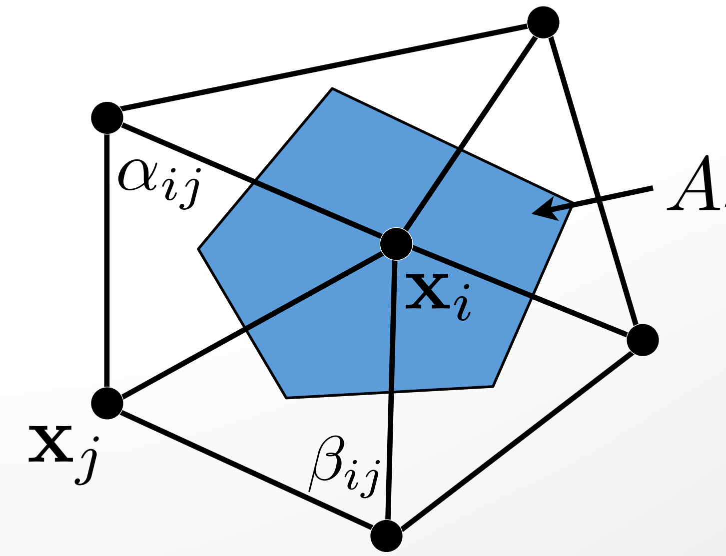
$$\Delta^2 \mathbf{d} = \mathbf{0}$$
$$\mathbf{d} = \mathbf{0}$$
$$\mathbf{d} = \delta \mathbf{h}$$



- Laplace discretization

$$\Delta \mathbf{d}_i = \frac{1}{2A_i} \sum_{j \in \mathcal{N}_i} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{d}_j - \mathbf{d}_i)$$

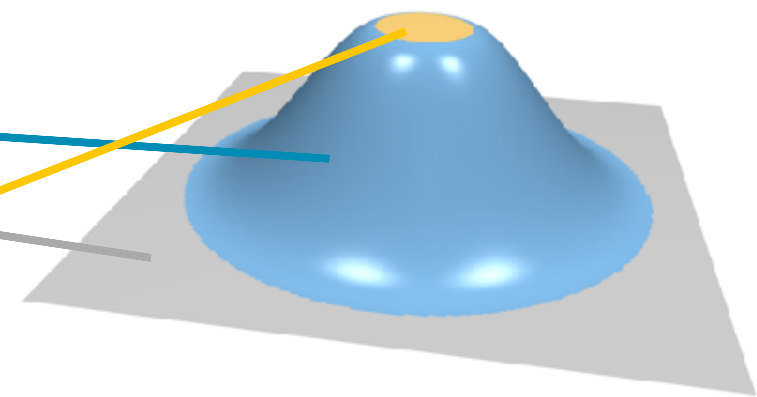
$$\Delta^2 \mathbf{d}_i = \Delta(\Delta \mathbf{d}_i)$$



Linear System

- Sparse linear system (19 nz/row)

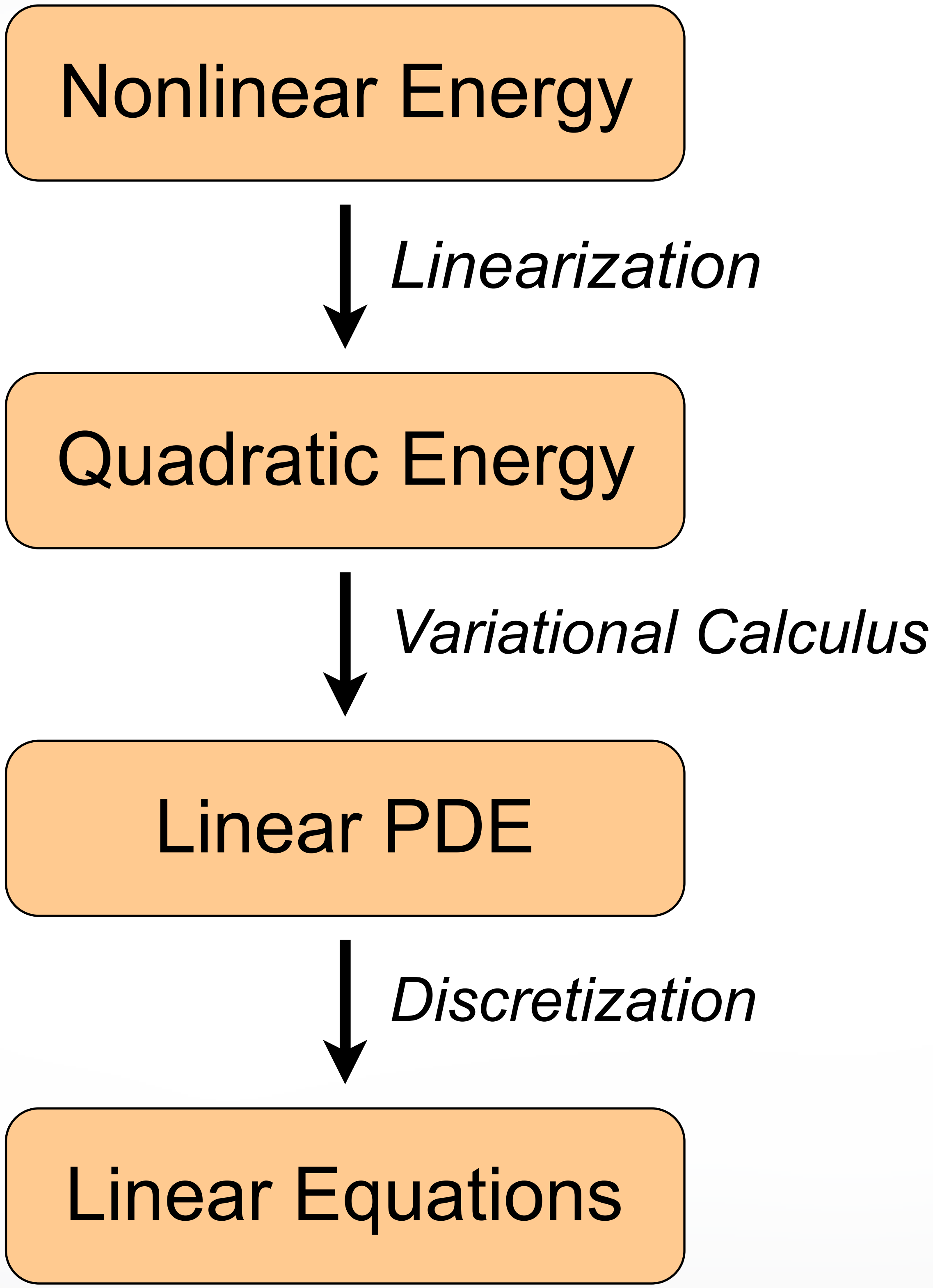
$$\begin{pmatrix} & \Delta^2 & \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{d}_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \delta\mathbf{h}_i \end{pmatrix}$$



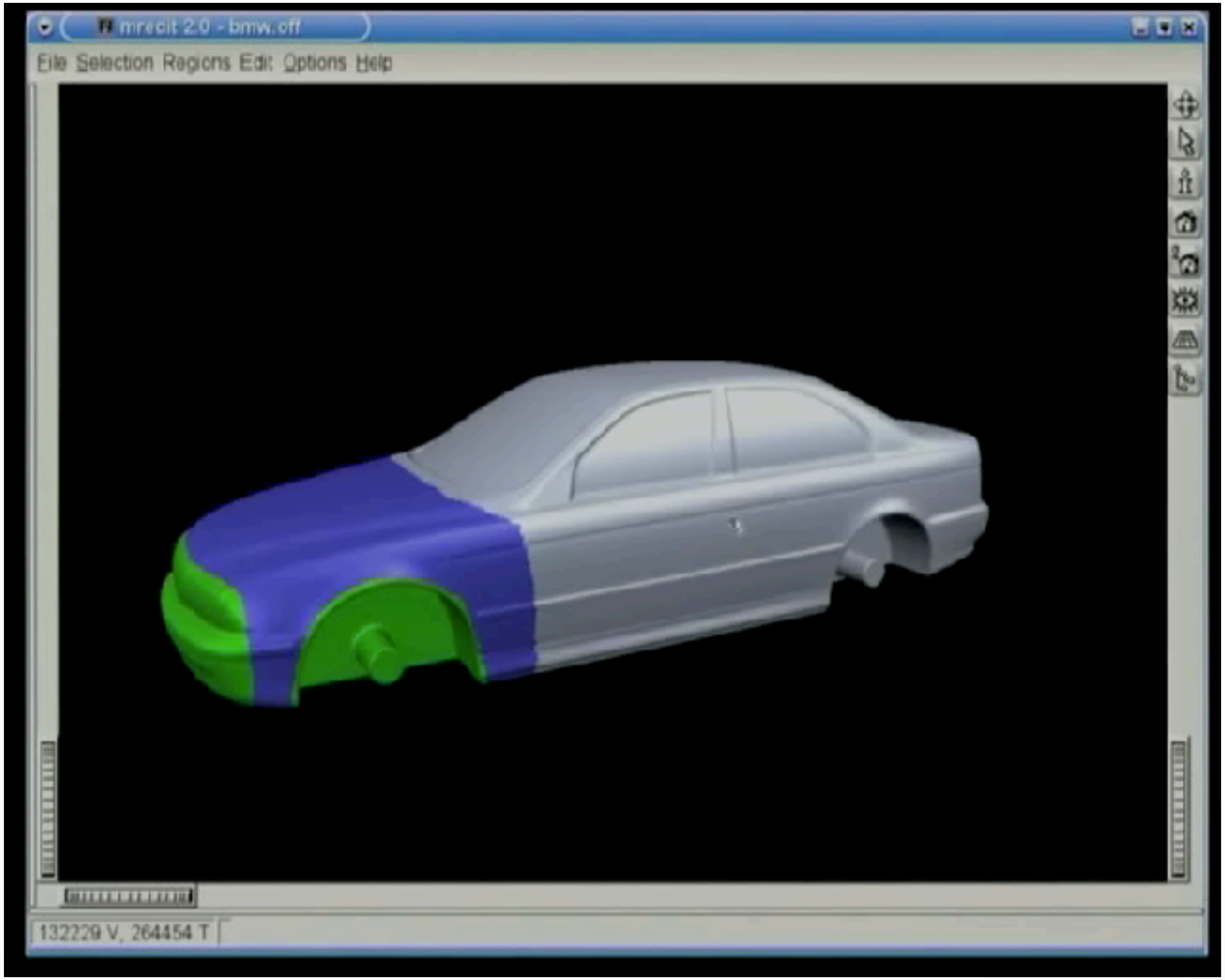
  - Turn into symmetric positive definite system

- Solve this system *each frame*
  - Use efficient linear solvers !!!
  - Sparse Cholesky factorization
  - **See book for details**

Derivation Steps



CAD-Like Deformation



[Botsch & Kobbelt, SIGGRAPH 04]

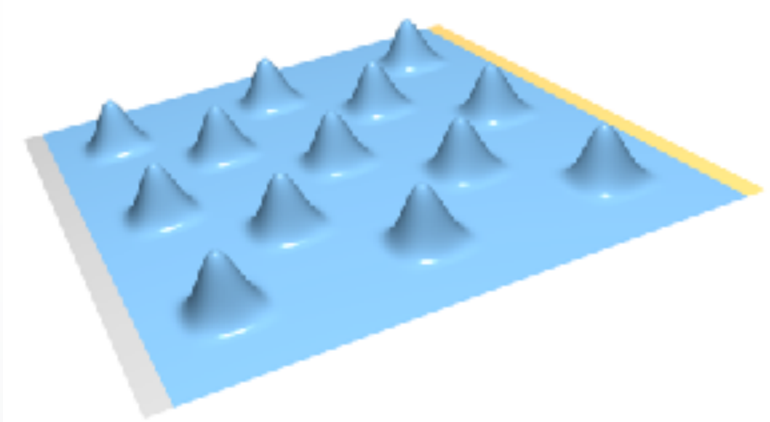
Facial Animation

Linear Surface-Based Deformation

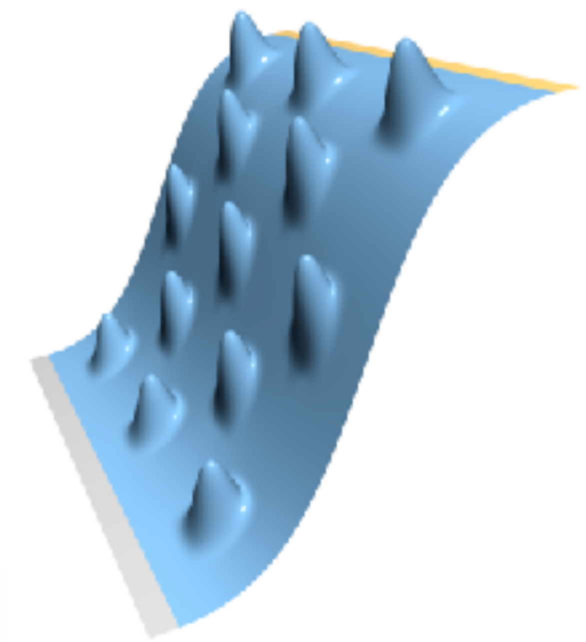
- Shell-Based Deformation
- **Multiresolution Deformation**
- Differential Coordinates

Multiresolution Modeling

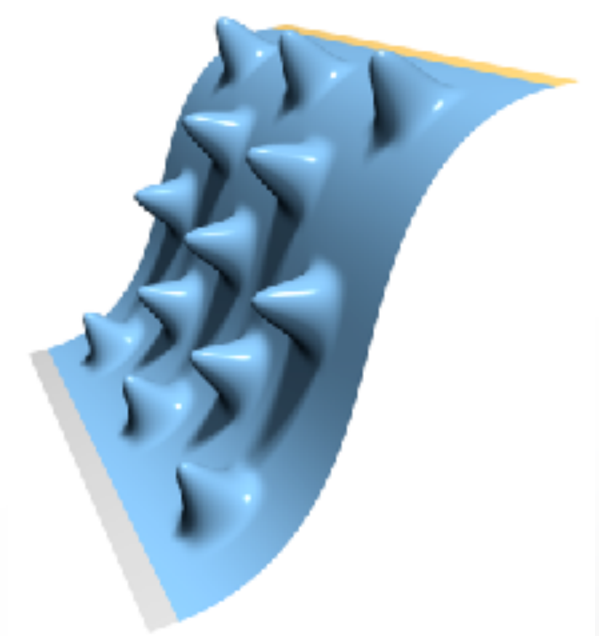
- Even pure translations induce local rotations!
  ➡ Inherently non-linear coupling
- Alternative approach
  – Linear deformation + multi-scale decomposition...



Original

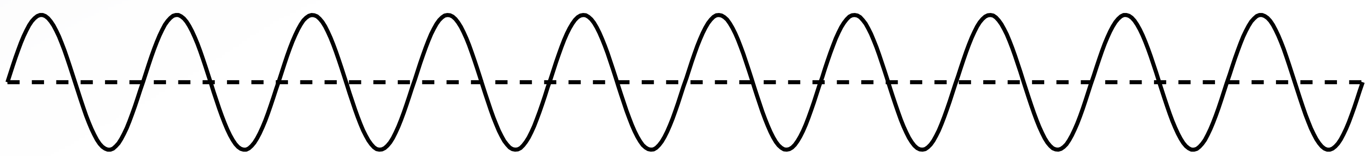


Linear



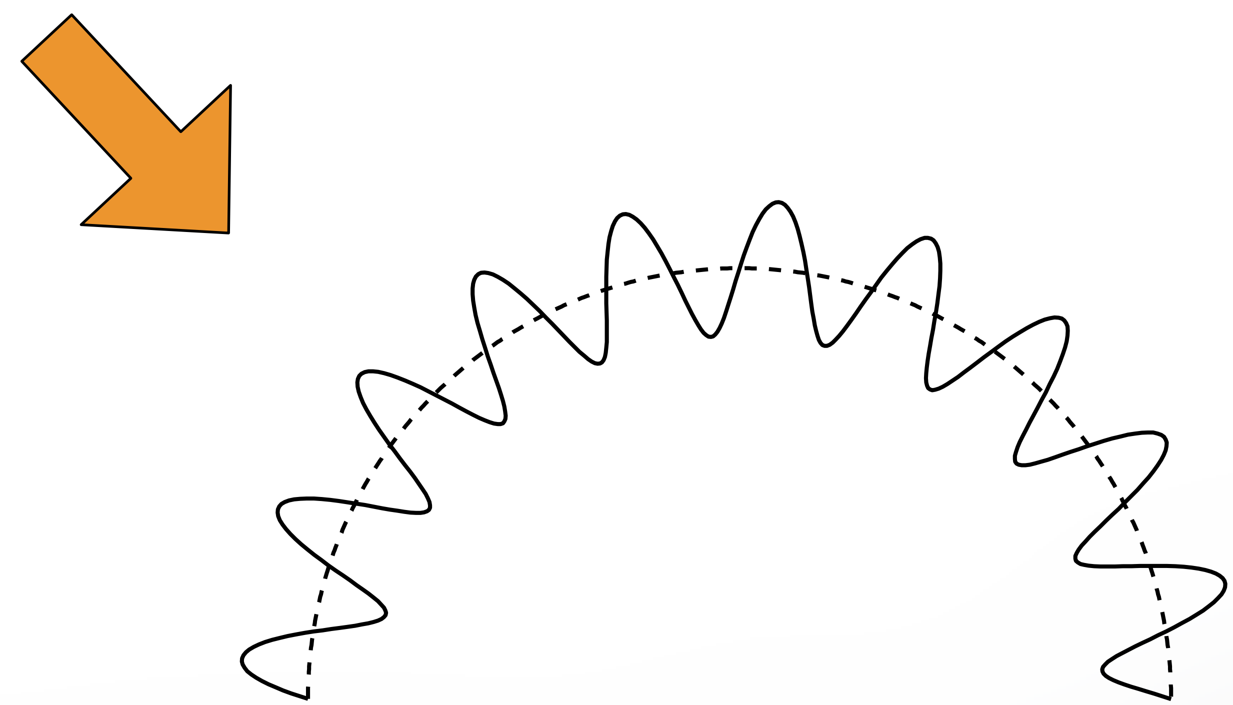
Nonlinear

Multiresolution Editing



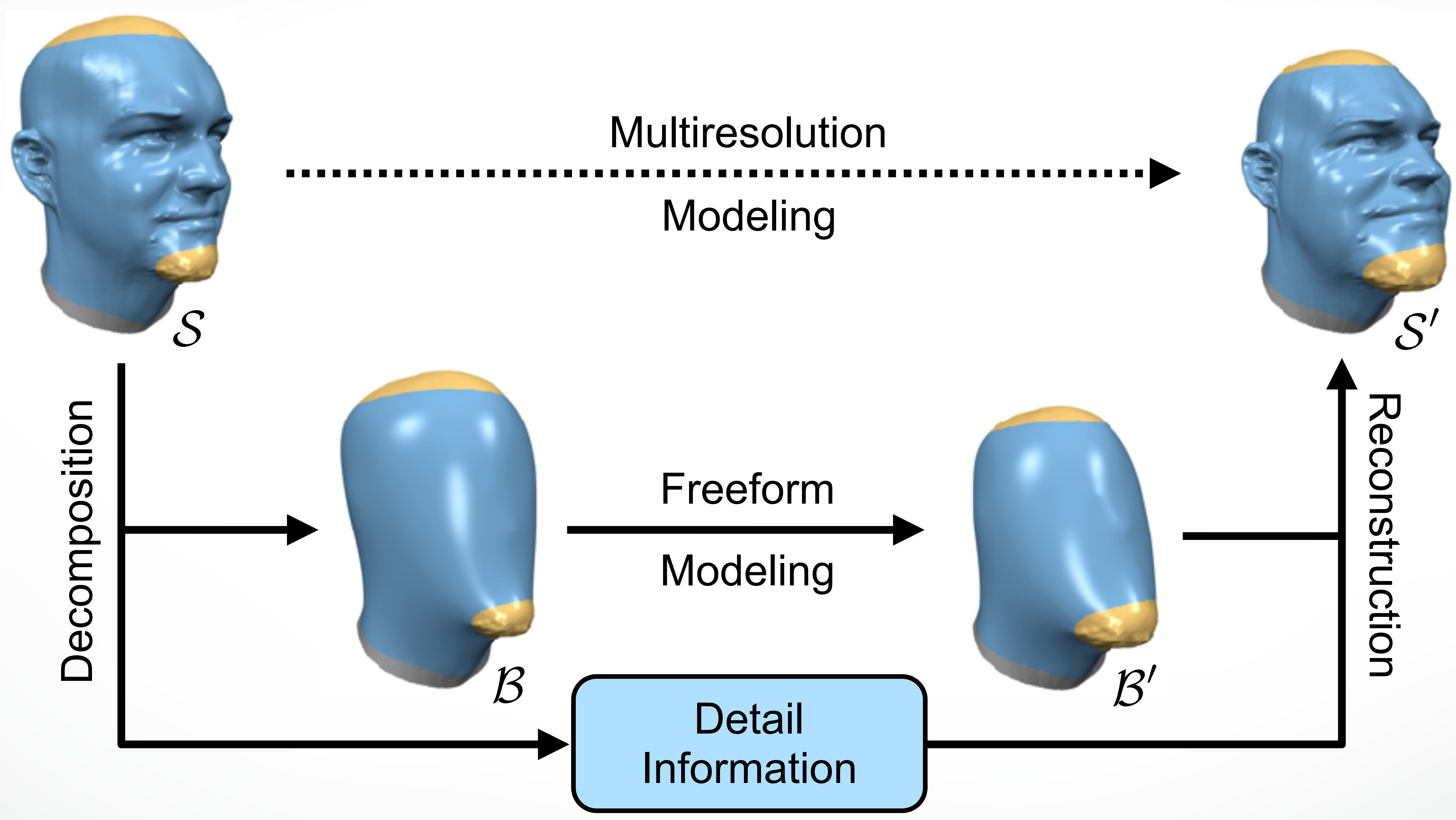
Frequency decomposition

Change low frequencies

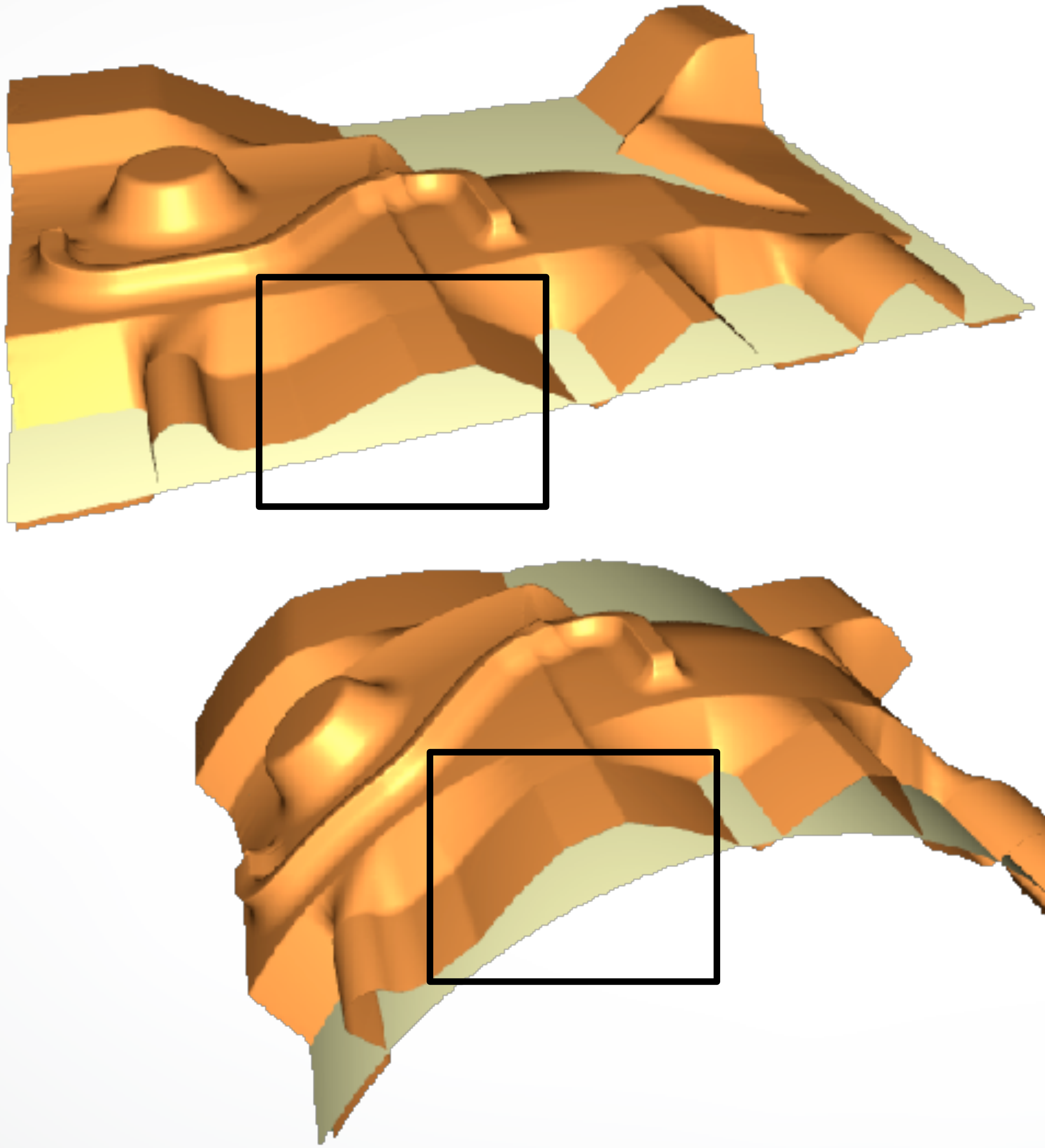


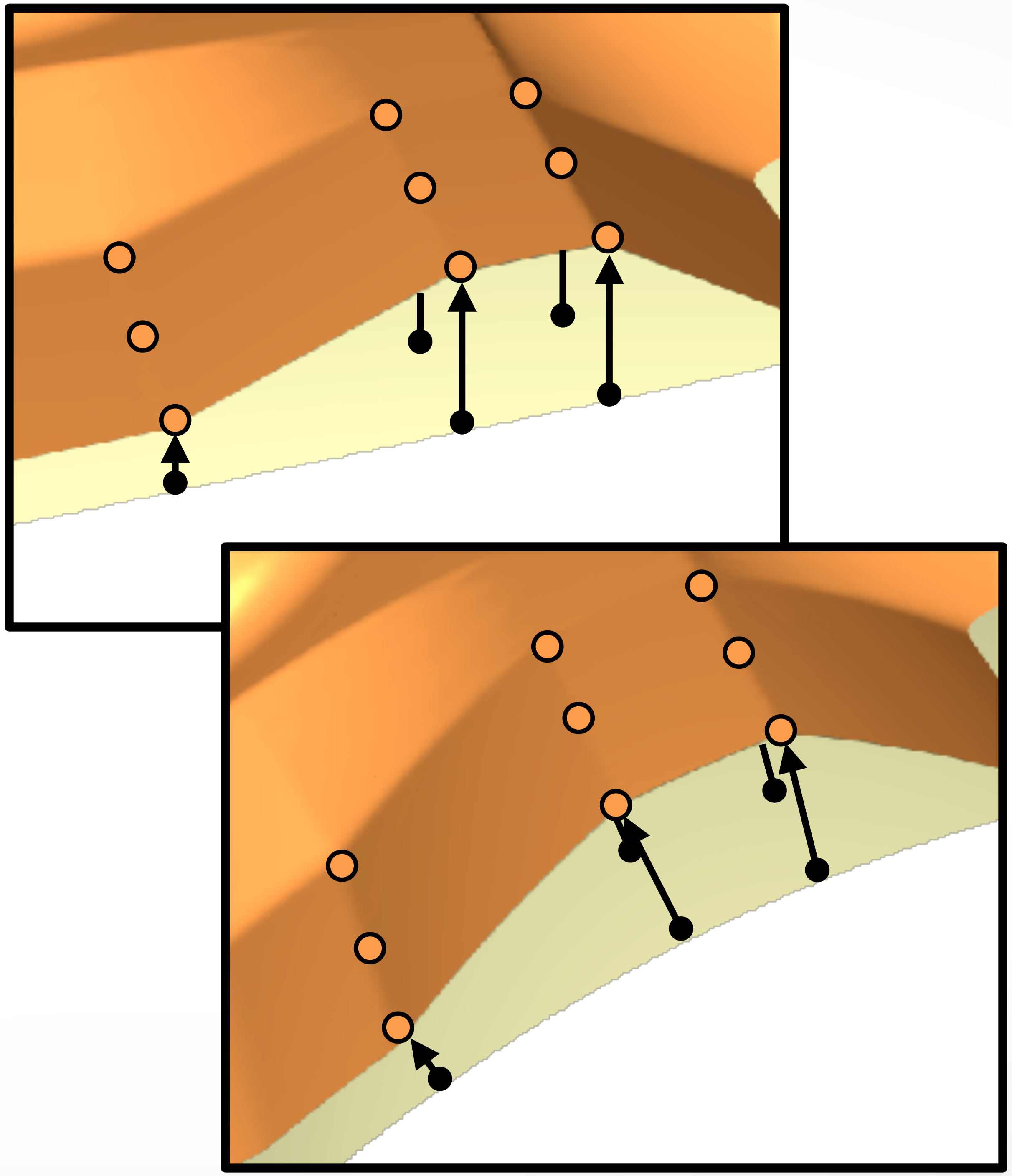
Add high frequency details, stored in local frames

Multiresolution Editing



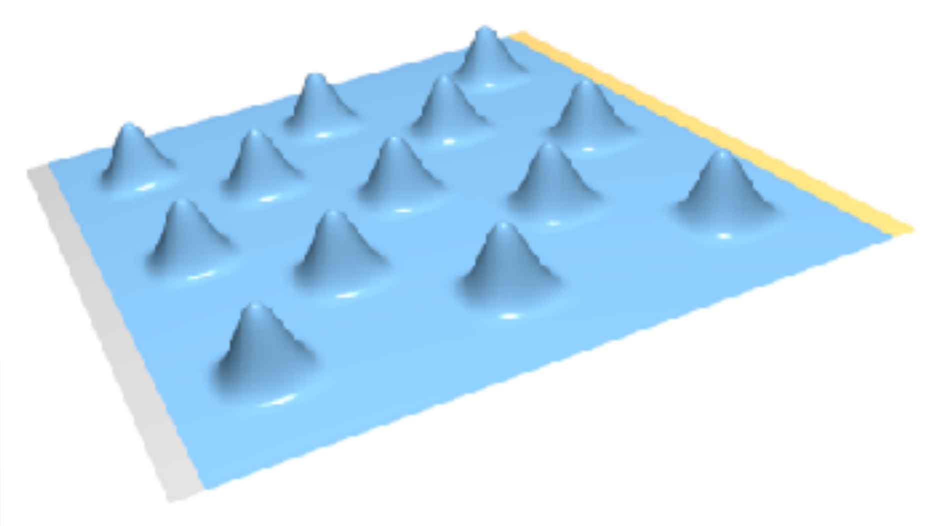
Normal Displacements



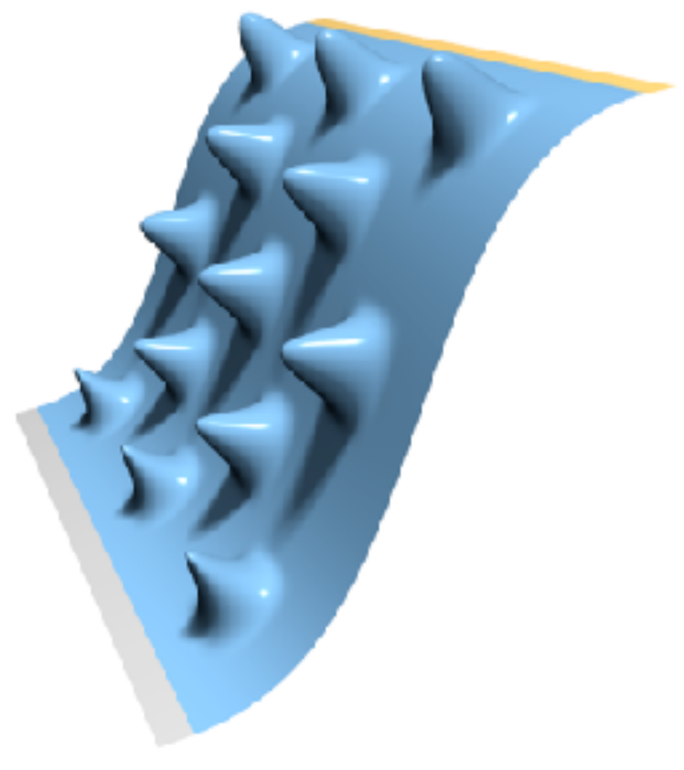


Limitations

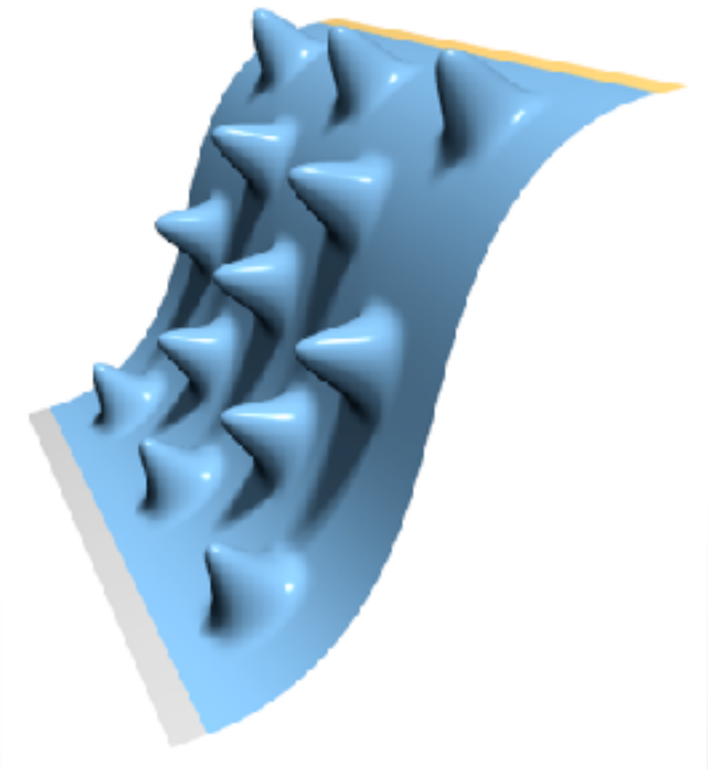
- Neighboring displacements are not coupled
  - Surface bending changes their angle
  - Leads to volume changes or self-intersections



Original



Normal Displ.



Nonlinear

Limitations

- Neighboring displacements are not coupled
  - Surface bending changes their angle
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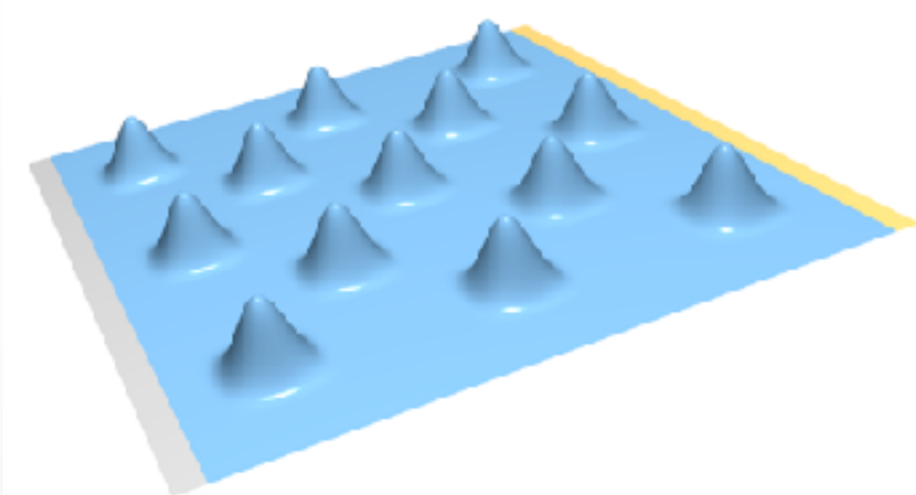
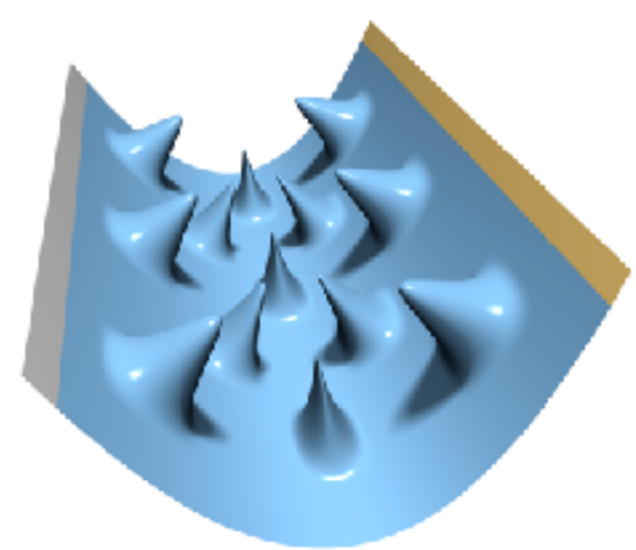
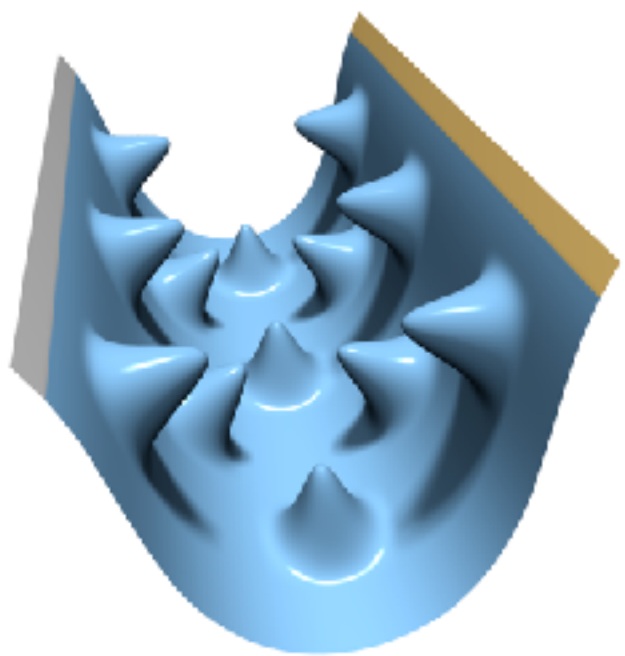


Original



Normal Displ.



Nonlinear

Limitations

- Neighboring displacements are not coupled
  - Surface bending changes their angle
  - Leads to volume changes or self-intersections
- Multiresolution hierarchy difficult to compute
  - Complex topology
  - Complex geometry
  - Might require more hierarchy levels

Linear Surface-Based Deformation

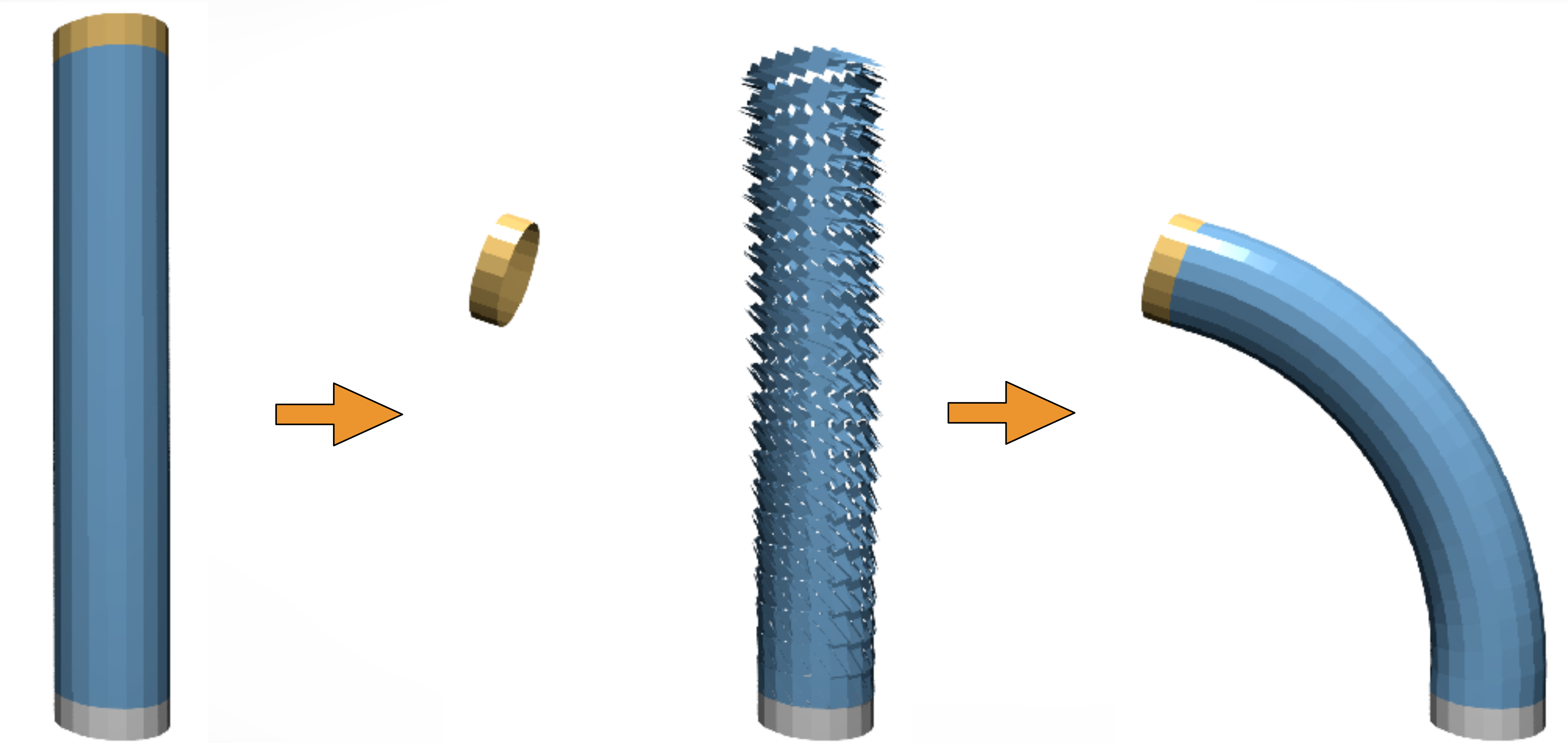
- Shell-Based Deformation
- Multiresolution Deformation
- **Differential Coordinates**

Differential Coordinates

1. Manipulate *differential coordinates* instead of *spatial* coordinates
   - Gradients, Laplacians, local frames
   - Intuition: Close connection to surface normal

2. Find mesh with desired differential coords
   - Cannot be solved exactly
   - Formulate as energy minimization

Differential Coordinates



Original Rotated Diff-Coords Reconstructed Mesh

Differential Coordinates

- **Which differential coordinate $\boldsymbol{\delta}_i$?**
  - Gradients
  - Laplacians
  - ...
- How to get local transformations $T_i(\boldsymbol{\delta}_i)$?
  - Smooth propagation
  - Implicit optimization
  - ...

Gradient-Based Editing

- Manipulate gradient of a function (e.g. a surface)

$$\mathbf{g} = \nabla \mathbf{f} \qquad \mathbf{g} \mapsto \mathbf{T}(\mathbf{g})$$

- Find function **f**' whose gradient is (close to) **g**'=T(**g**)

$$\mathbf{f}' = \operatorname*{argmin}_{\mathbf{f}} \int_{\Omega} \|\nabla \mathbf{f} - \mathbf{T}(\mathbf{g})\|^2 \, \mathrm{d}u\mathrm{d}v$$

- Variational calculus → Euler-Lagrange PDE

$$\Delta \mathbf{f}' = \operatorname{div} \mathbf{T}(\mathbf{g})$$

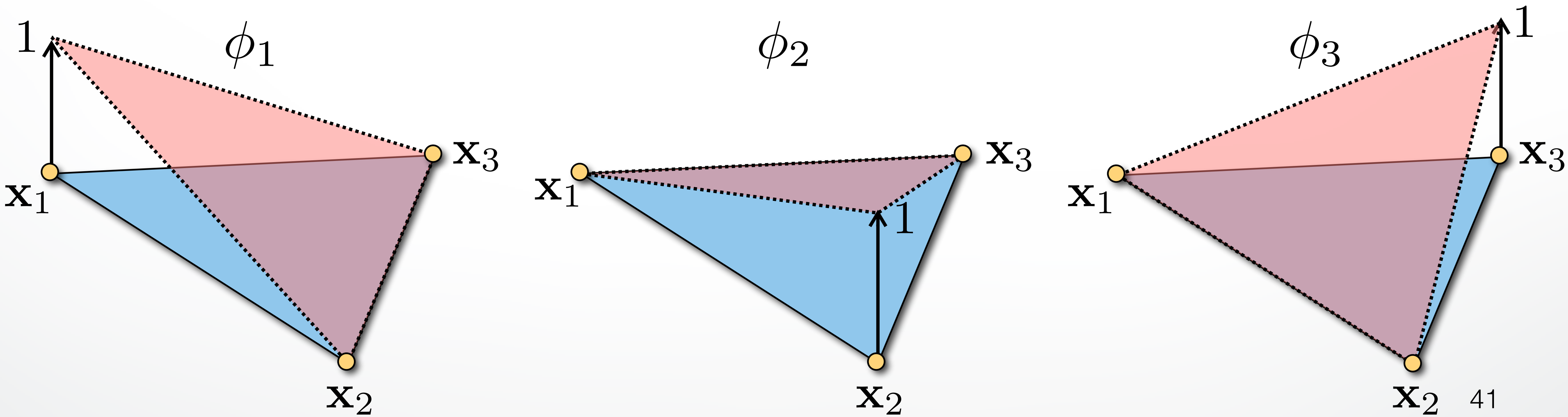
Gradient-Based Editing

- Consider piecewise linear ***coordinate function***

$$\mathbf{p}(u,v) = \sum_{v_i} \mathbf{p}_i \cdot \phi_i(u,v)$$

- Its gradient is

$$\nabla\mathbf{p}(u,v) = \sum_{v_i} \mathbf{p}_i \cdot \nabla\phi_i(u,v)$$



Gradient-Based Editing

- Consider piecewise linear ***coordinate function***

$$\mathbf{p}(u,v) = \sum_{v_i} \mathbf{p}_i \cdot \phi_i(u,v)$$

- Its gradient is

$$\nabla \mathbf{p}(u,v) = \sum_{v_i} \mathbf{p}_i \cdot \nabla \phi_i(u,v)$$

- It is constant per triangle

$$\nabla \mathbf{p}|_{f_j} \;=:\; \mathbf{g}_j \in \mathbb{R}^{3\times 3}$$

Gradient-Based Editing

- Gradient of coordinate function $\mathbf{p}$

$$\begin{pmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_F \end{pmatrix} = \underbrace{\mathbf{G}}_{(3F \times V)} \begin{pmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_V^T \end{pmatrix}$$

- Manipulate per-face gradients

$$\mathbf{g}_j \;\mapsto\; \mathbf{T}_j(\mathbf{g}_j)$$

Gradient-Based Editing

- Reconstruct mesh from new gradients
  - Overdetermined $(3F \times V)$ system
  - Weighted least squares system
  - ➡Linear Poisson system $\Delta \mathbf{p}' = \operatorname{div} \mathbf{T}(\mathbf{g})$

$$\underbrace{\mathbf{G}}_{\operatorname{div}\nabla = \Delta} \cdot \begin{pmatrix} {\mathbf{p}'_1}^T \\ \vdots \\ {\mathbf{p}'_V}^T \end{pmatrix} = \operatorname{div} \begin{pmatrix} \mathbf{T}_1(\mathbf{g}_1) \\ \vdots \\ \mathbf{T}_F(\mathbf{g}_F) \end{pmatrix}$$

Laplacian-Based Editing

- Manipulate Laplacians field of a surface

$$\mathbf{l} = \Delta(\mathbf{p}) \ , \quad \mathbf{l} \mapsto \mathbf{T}(\mathbf{l})$$

- Find surface whose Laplacian is (close to) $\boldsymbol{\delta}'=\mathbf{T}(\mathbf{l})$

$$\mathbf{p}' = \underset{\mathbf{p}}{\operatorname{argmin}} \int_{\Omega} \left\| \Delta\mathbf{p} - \mathbf{T}(\mathbf{l}) \right\|^2 \mathrm{d}u\mathrm{d}v$$

- Variational calculus yields Euler-Lagrange PDE

$$\Delta^2\mathbf{p}' = \Delta\mathbf{T}(\mathbf{l})$$

soft constraints

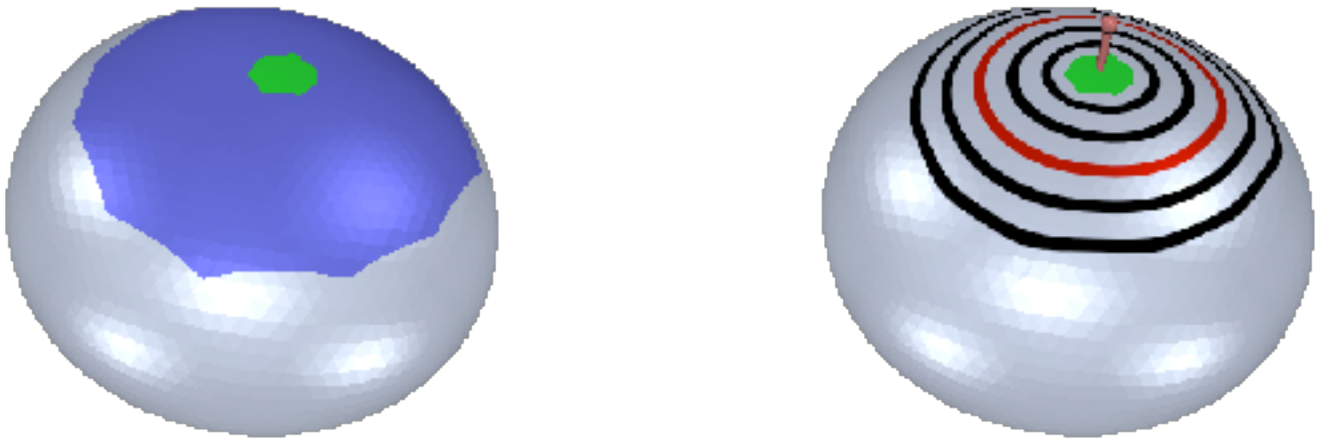
Differential Coordinates

- Which differential coordinate $\boldsymbol{\delta}_i$?
  - Gradients
  - Laplacians
  - ...

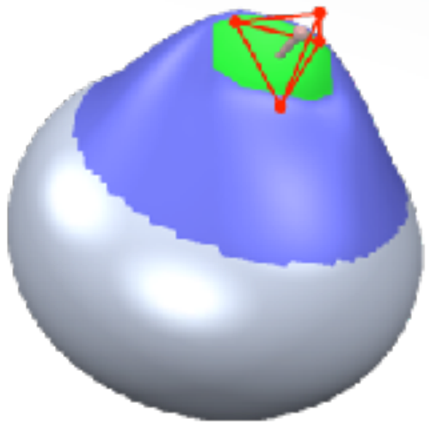
- **How to get local transformations $T_i(\boldsymbol{\delta}_i)$?**
  - Smooth propagation
  - Implicit optimization
  - ...

Smooth Propagation

1. Compute handle's deformation gradient
2. Extract rotation and scale/shear components
3. Propagate damped rotations over ROI



Deformation Gradient



- Handle has been transformed *affinely*

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

- Deformation gradient is

$$\nabla\mathbf{T}(\mathbf{x}) = \mathbf{A}$$

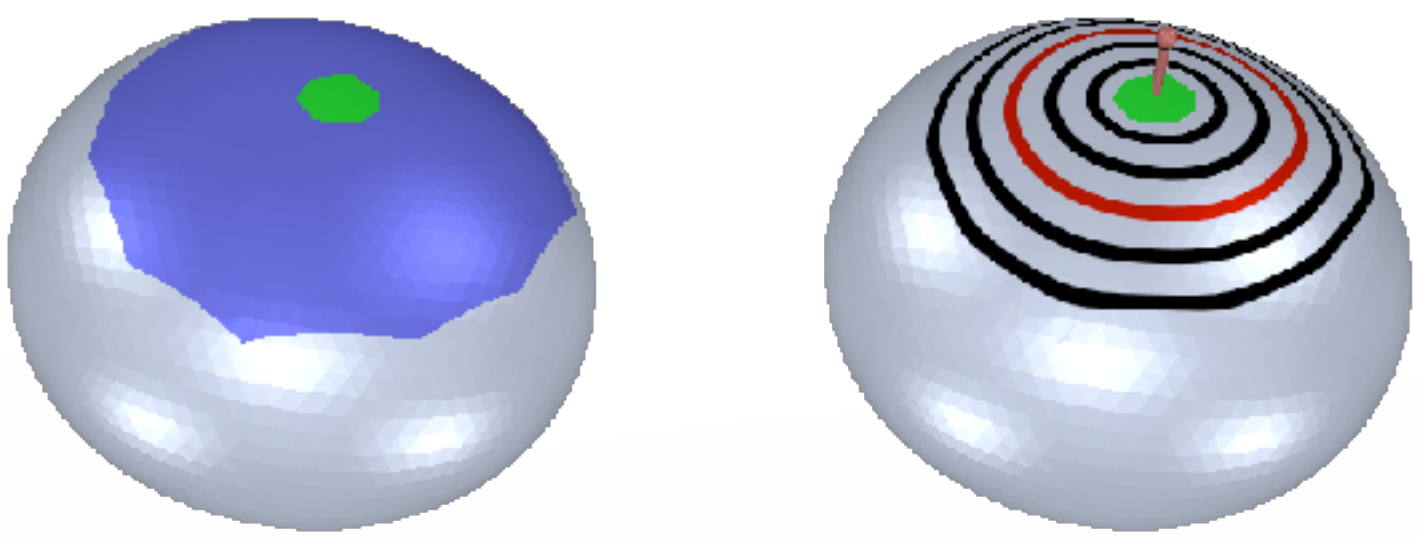
- Extract rotation $\mathbf{R}$ and scale/shear $\mathbf{S}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad \Rightarrow \quad \mathbf{R} = \mathbf{U}\mathbf{V}^T,\ \mathbf{S} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^T$$

SVD

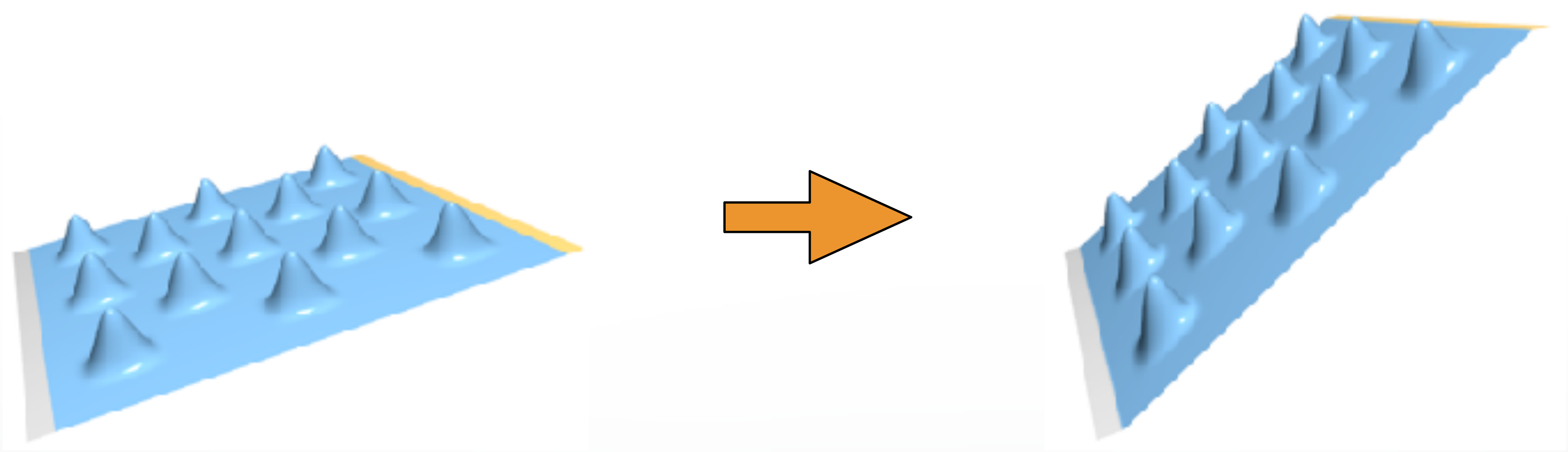
Smooth Propagation

- Construct smooth scalar field [0,1]
  - $s(\mathbf{x})=1$: Full deformation (handle)
  - $s(\mathbf{x})=0$: No deformation (fixed part)
  - $s(\mathbf{x})\in(0,1)$: Damp handle transformation (in between)



Limitations

- Differential coordinates work well for **rotations**
  - Represented by deformation gradient
- **Translations** don't change deformation gradient
  - Translations don't change differential coordinates
  - *"Translation insensitivity"*



Implicit Optimization

- Optimize for positions $\mathbf{p}_i$' & transformations $\mathbf{T}_i$

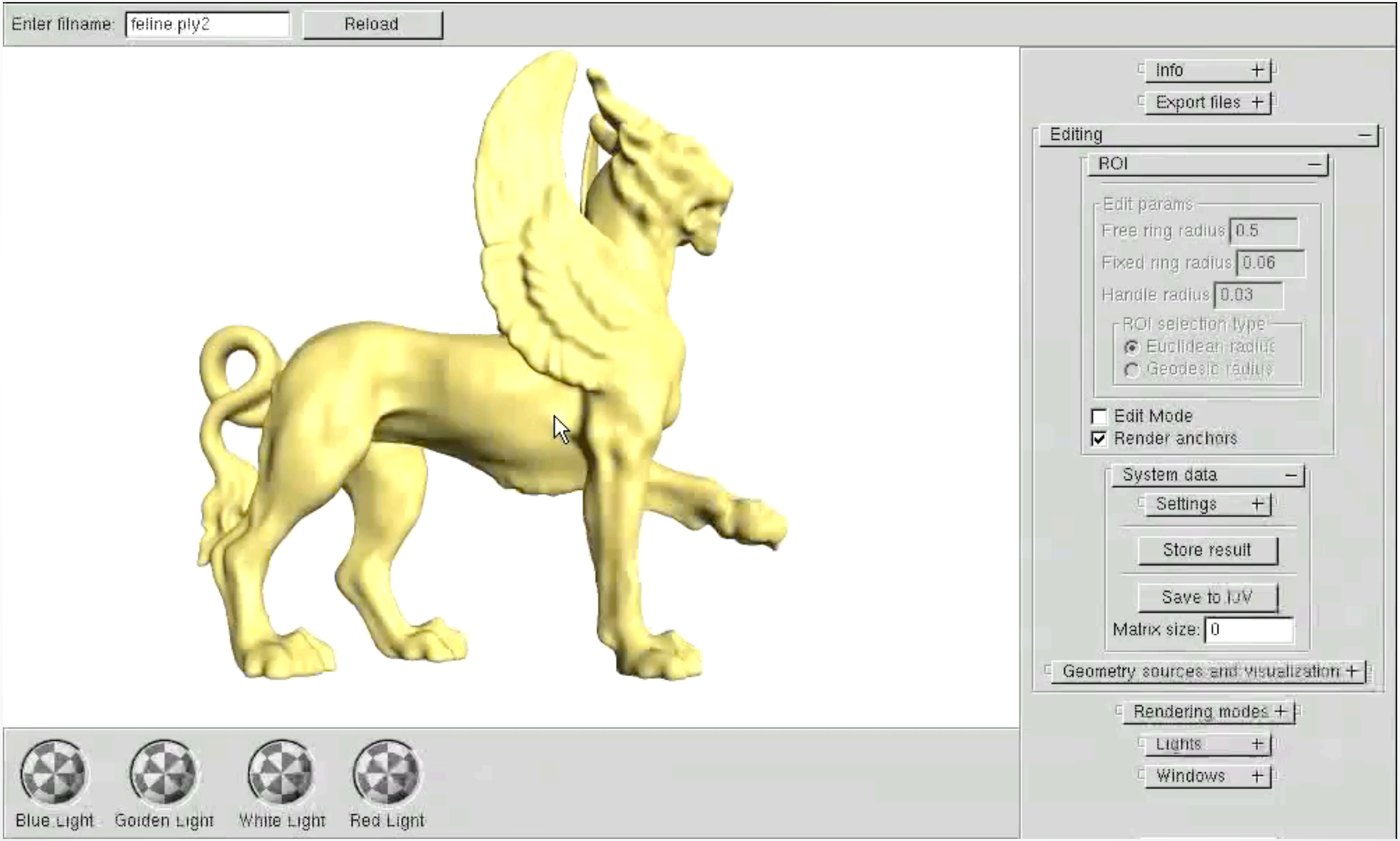
$$\Delta^2 \begin{pmatrix} \vdots \\ \mathbf{p}'_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \Delta \mathbf{T}_i(\mathbf{l}_i) \\ \vdots \end{pmatrix} \quad \leftrightarrow \quad \mathbf{T}_i(\mathbf{p}_i - \mathbf{p}_j) \;=\; \mathbf{p}'_i - \mathbf{p}'_j$$

- Linearize rotation/scale → one linear system

$$\mathbf{R}\mathbf{x} \;\approx\; \mathbf{x} + \mathbf{r} \times \mathbf{x} = \begin{pmatrix} 1 & -r_3 & r_2 \\ r_3 & 1 & -r_1 \\ -r_2 & r_1 & 1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{T}_i = \begin{pmatrix} s & -r_3 & r_2 \\ r_3 & s & -r_1 \\ -r_2 & r_1 & s \end{pmatrix}$$

Laplacian Surface Editing



Connection to Shells?

- Neglect local transformations $\mathbf{T}_i$ for a moment...

$$\int \|\Delta \mathbf{p}' - \mathbf{l}\|^2 \rightarrow \min \quad \longrightarrow \quad \Delta^2 \mathbf{p}' = \Delta \mathbf{l}$$

$$\downarrow \quad \begin{aligned} \mathbf{p}' &= \mathbf{p} + \mathbf{d} \\ \mathbf{l} &= \Delta \mathbf{p} \end{aligned}$$

$$\Delta^2(\mathbf{p} + \mathbf{d}) = \Delta^2 \mathbf{p}$$

$$\downarrow$$

$$\int \|\mathbf{d}_{uu}\|^2 + 2\|\mathbf{d}_{uv}\|^2 + \|\mathbf{d}_{vv}\|^2 \rightarrow \min \quad \longleftarrow \quad \Delta^2 \mathbf{d} = 0$$

- Basic formulations equivalent!
- Differ in detail preservation
  - Rotation of Laplacians
  - Multi-scale decomposition

Linear Surface-Based Deformation

- Shell-Based Deformation
- Multiresolution Deformation
- Differential Coordinates

Next Time

Non-Linear

Surface Deformations

http://cs621.hao-li.com

Thanks!