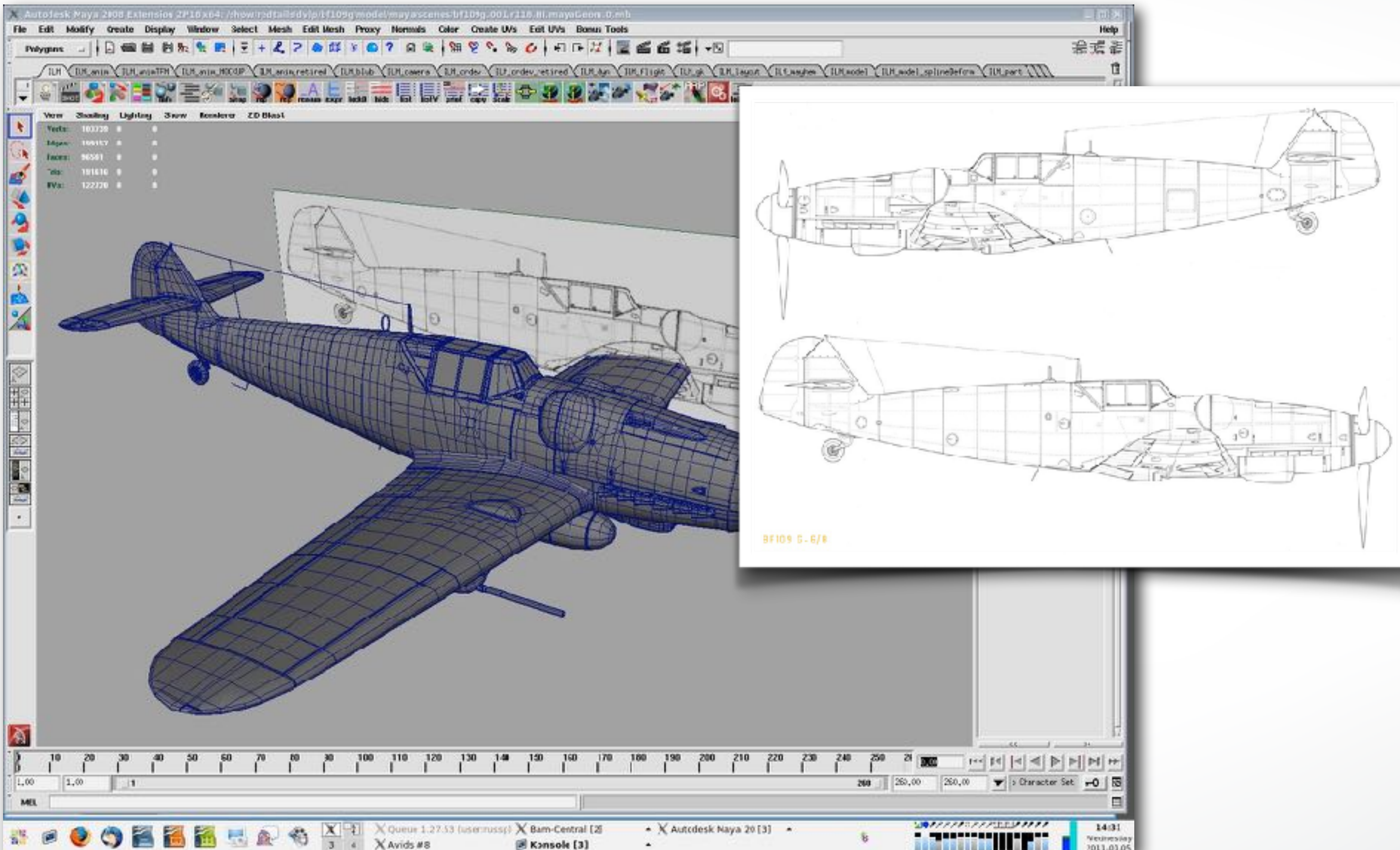
9.1 **Surface Parameterization**

Hao Li

http://cs621.hao-li.com

Modeling

Modeling

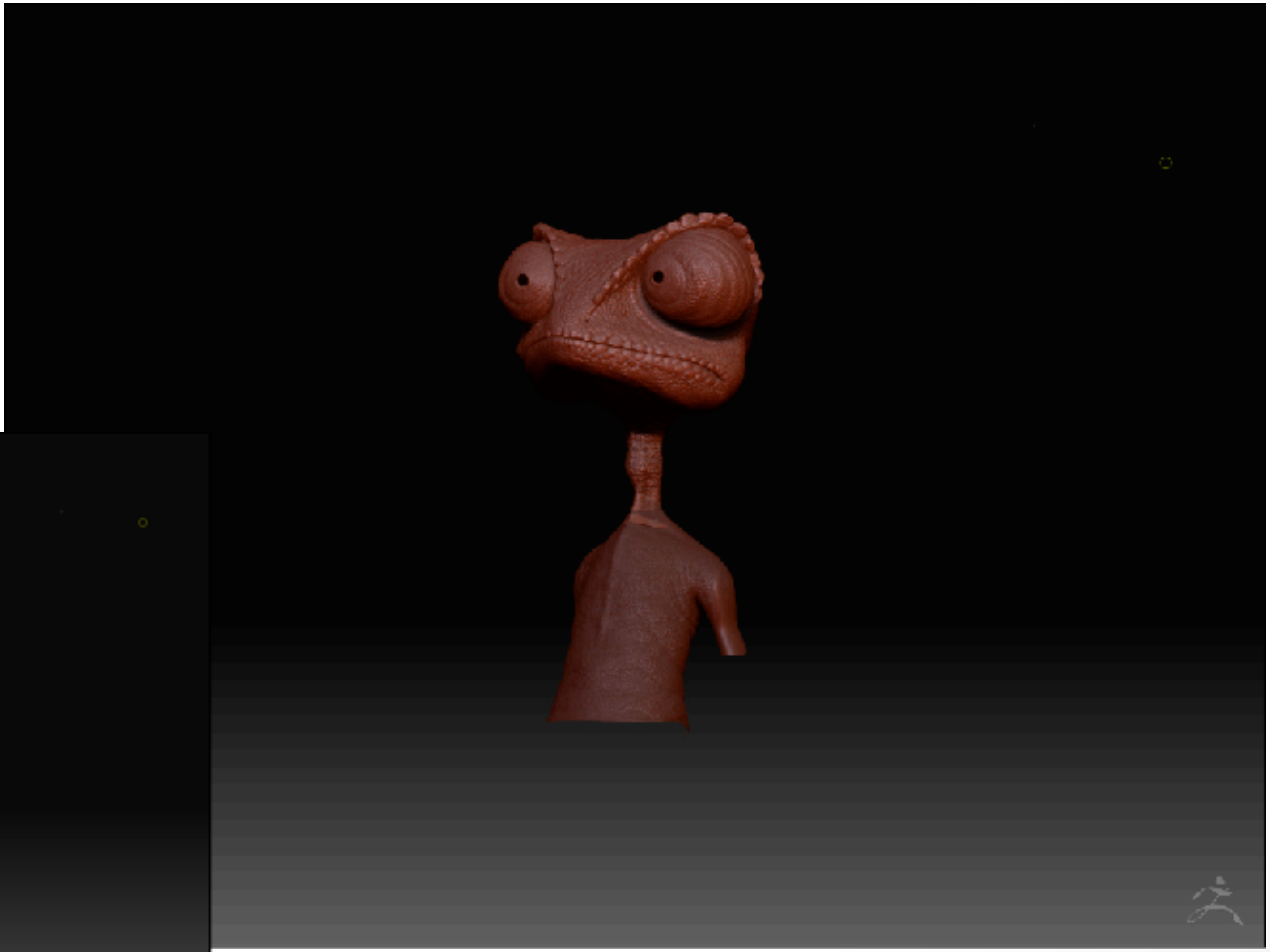


Viewpaint

The creation of a 3D assets surface, including that surface's color, texture, opacity, and reflectivity (or specularity).

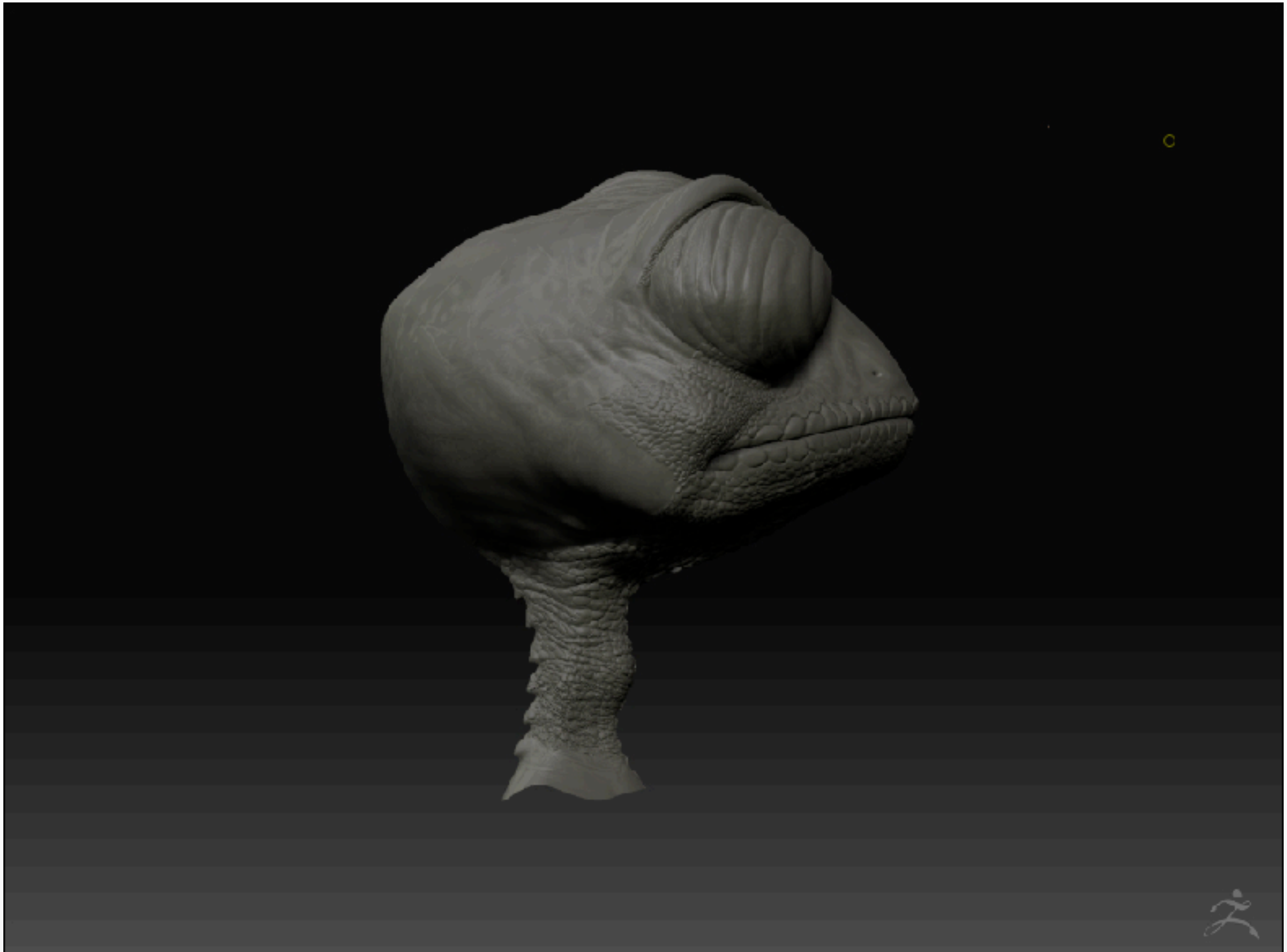
Viewpaint

Rango: Creating creature scale textures in ZBrush...

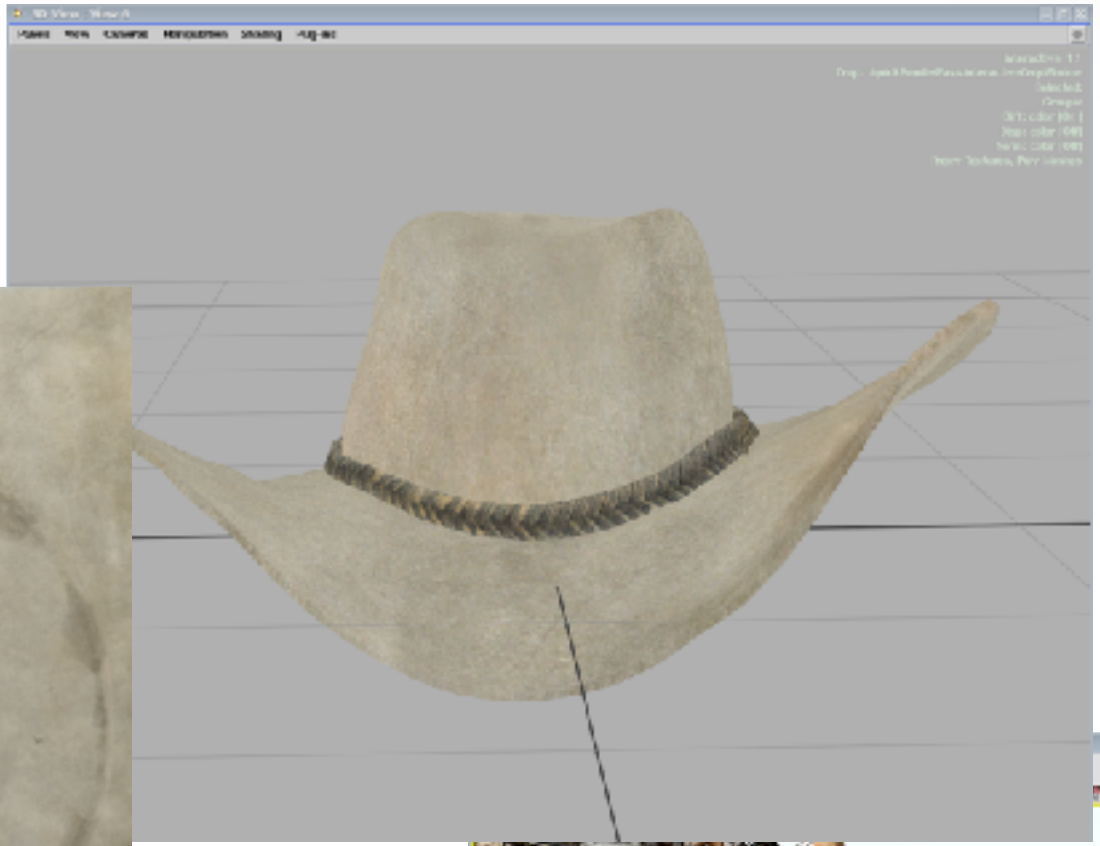


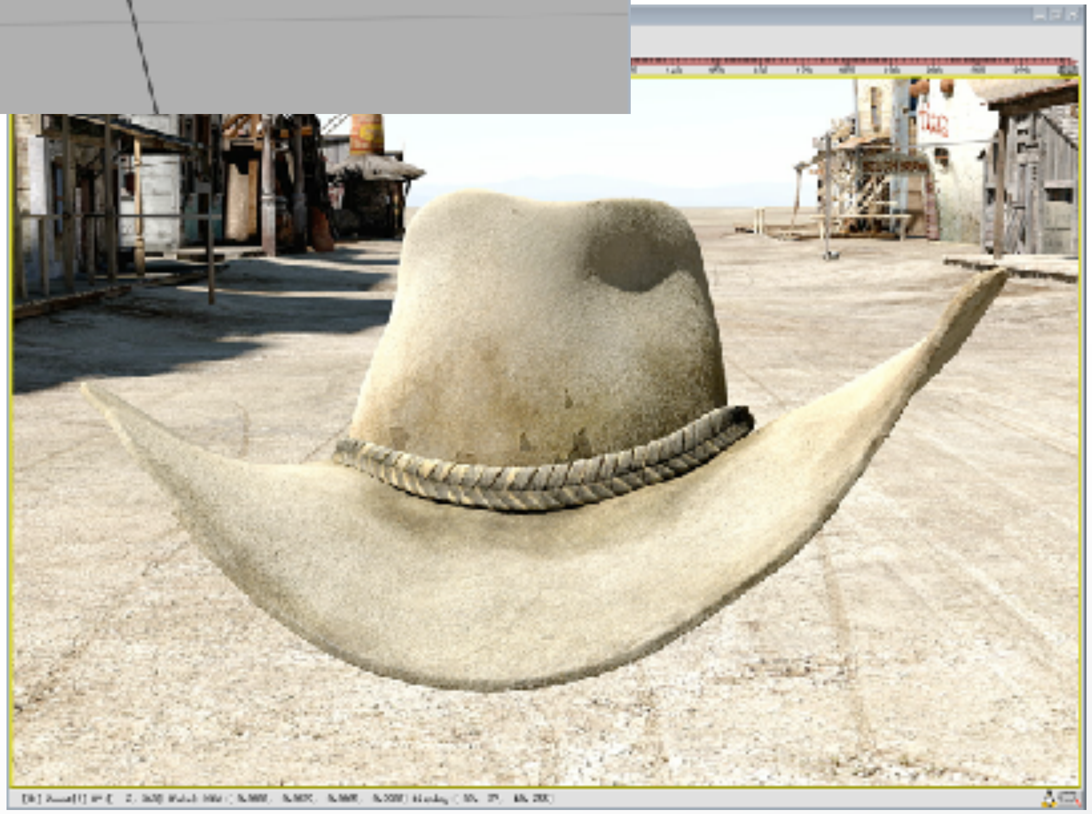
Viewpaint

(Wrinkle Pass)

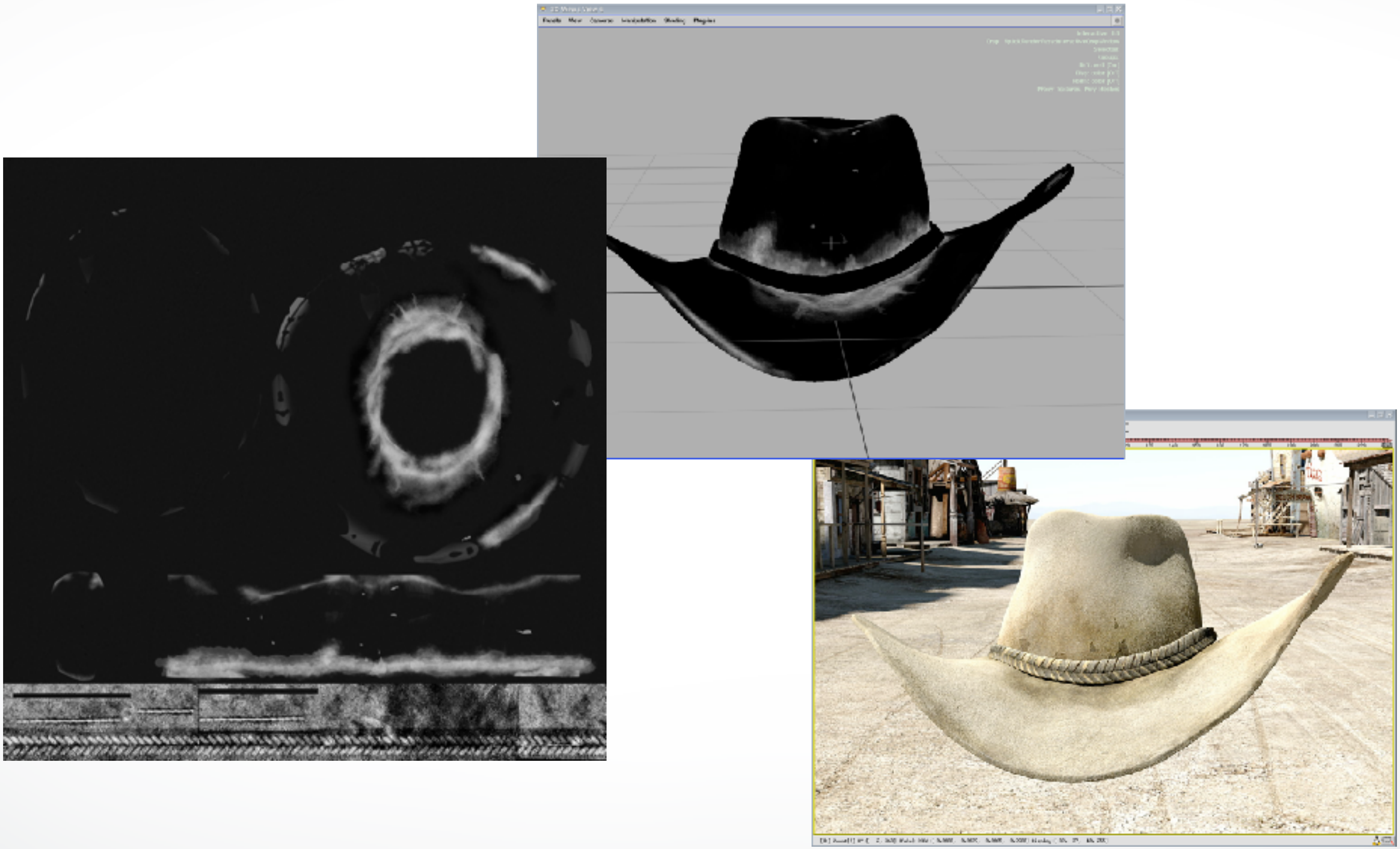


Color **Maps**

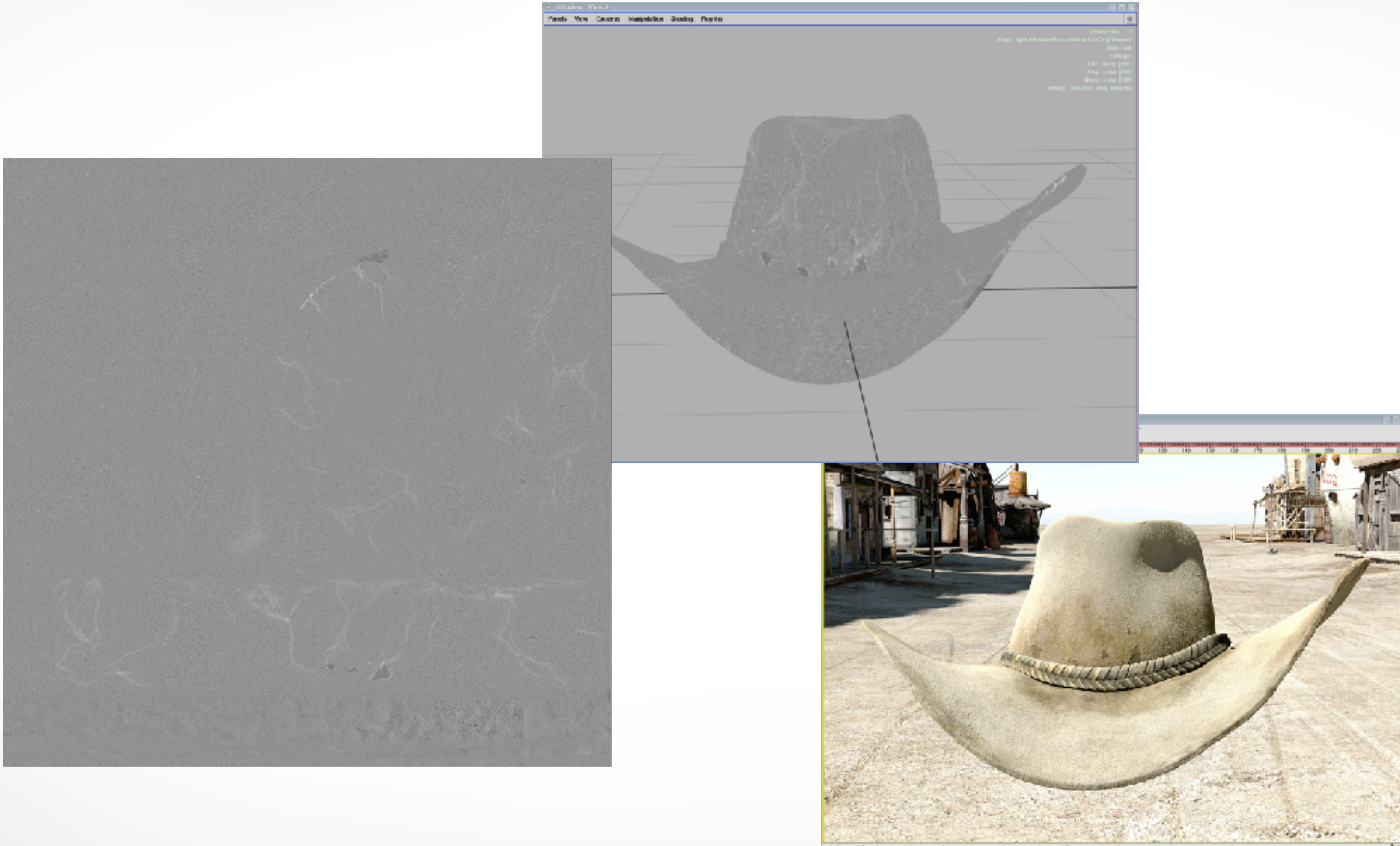




Wet **Maps**

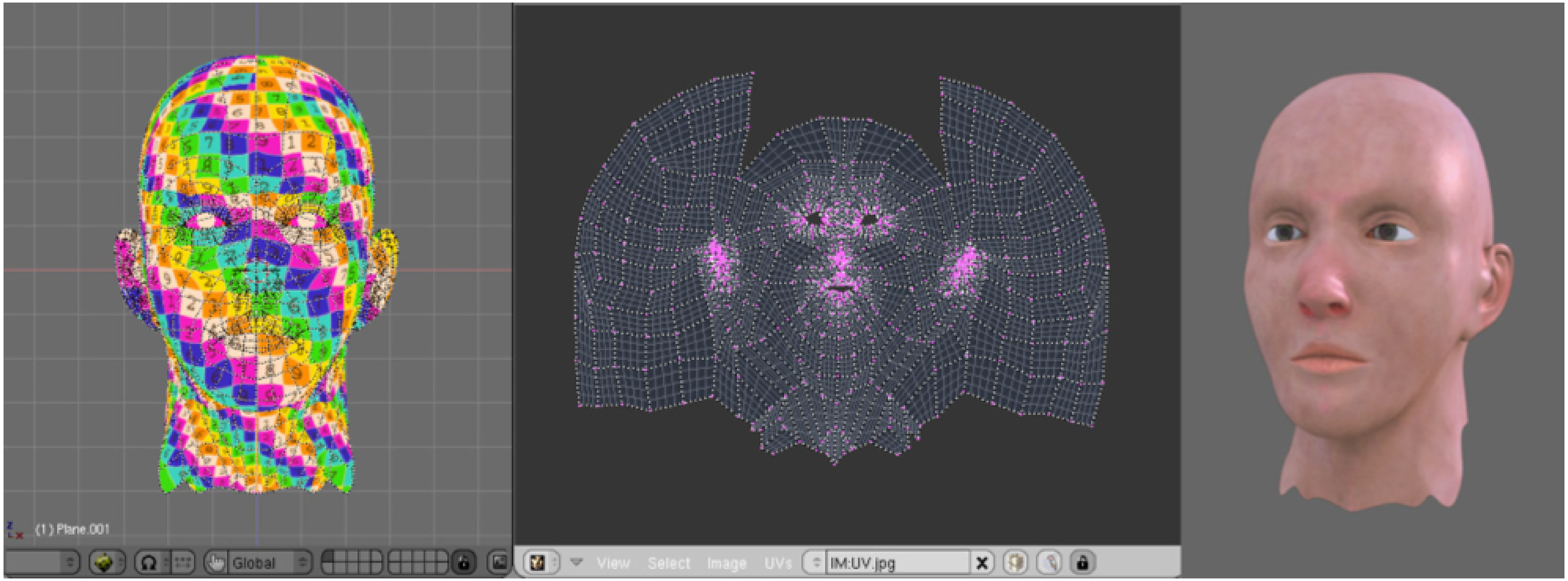


bump **Maps**



Motivation

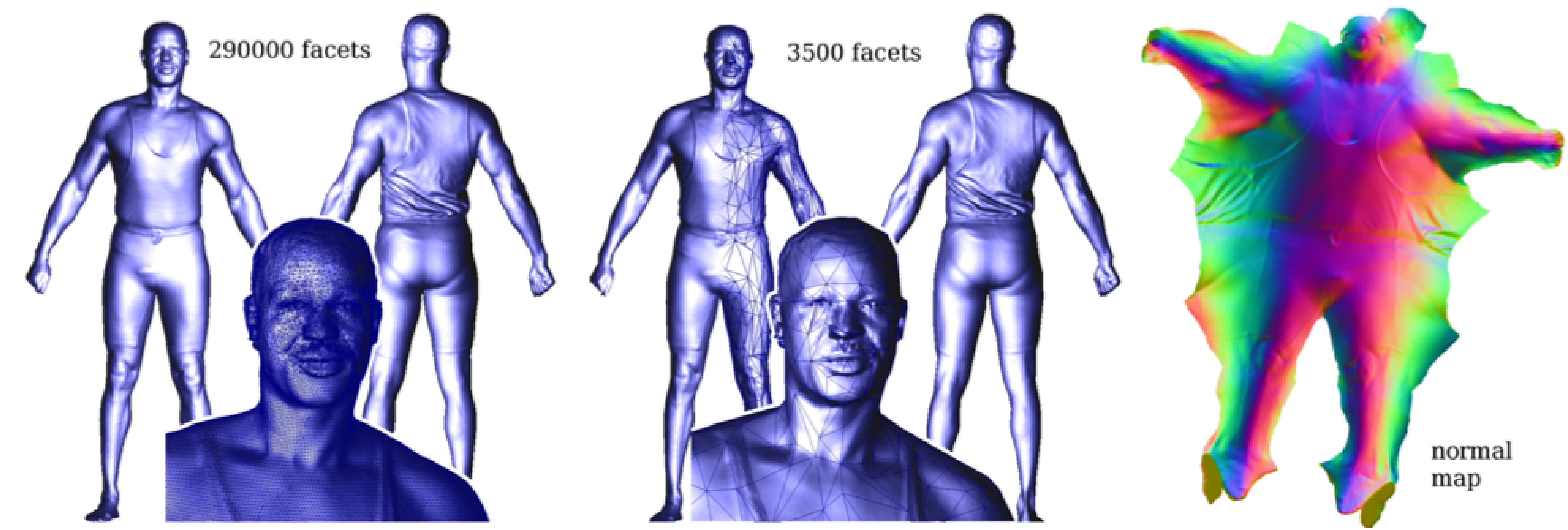
Texture Mapping



Levy et al.: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002.

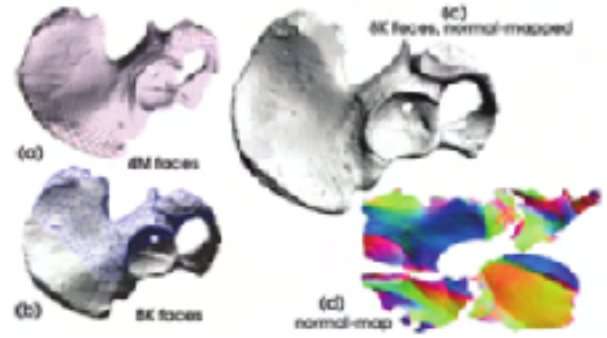
Motivation

Normal Mapping

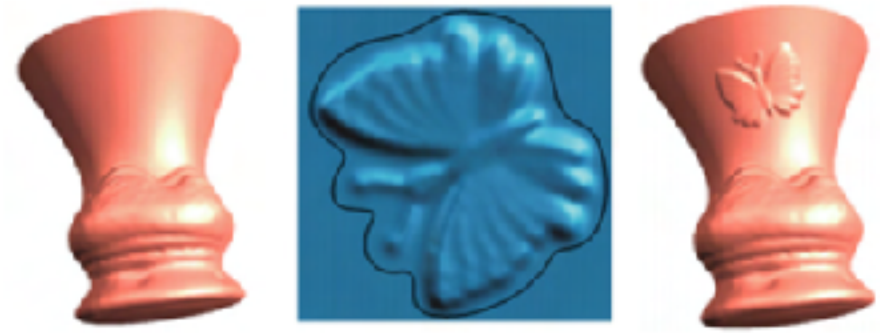


Motivation

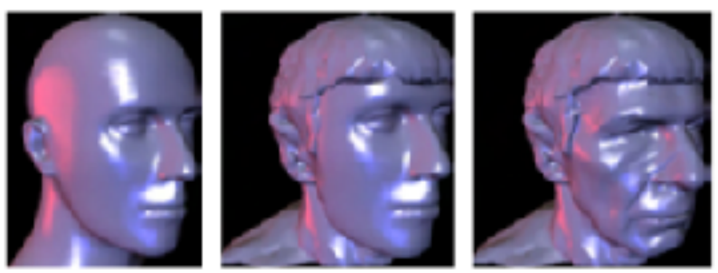
Texture Mapping



Normal Mapping

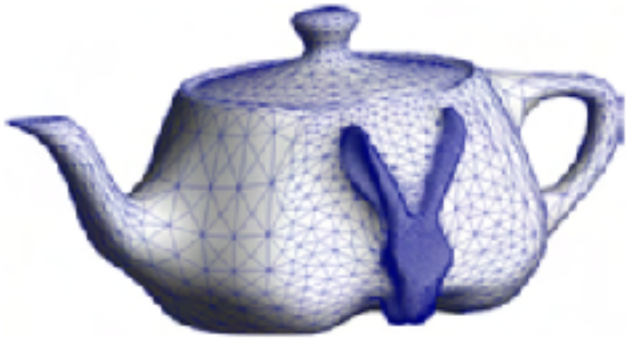


Detail Transfer



Morphing

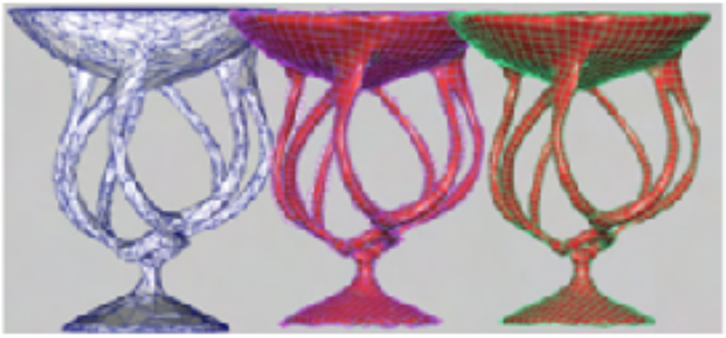
Mesh Completion



Editing

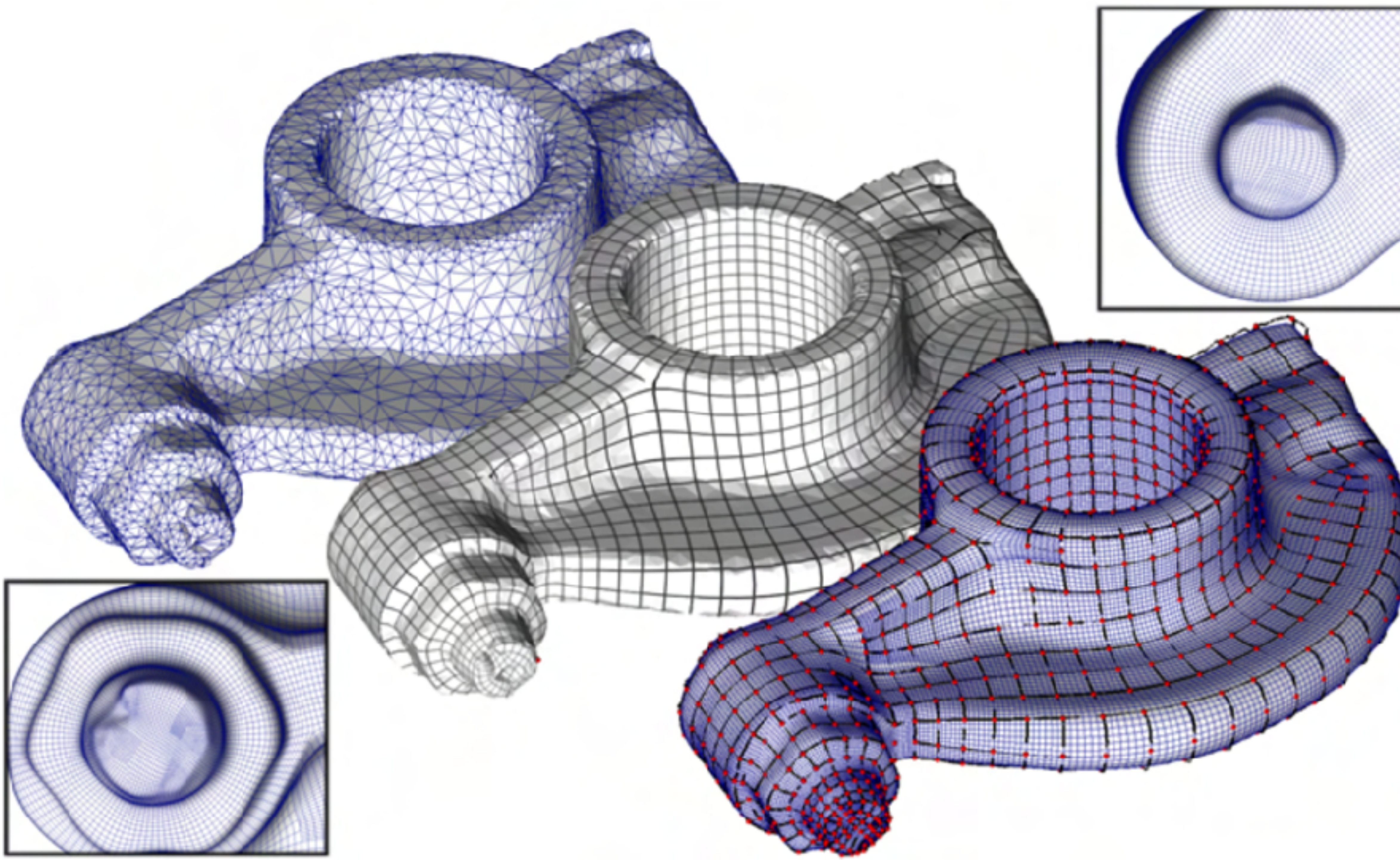
Databases

Remeshing

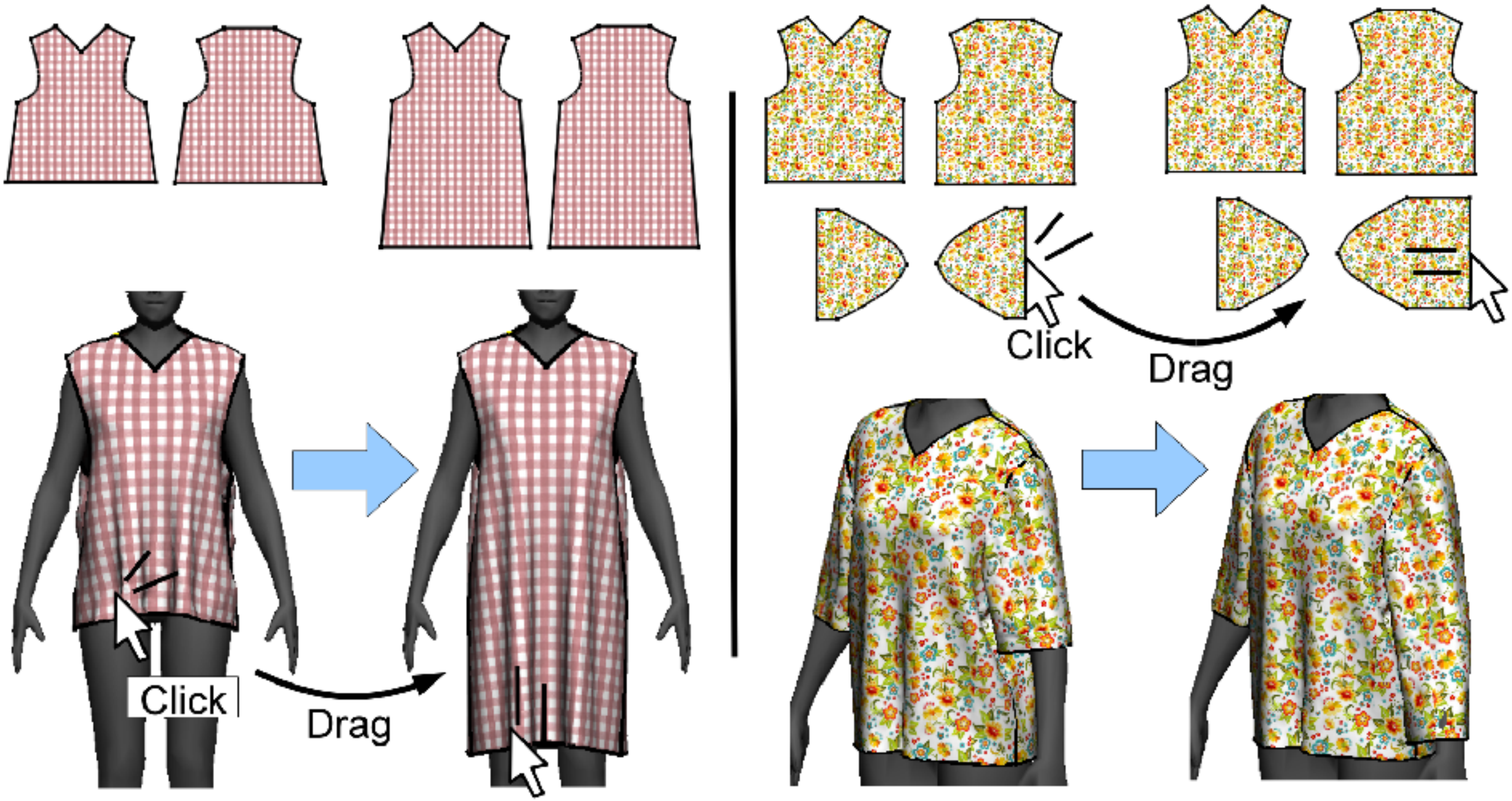


Surface Fitting

Motivation

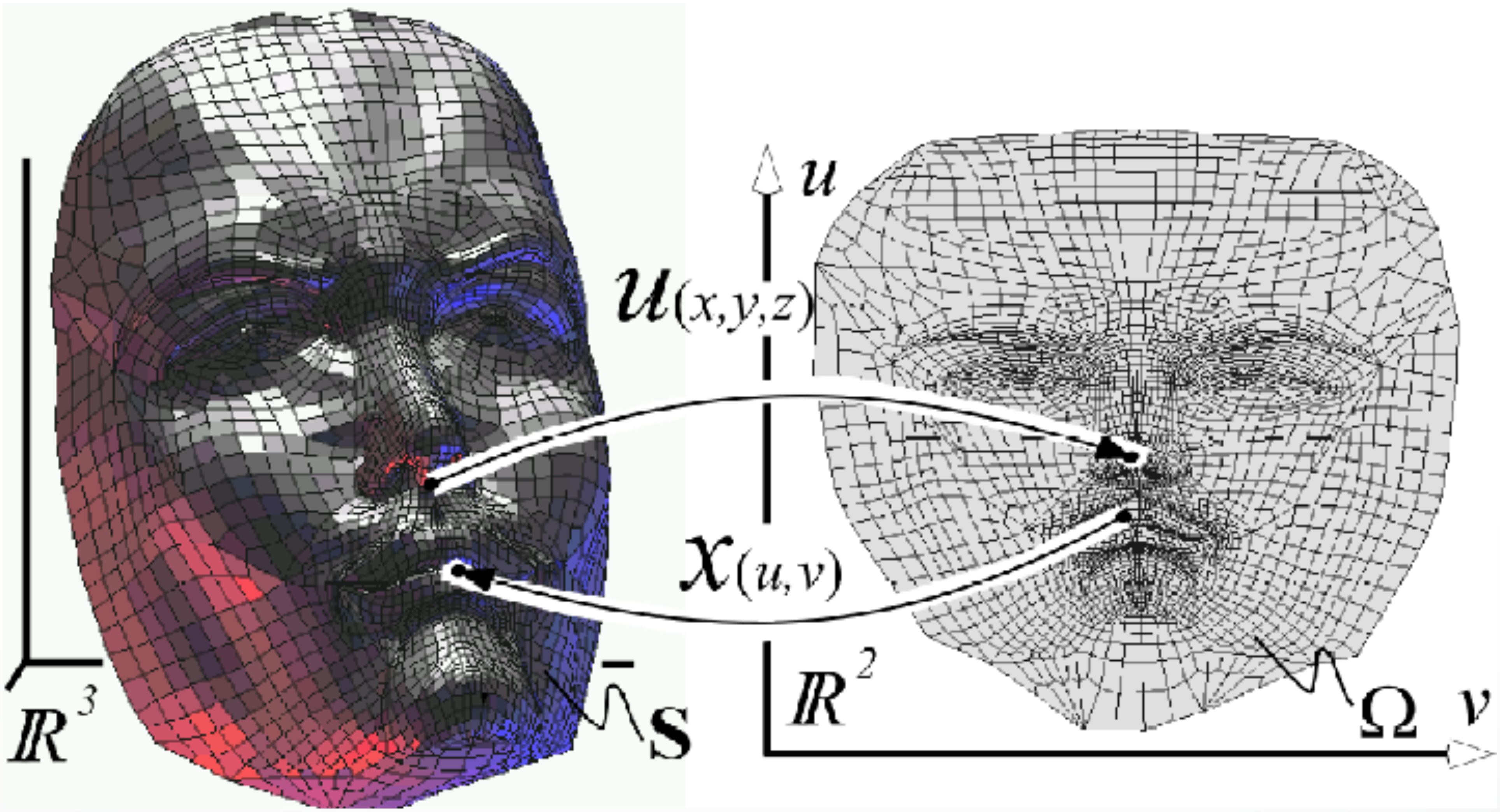


Motivation

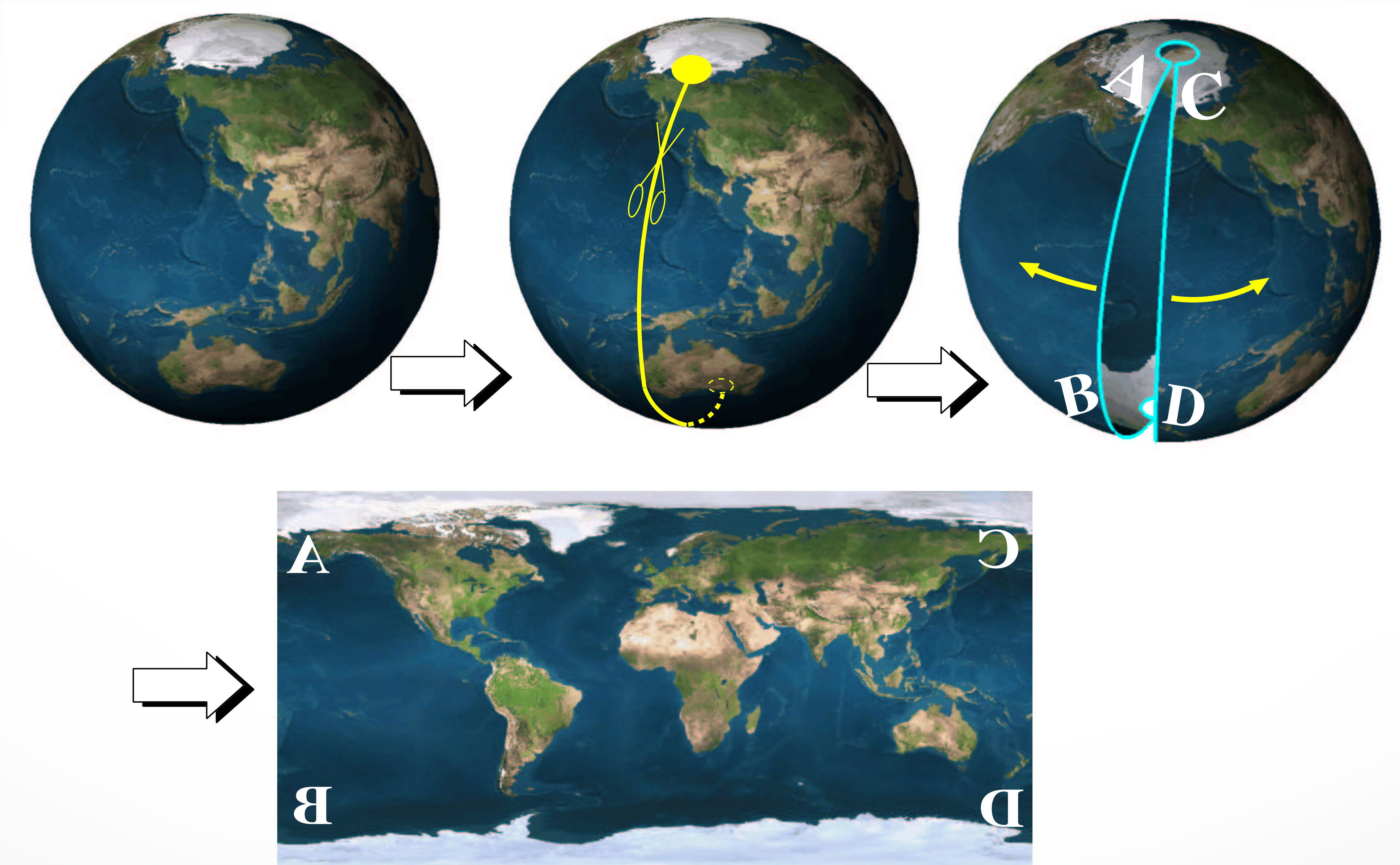


Mesh Parameterization

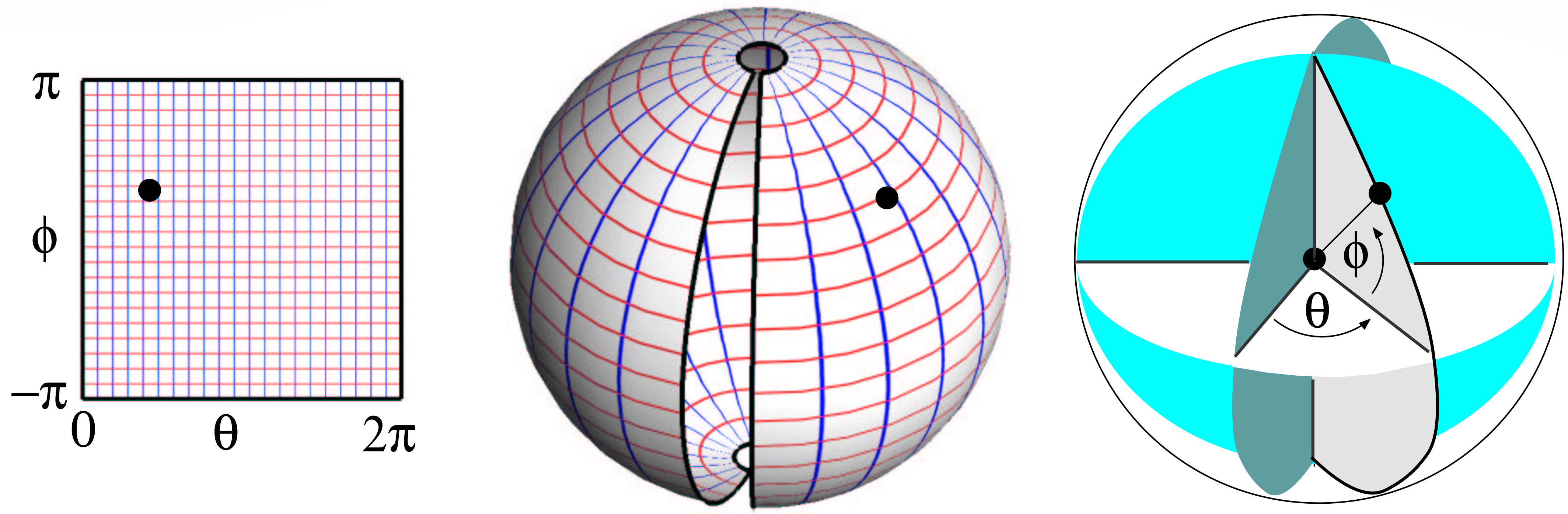
Find a 1-to-1 mapping between given surface mesh and 2D parameter domain



Unfolding Earth



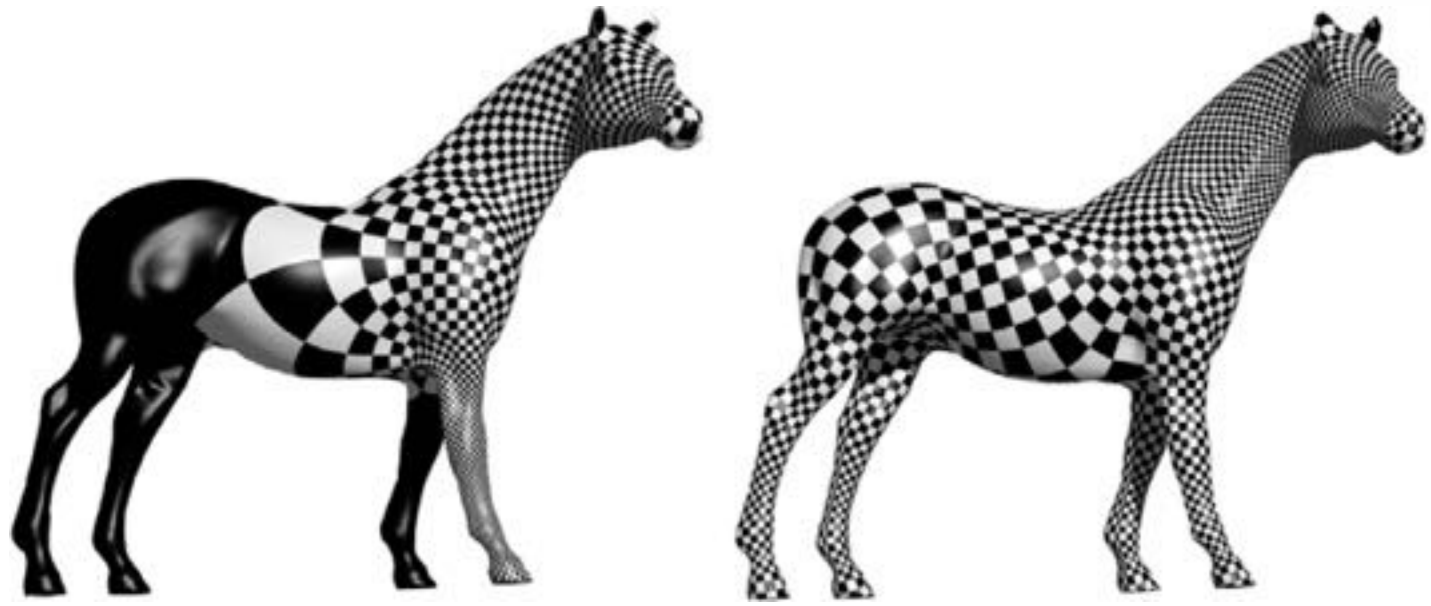
Spherical Coordinates



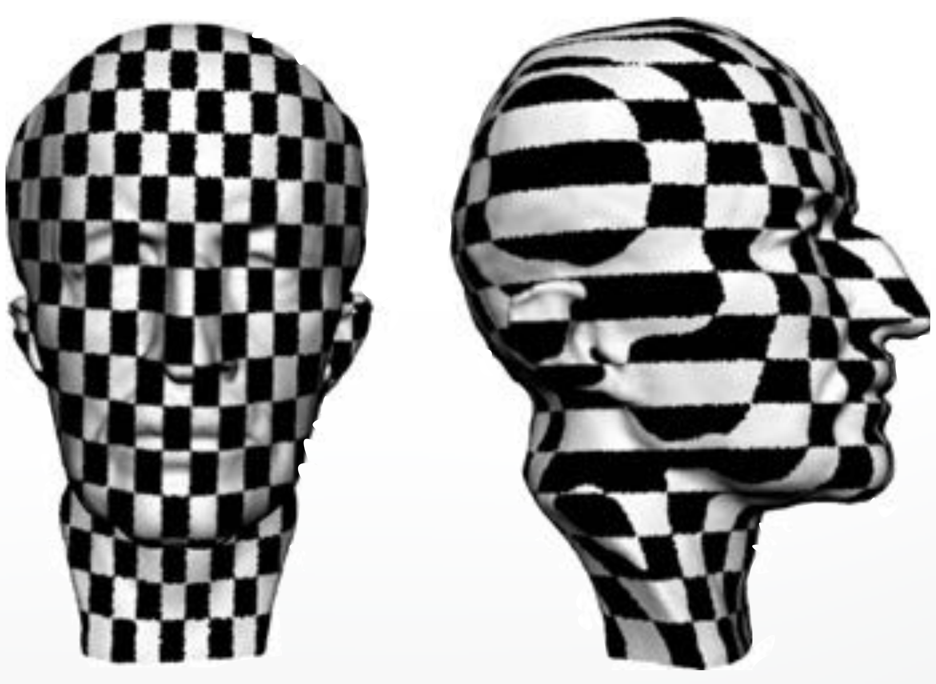
$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} \sin\theta\sin\phi \\ \cos\theta\ \sin\phi \\ \cos\phi \end{bmatrix}$$

Desirable Properties

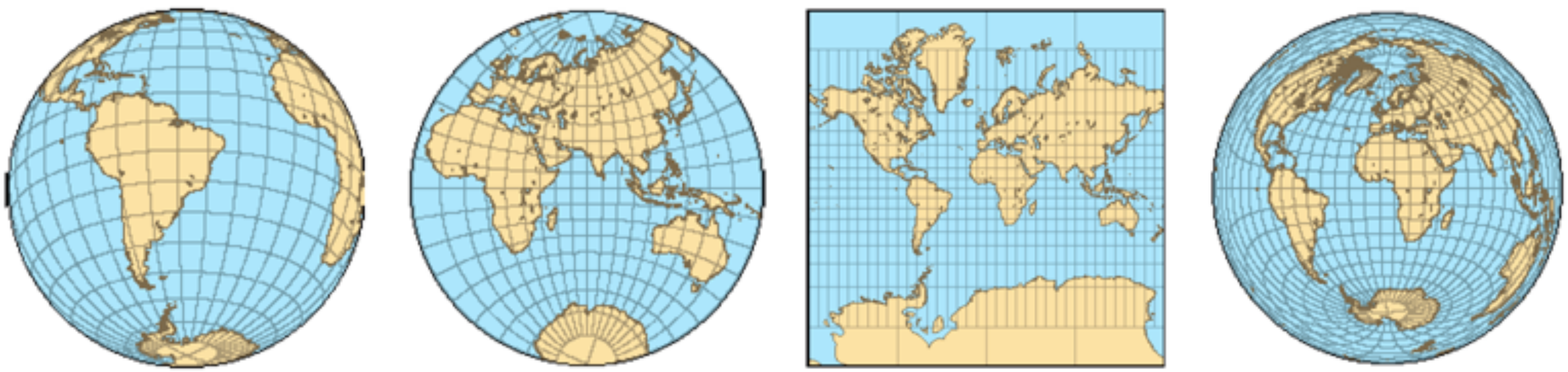
Low distortion



Bijective mapping



Cartography



orthographic stereographic Mercator Lambert

preserves angles

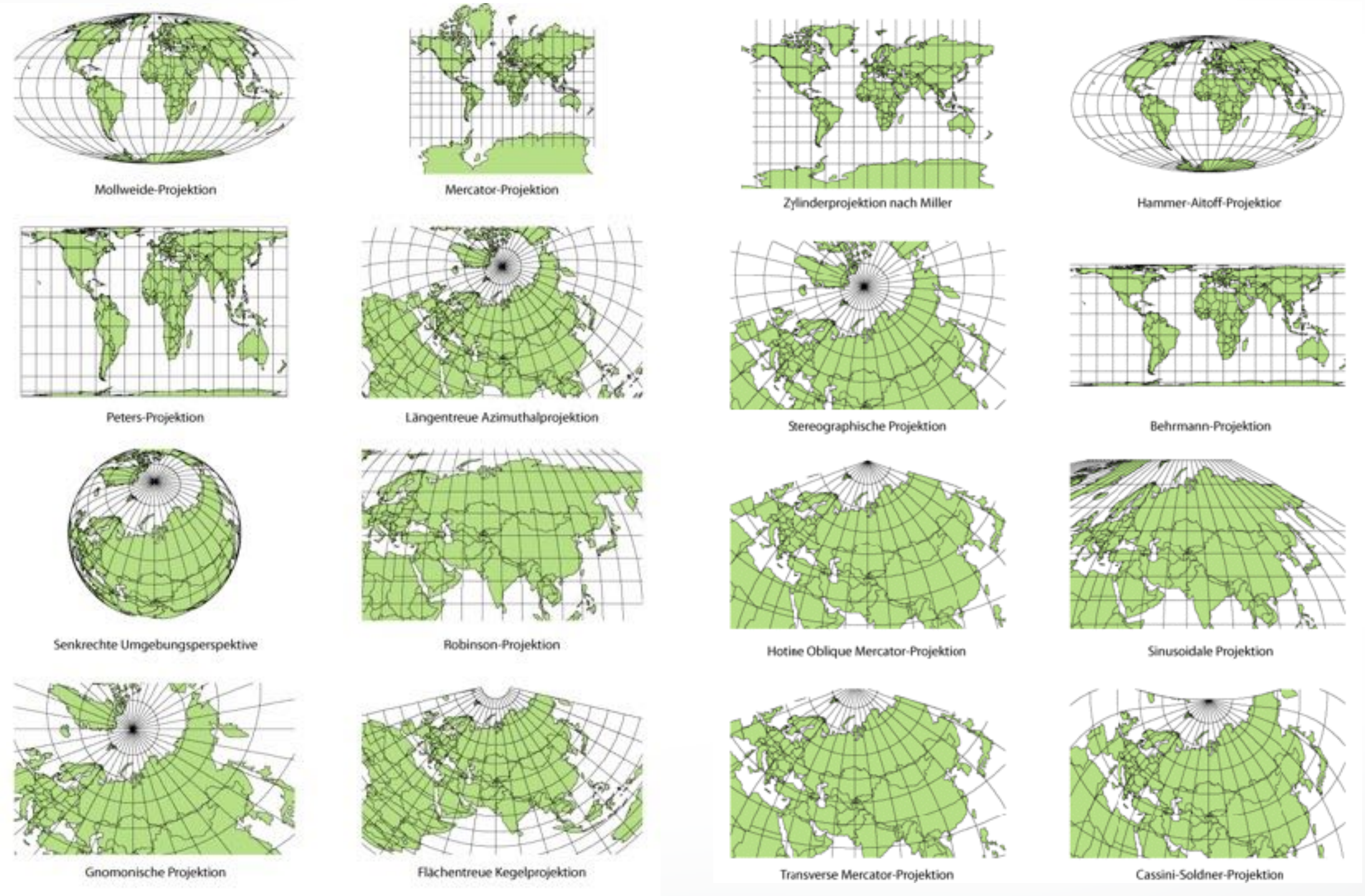
= conformal

preserves area

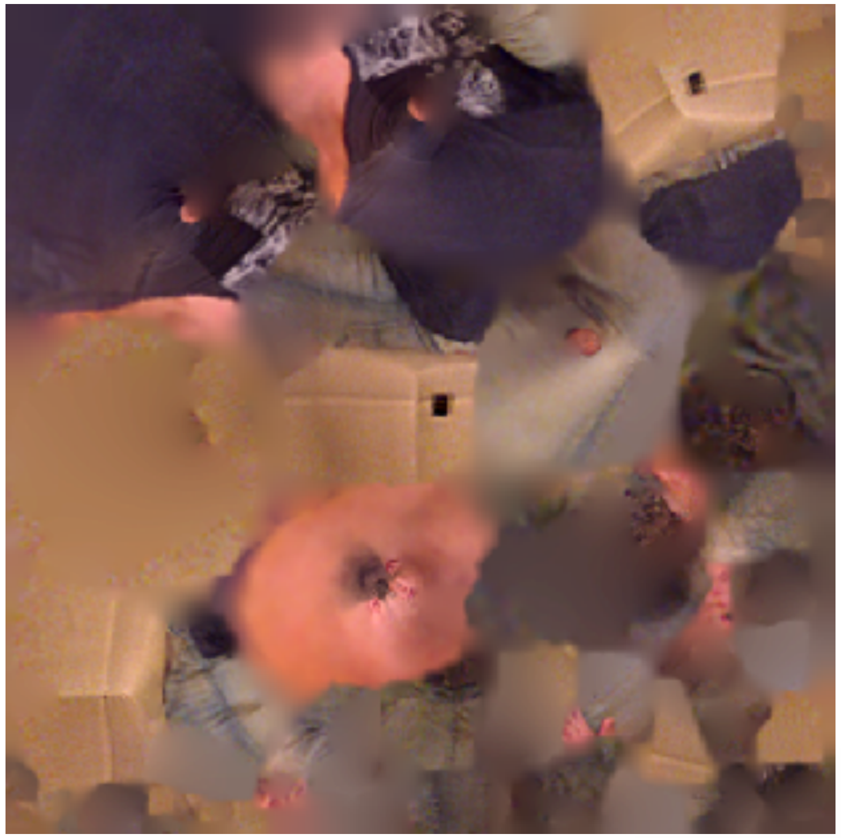
= equiareal

Floater, Hormann: *Surface Parameterization: A Tutorial and Survey*, Advances in Multiresolution for Geometric Modeling, 2005

More Maps


Mollweide-Projektion
Mercator-Projektion
Zylinderprojektion nach Miller
Hammer-Aitoff-Projektior
Peters-Projektion
Längentreue Azimuthalprojektion
Stereographische Projektion
Behrmann-Projektion
Senkrechte Umgebungsperspektive
Robinson-Projektion
Hotine Oblique Mercator-Projektion
Sinusoidale Projektion
Gnomonische Projektion
Flächentreue Kegelprojektion
Transverse Mercator-Projektion
Cassini-Soldner-Projektion

Demo: **Parameterization**

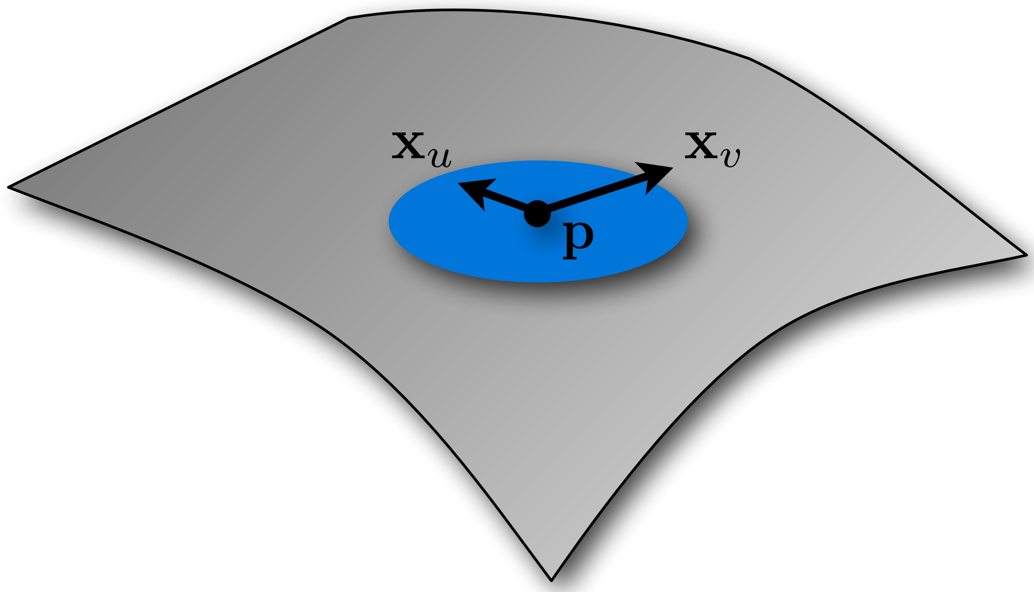


Recall: **Differential Geometry**

Parametric surface representation

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \quad \rightarrow \quad \mathcal{S} \subset \mathbb{R}^3$$

$$(u, v) \quad \mapsto \quad \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



Regular if

- Coordinate functions x,y,z are smooth
- Tangents are linearly independent

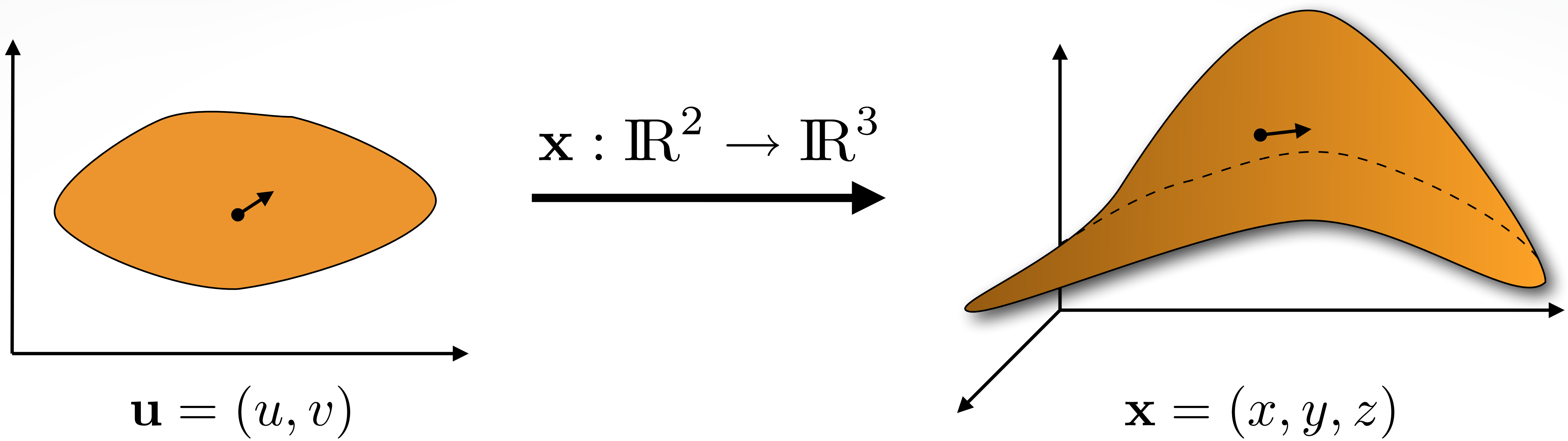
$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Definitions

A regular parameterization $\mathbf{x} : \Omega \to S$ is

- **Conformal** (angle preserving), if the angle of every pair of intersecting curves on S is the same as that of the corresponding pre-images in Ω.
- **Equiareal** (area preserving) if every part of Ω is mapped onto a part of S with the same area
- **Isometric** (length preserving), if the length of any arc on S is the same as that of its pre-image in Ω.

Distortion Analysis



Jacobian transforms infinitesimal vectors

$$\mathrm{d}\mathbf{x} = \mathbf{J}\,\mathrm{d}\mathbf{u} \qquad \mathbf{J} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

$$\|\mathrm{d}\mathbf{x}\|^2 = (\mathrm{d}\mathbf{u})^T \, \mathbf{J}^T \mathbf{J} \, \mathrm{d}\mathbf{u} = (\mathrm{d}\mathbf{u})^T \, \mathbf{I} \, \mathrm{d}\mathbf{u}$$

First Fundamental Form

Characterizes the surface locally

$$\mathbf{I} \;=\; \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Allows to measure on the surface

- Angles $\cos\theta \;=\; \left(\mathrm{d}\mathbf{u}_1^T \,\mathbf{I}\, \mathrm{d}\mathbf{u}_2\right) / \left(\|\mathrm{d}\mathbf{u}_1\| \cdot \|\mathrm{d}\mathbf{u}_2\|\right)$
- Length $\mathrm{d}s^2 \;=\; \mathrm{d}\mathbf{u}^T \,\mathbf{I}\, \mathrm{d}\mathbf{u}$
- Area $\mathrm{d}A \;=\; \det(\mathbf{I})\, \mathrm{d}u\, \mathrm{d}v$

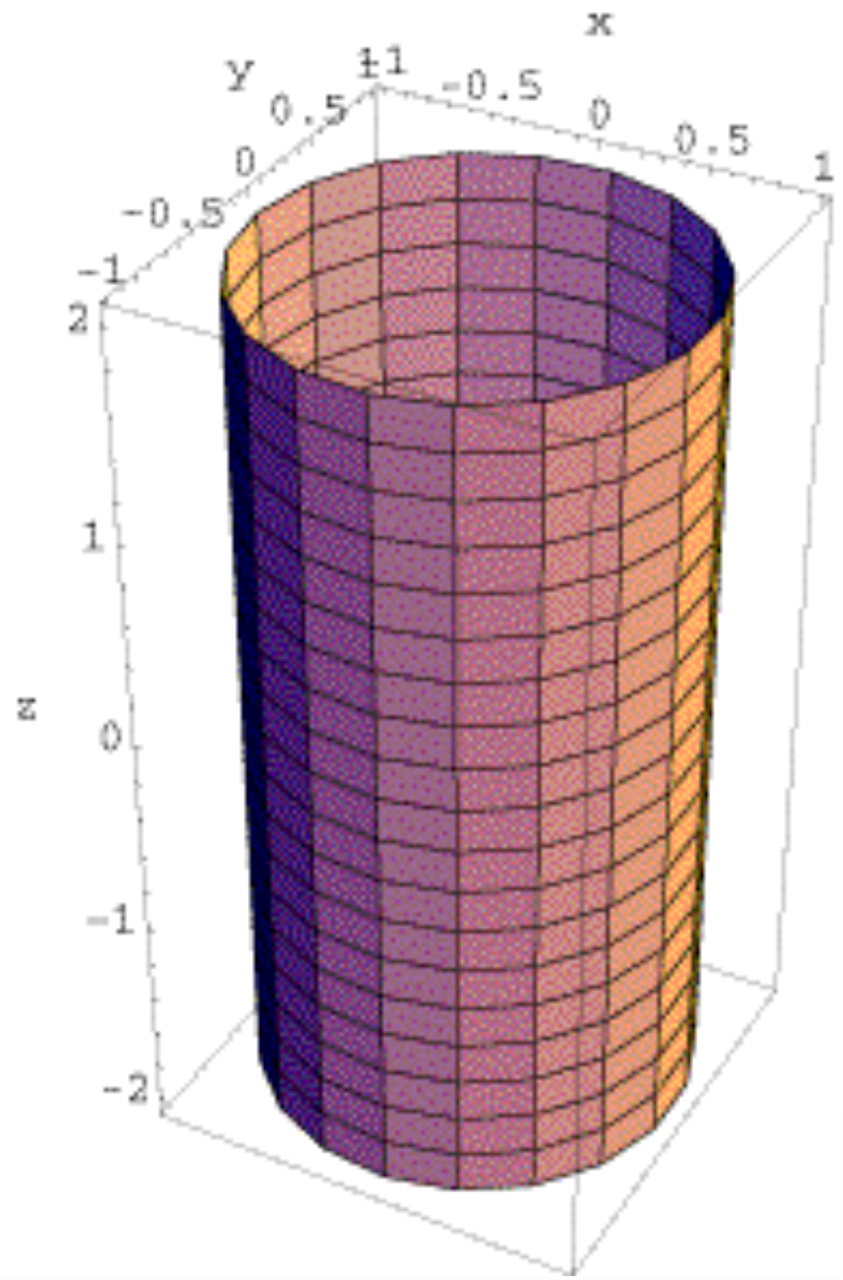
Isometric Maps

A regular parameterization $\mathbf{x}(u, v)$ is isometric, iff its first fundamental form is the identity:

$$\mathbf{I}(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A surface has an isometric parameterization iff it has zero Gaussian curvature

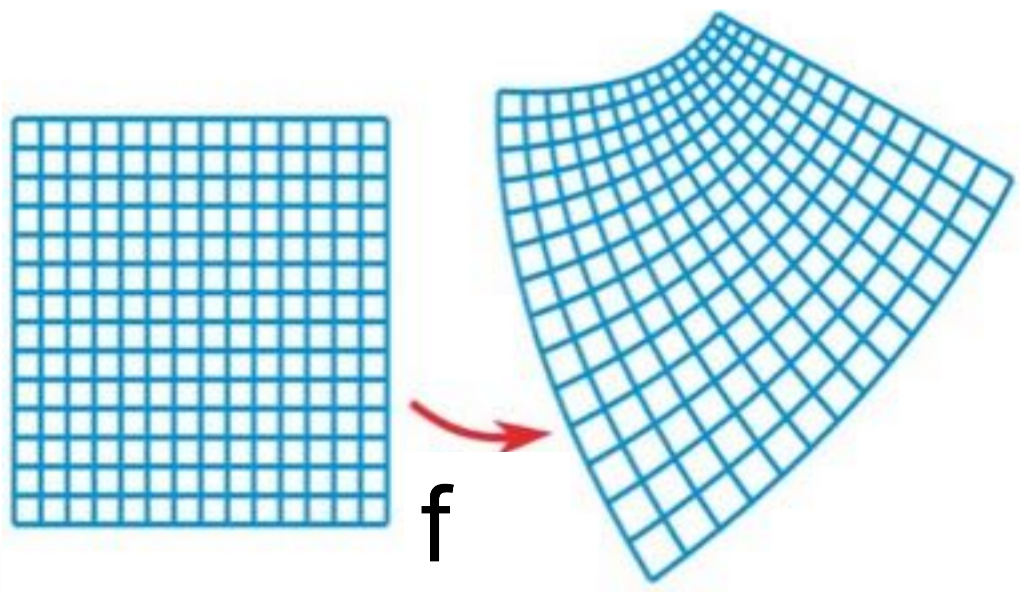
Cylinder

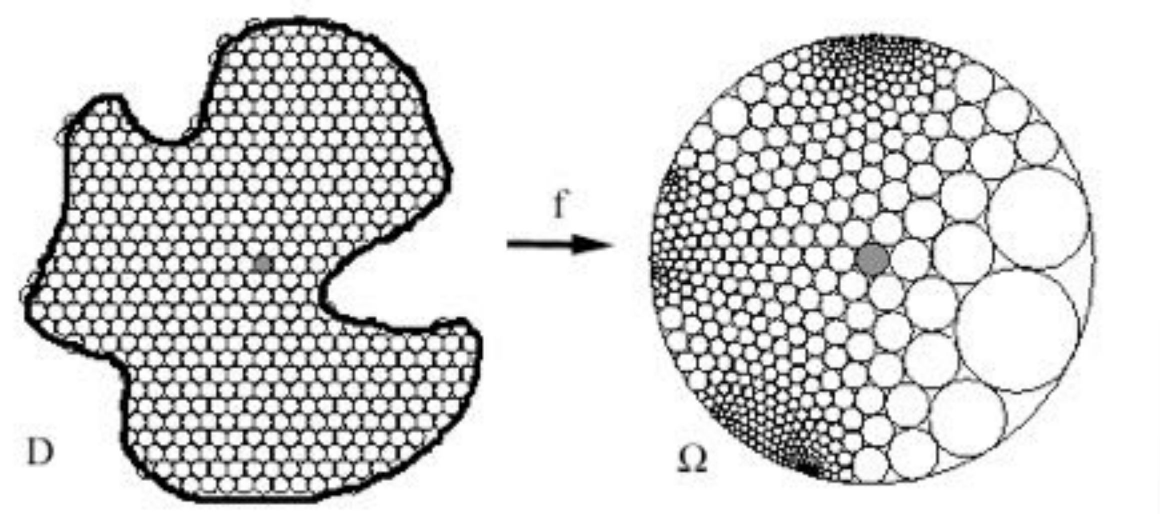


Conformal Maps (A-Similar-AP)

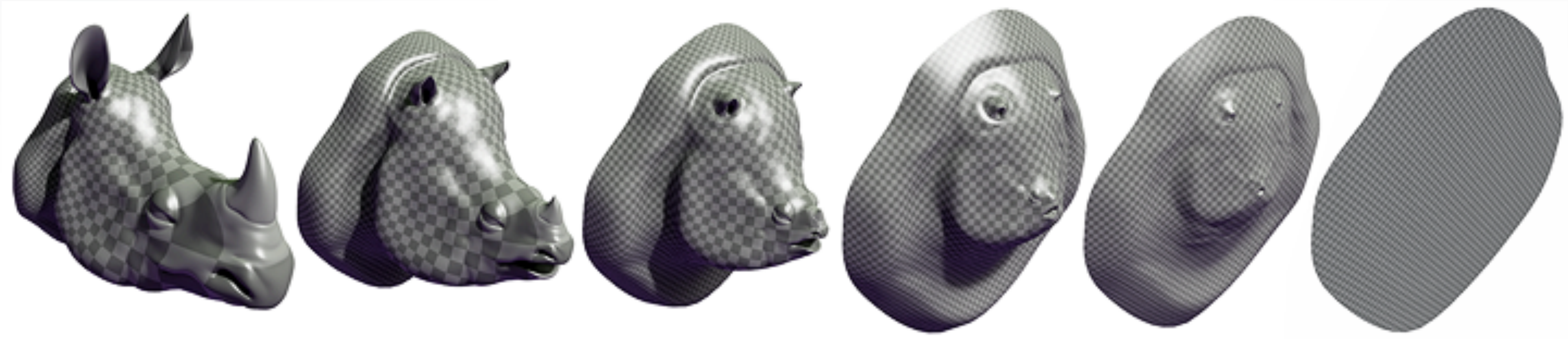
A regular parameterization $\mathbf{x}(u,v)$ is conformal, iff its first fundamental form is a scalar multiple of the identity:

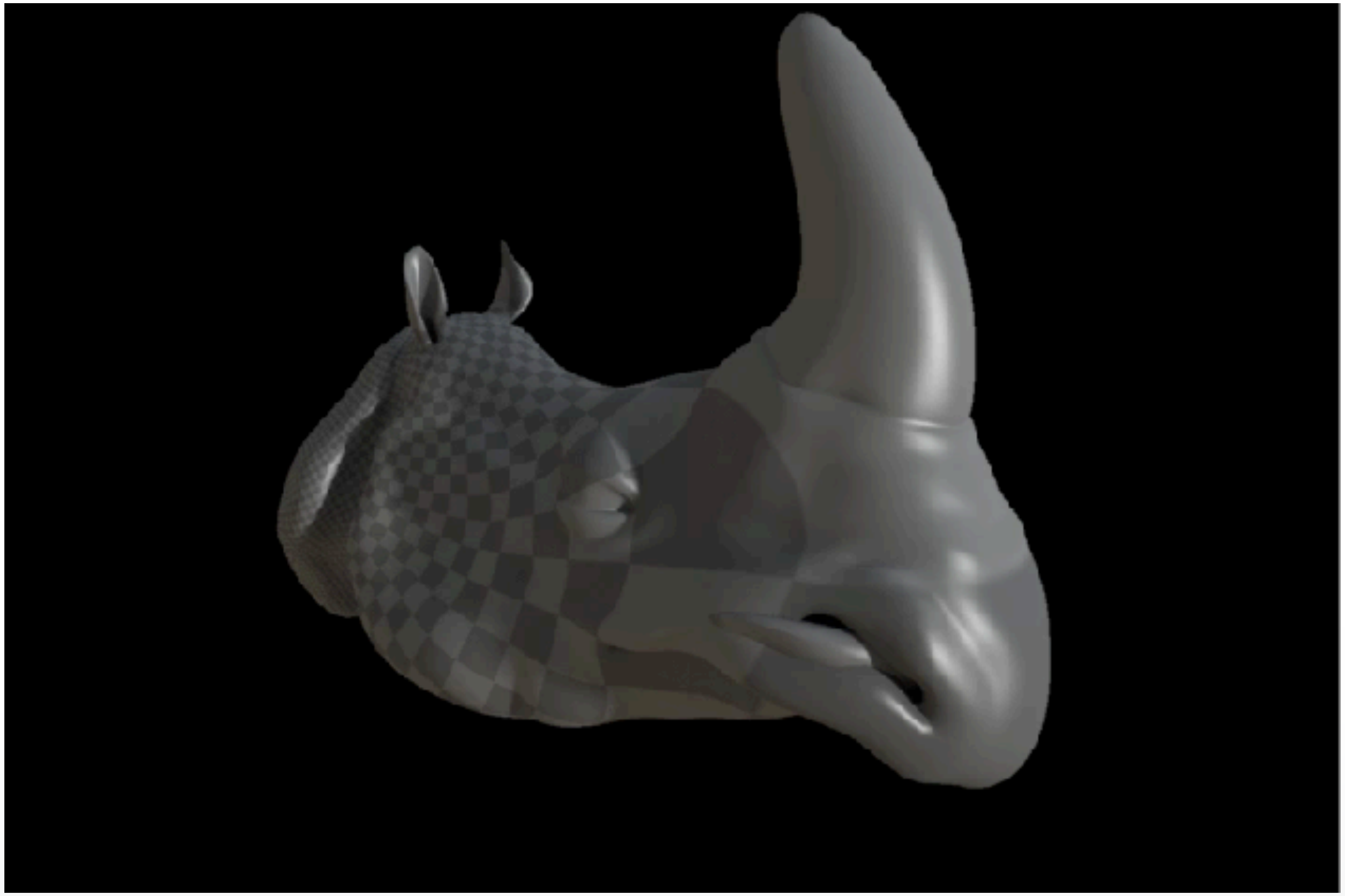
$$\mathbf{I}(u,v) \;=\; s(u,v)\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$





Conformal Flow



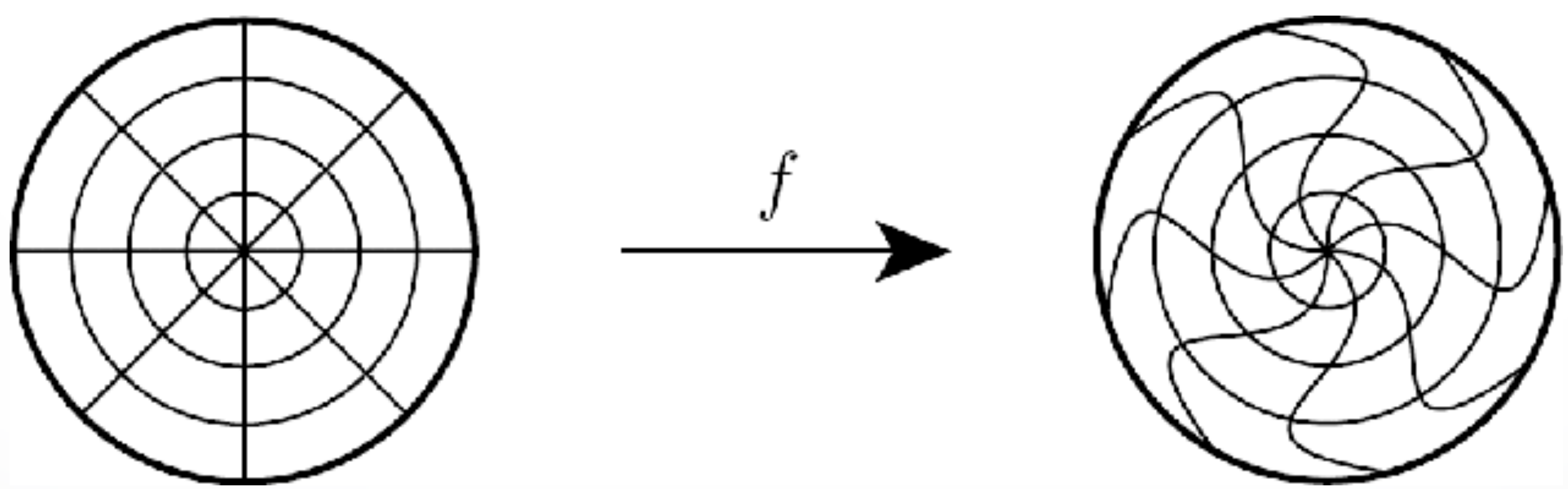


Crane et al. Spin Transformations of Discrete Surfaces, ACM Siggraph 2011

Equiareal Maps

A regular parameterization $\mathbf{x}(u, v)$ is equiareal, iff the determinant of its first fundamental form is 1:

$$\det(\mathbf{I}(u, v)) \;=\; 1$$



Relationships

An isometric parameterization is conformal and equiareal, and vice versa:

isometric ⇔ conformal + equiareal

Isometric is ideal, but rare. In practice, people try to compute:

- Conformal
- Equiareal
- Some balance between the two

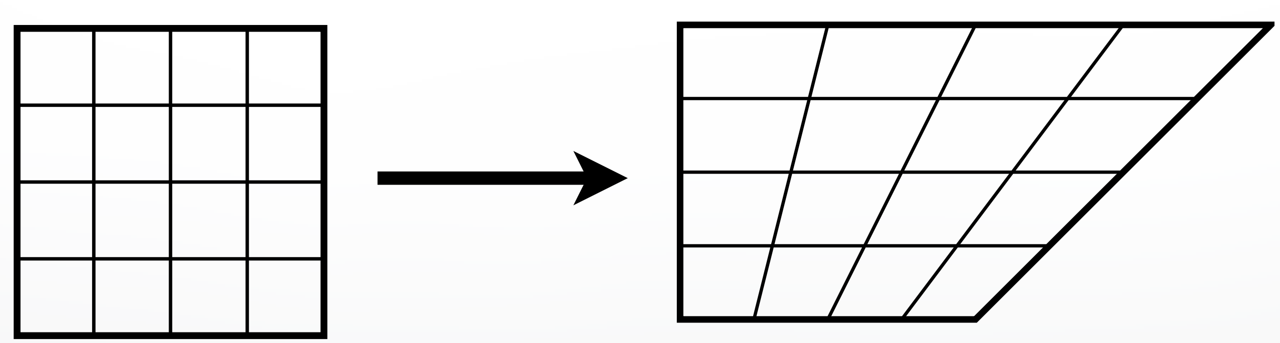
Harmonic Maps

- A regular parameterization $\mathbf{x}(u, v)$ is harmonic, iff it satisfies

$$\Delta \mathbf{x}(u, v) = 0$$

- isometric ⇒ conformal ⇒ harmonic

- Easier to compute than conformal, but does not preserve angles



Harmonic Maps

- A harmonic map minimizes the Dirichlet energy

$$\int_\Omega \|\nabla \mathbf{x}\|^2 \;=\; \int_\Omega \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \;\mathrm{d}u\,\mathrm{d}v$$

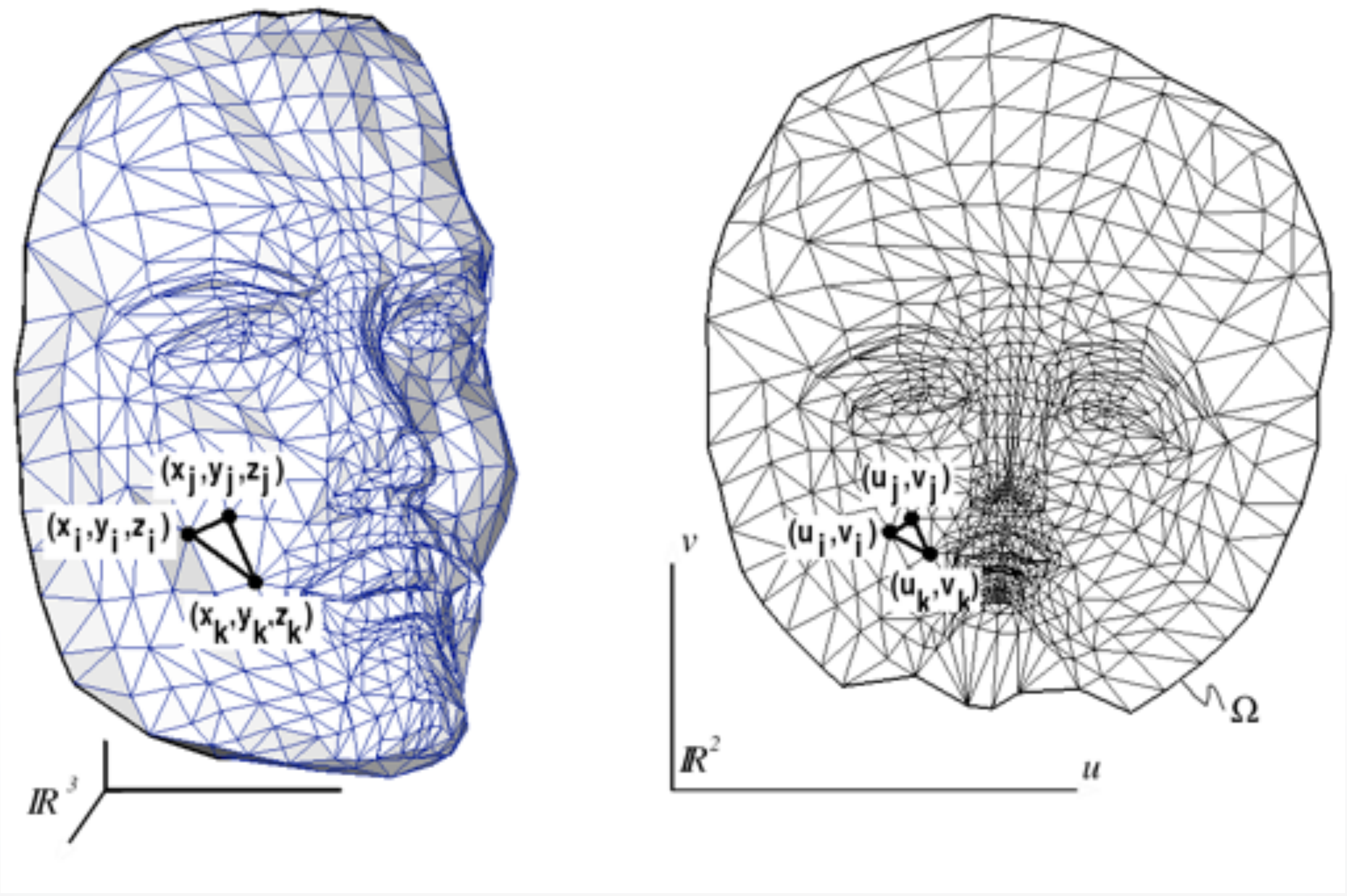
- Variational calculus then tells us that

$$\Delta \mathbf{x}(u,v) \;=\; 0$$

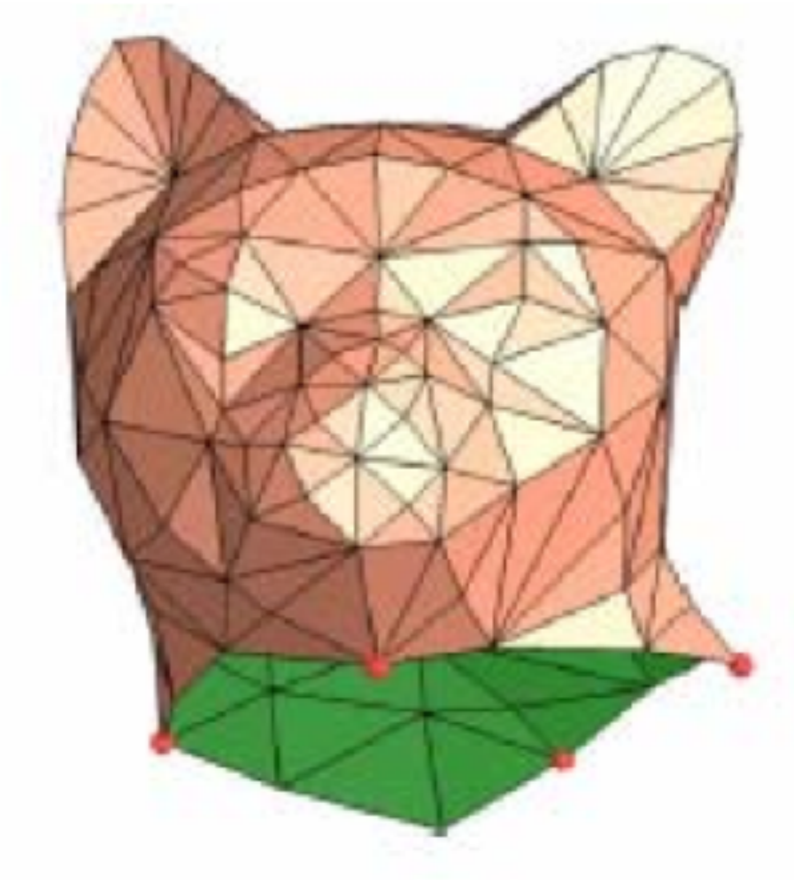
- If $\mathbf{x} : \Omega \to S$ is harmonic and maps the boundary $\partial\Omega$ of a convex region $\Omega \subset \mathbb{R}^2$ homeomorphically onto the boundary ∂S, then $\mathbf{x}$ is one-to-one.

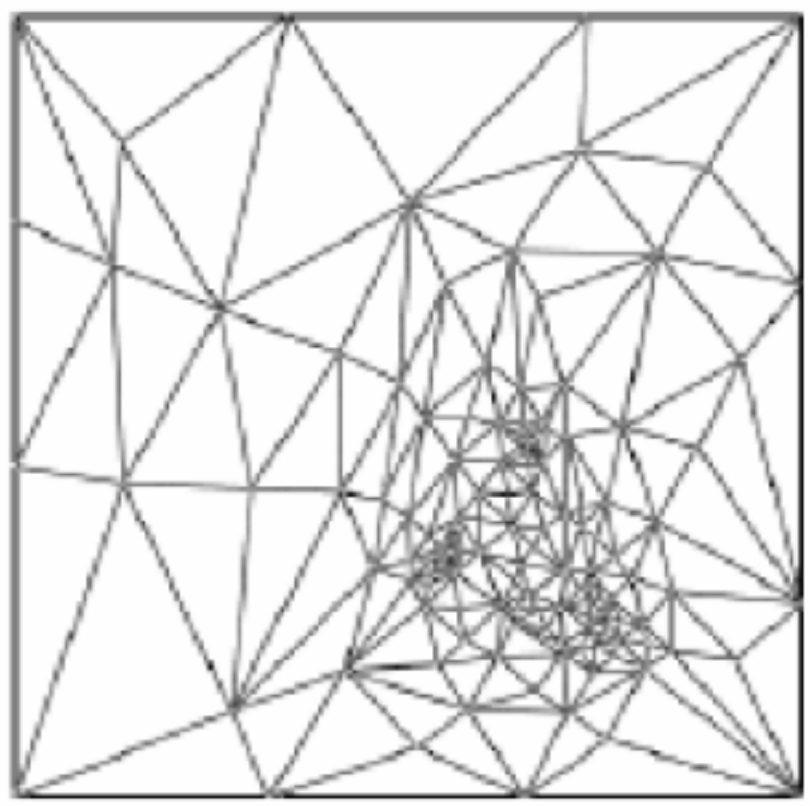
Parameterization Goal

- Piecewise linear mapping of a discrete 3D triangle mesh onto a planar 2D polygon
- Slightly different situation: Given a 3D mesh, compute the inverse parameterization



Floater's Parameterization





Floater's Parameterization

- For Quadrilateral Patch
- Fix the parameters of the boundary vertices on a unit square
- Derive the bijection $\mathbf{u}$ for each of the interior vertices $\mathbf{v}_i$ by solving

$$u(v_i) = \sum_{k \in \nu(i)} \lambda_{i,k} u(v_k)$$

where $\lambda_{i,k}$ satisfies shape preserving criteria

and $\sum_{k \in \nu(i)} \lambda_{i,k} = 1, \quad i = 1,2,\ldots,n$

Floater's Algorithm

- Compute for each i the $\lambda_{i,k},\ k \in v(i)$
  - Compute a local parameterization for $v(i)$ that preserves the aspect ratio of the angle and length
  - Compute $\lambda_{i,k},\ k \in v(i)$ that satisfies

Shape preserving criteria

and $\sum_{k \in v(i)} \lambda_{i,k} = 1, \quad i = 1,2,\ldots,n$

- Solve the sparse equation for $u(v_i), i = 1 \ldots n$

$$u(v_i) = \sum_{k \in v(i)} \lambda_{i,k} u(v_k)$$

Discrete Harmonic Maps

- Map the boundary ∂S homeomorphically to some (convex) polygon $\partial\Omega$ in the parameter plane

- Minimize the Dirichlet energy of $\mathbf{u}$ by solving the corresponding Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}\,\mathbf{u} \;=\; 0$$

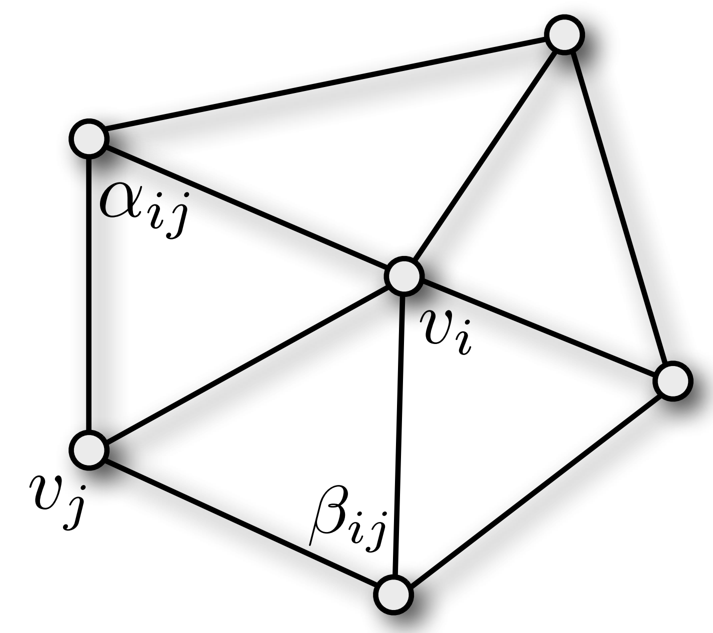
  - Requires discretization of Laplace-Beltrami
  - Compare to surface fairing

Discrete Harmonic Maps

- System of linear equations

$$\forall v_i \in \mathcal{S} \; : \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \left(\mathbf{u}(v_j) - \mathbf{u}(v_i)\right)$$

$$w_{ij} \;=\; \cot\alpha_{ij} + \cot\beta_{ij}$$



- Properties of system matrix:
  - Symmetric + positive definite → unique solution
  - Sparse → efficient solvers

Discrete Harmonic Maps

- But…
    - Does the same theory hold for discrete harmonic maps as for harmonic maps?
    - In other words, i[illegible] triangles to flip or become degene[illegible]

Convex Combination Maps

- If the linear equations are satisfied

$$\sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \left(\mathbf{u}(v_j) - \mathbf{u}(v_i) \right)$$

and if the weights satisfy

$$w_{ij} > 0 \quad \wedge \quad \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} = 1$$

then we get a convex combination mapping.

Convex Combination Maps

- Each $\mathbf{u}(v_i)$ is a convex combination of $\mathbf{u}(v_j)$

$$\mathbf{u}(v_i) = \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij}\mathbf{u}(v_j)$$

- If $\mathbf{u} : S \to \Omega$ is a convex combination map that maps the boundary ∂S homeomorphically to the boundary $\partial\Omega$ of a convex region $\Omega \subset \mathbb{R}^2$, then $\mathbf{u}$ is one-to-one.

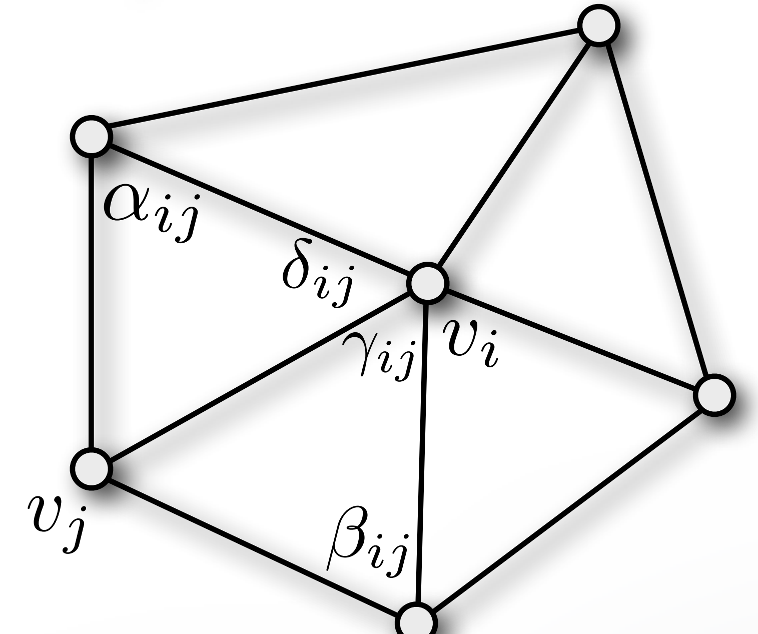
Convex Combination Maps

- Uniform barycentric weights

$$w_{ij} \;=\; 1/\text{valence}(v_i)$$

- Cotangent weights (> 0 if $\alpha_{ij} + \beta_{ij} < \pi$)

$$w_{ij} \;=\; \cot(\alpha_{ij}) + \cot(\beta_{ij})$$



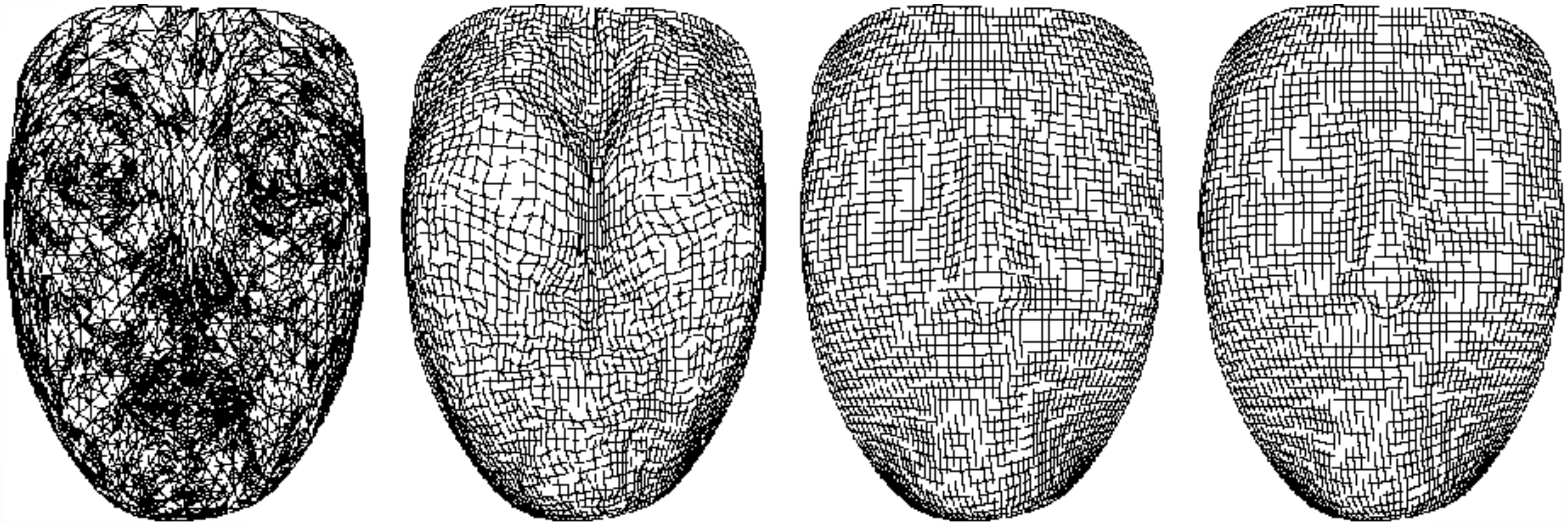
- Mean value weights

$$w_{ij} \;=\; \frac{\tan(\delta_{ij}/2) + \tan(\gamma_{ij}/2)}{\|\mathbf{p}_j - \mathbf{p}_i\|}$$

(no negative weights, even for obtuse angles)

Convex Combination Maps

- Comparison

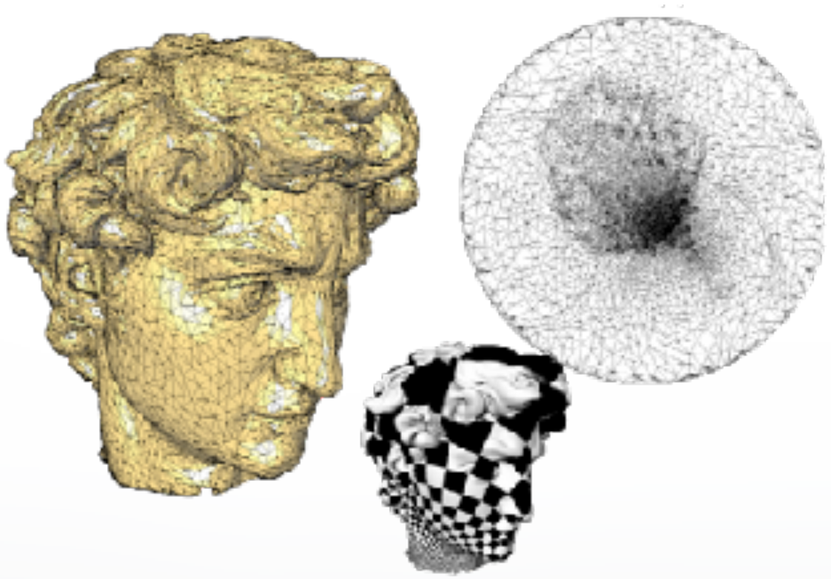


original mesh | uniform weights | cotan weights (shape preserving) | mean value

Fixing the Boundary

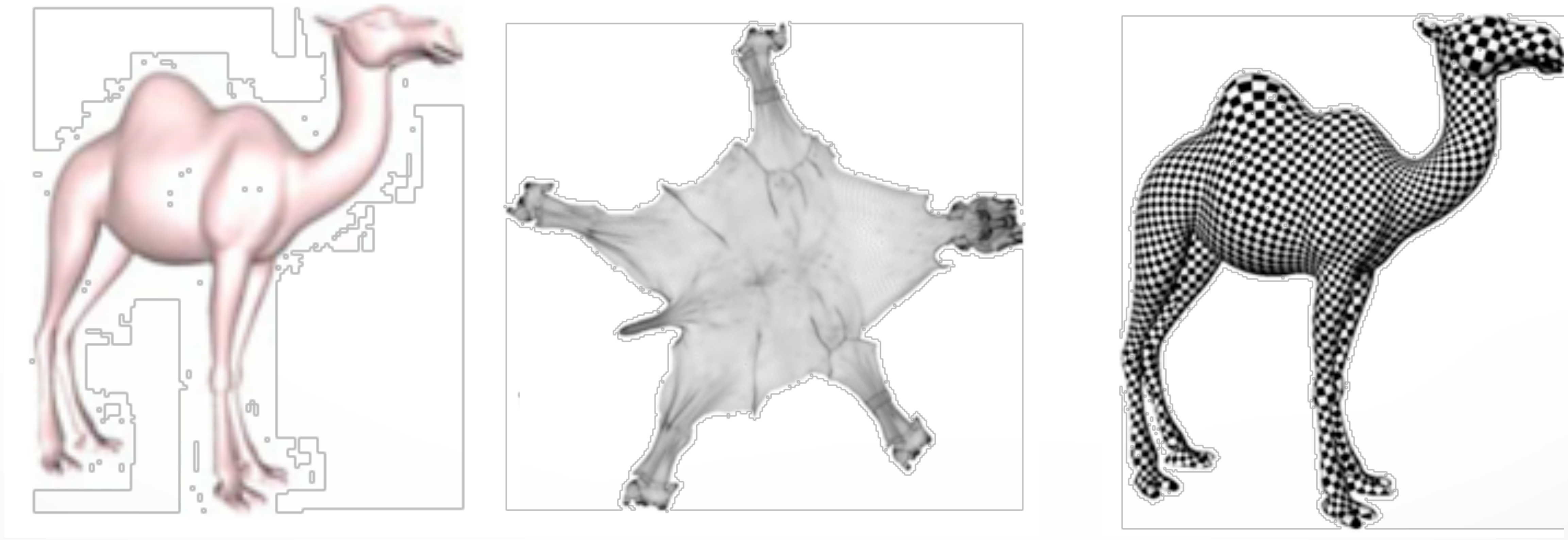
- Choose a simple convex shape
  - Triangle, square, circle
- Distribute points on boundary
  - Use chord length parameterization

Fixed boundary can create high distortion

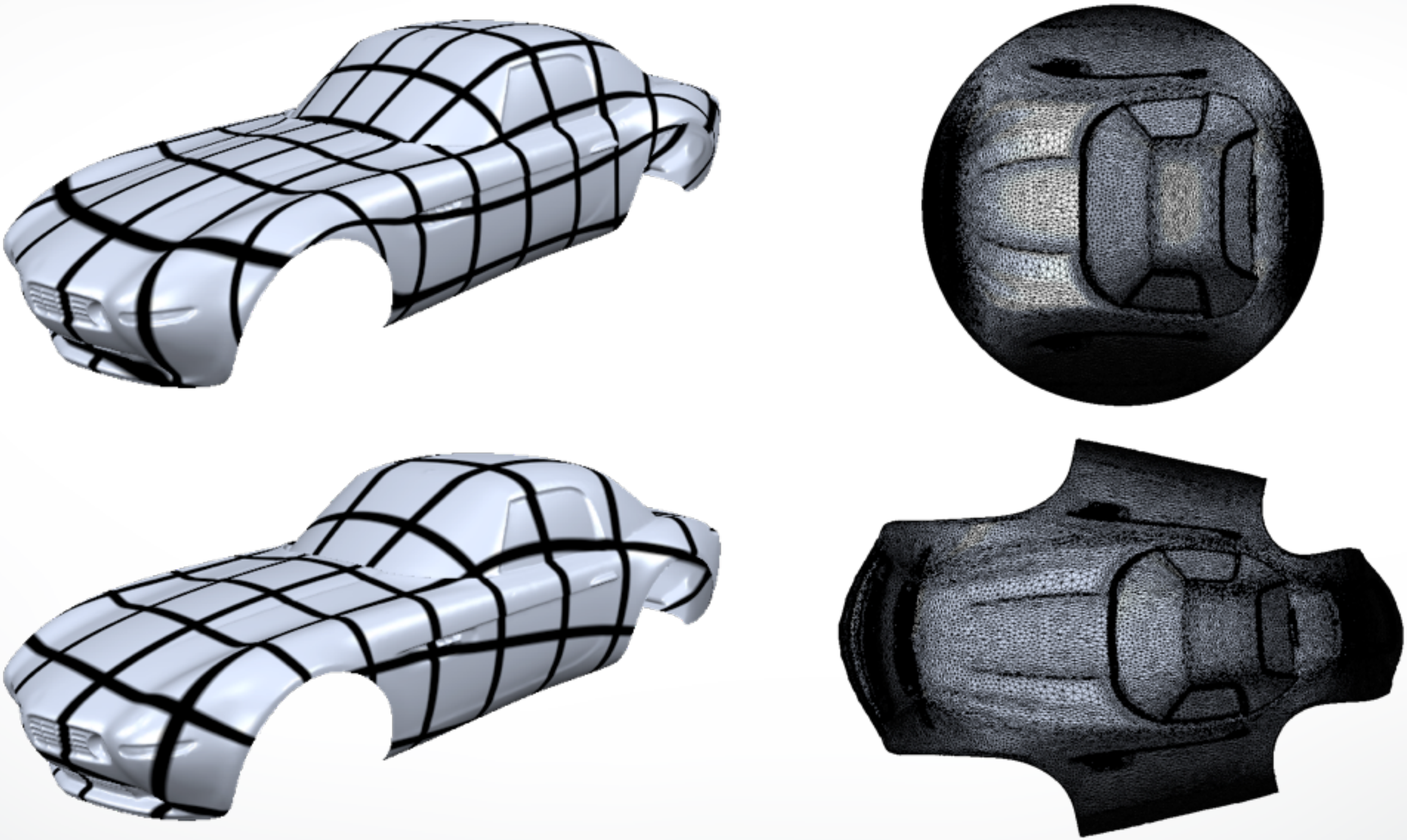


Open Boundary Mappings

- Include boundary vertices in the optimization
- Produces mappings with lower distortion

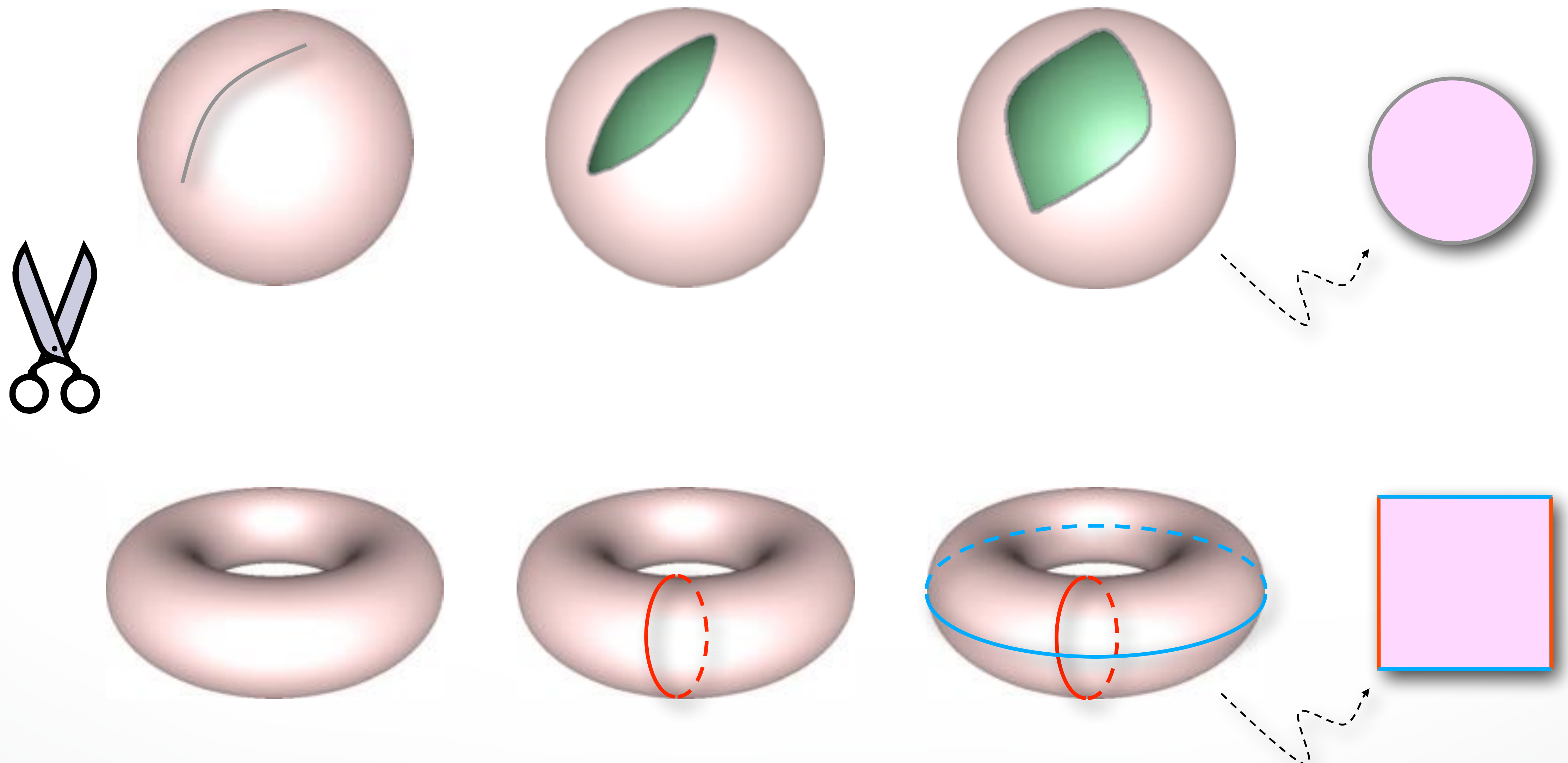


Open Boundary Mappings

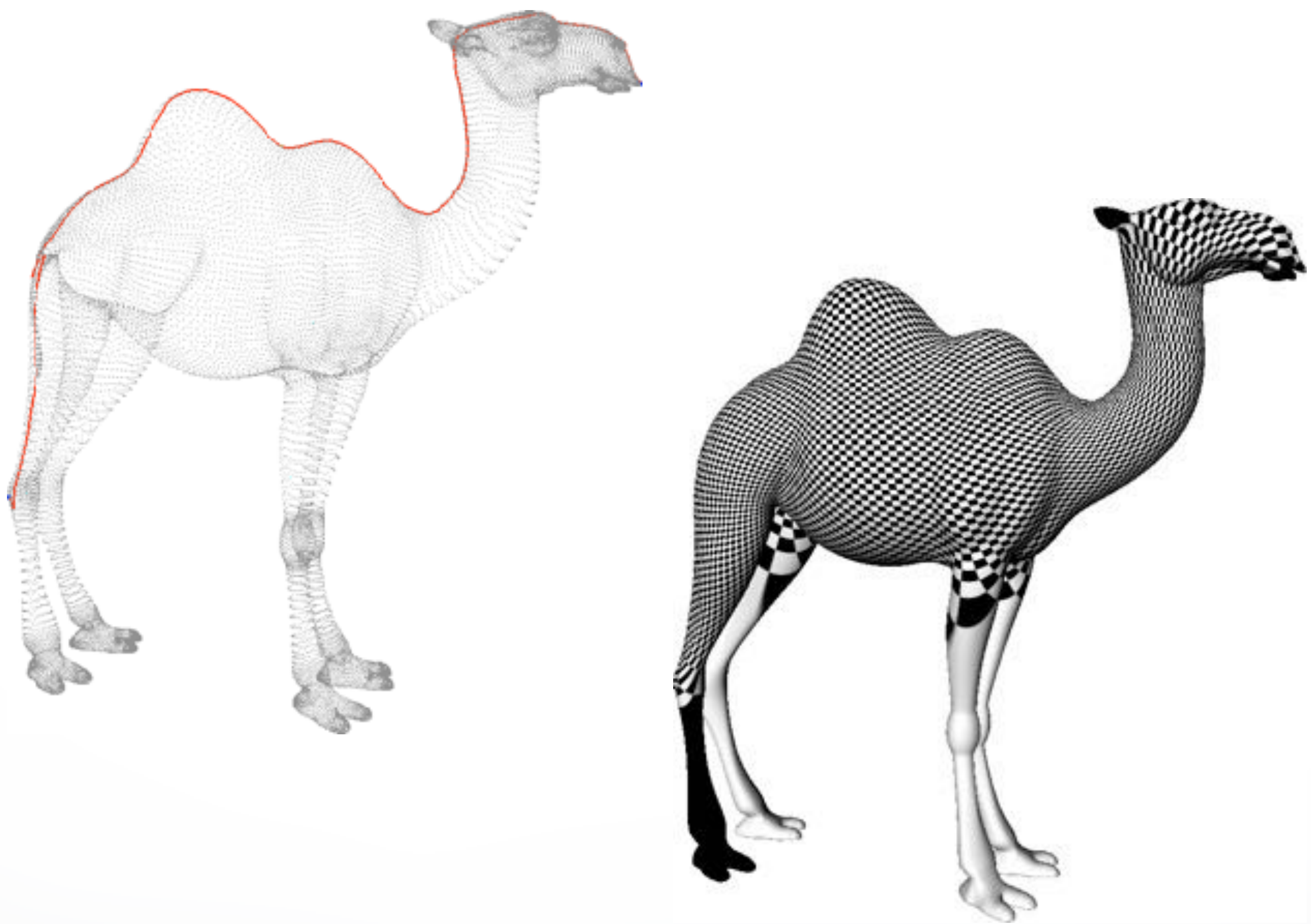


Need disk-like topology

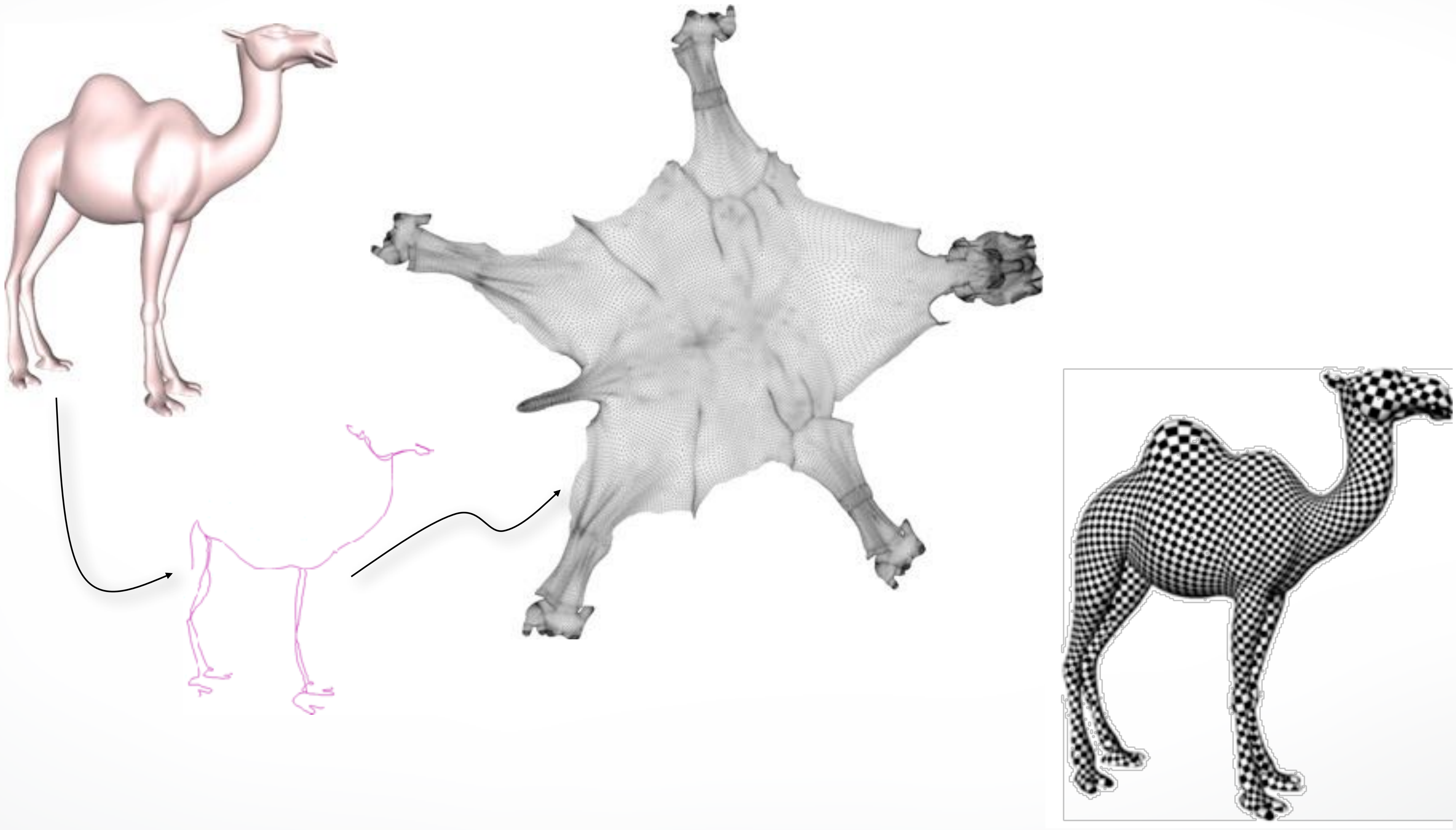
- Introduce cuts on the mesh



Naive Cut, Numerical Problems

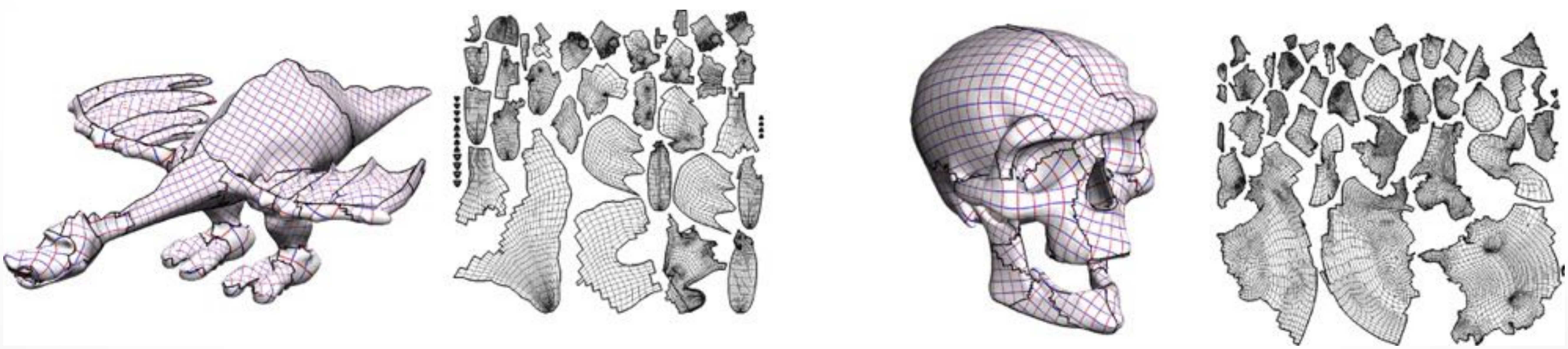


Smart Cut, Free Boundary



Texture Atlas Generation

- Split model into number of patches (atlas)
  - because higher genus models cannot be mapped onto plane and/or
  - because distortion, the number of patches will be too high eventually



Levy, Petitjean, Ray, Maillot: *Least Squares Conformal Maps for Automatic Texture Atlas Generation*, SIGGRAPH, 2002

Texture Atlas Generation

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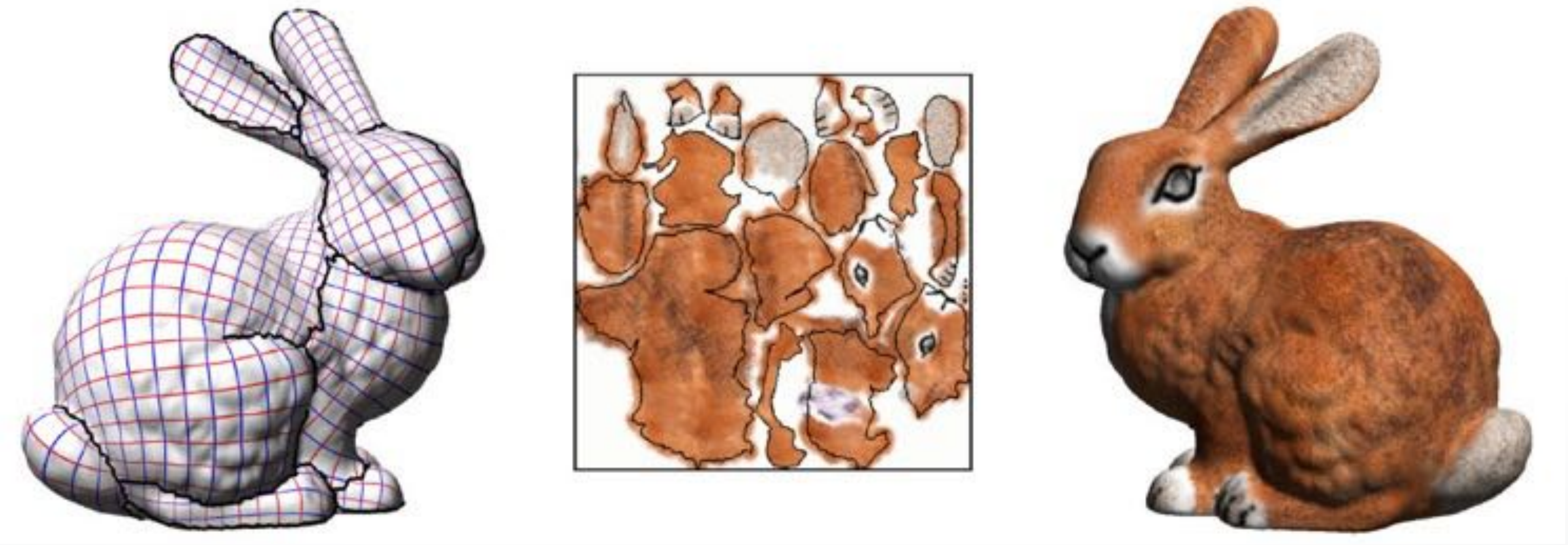
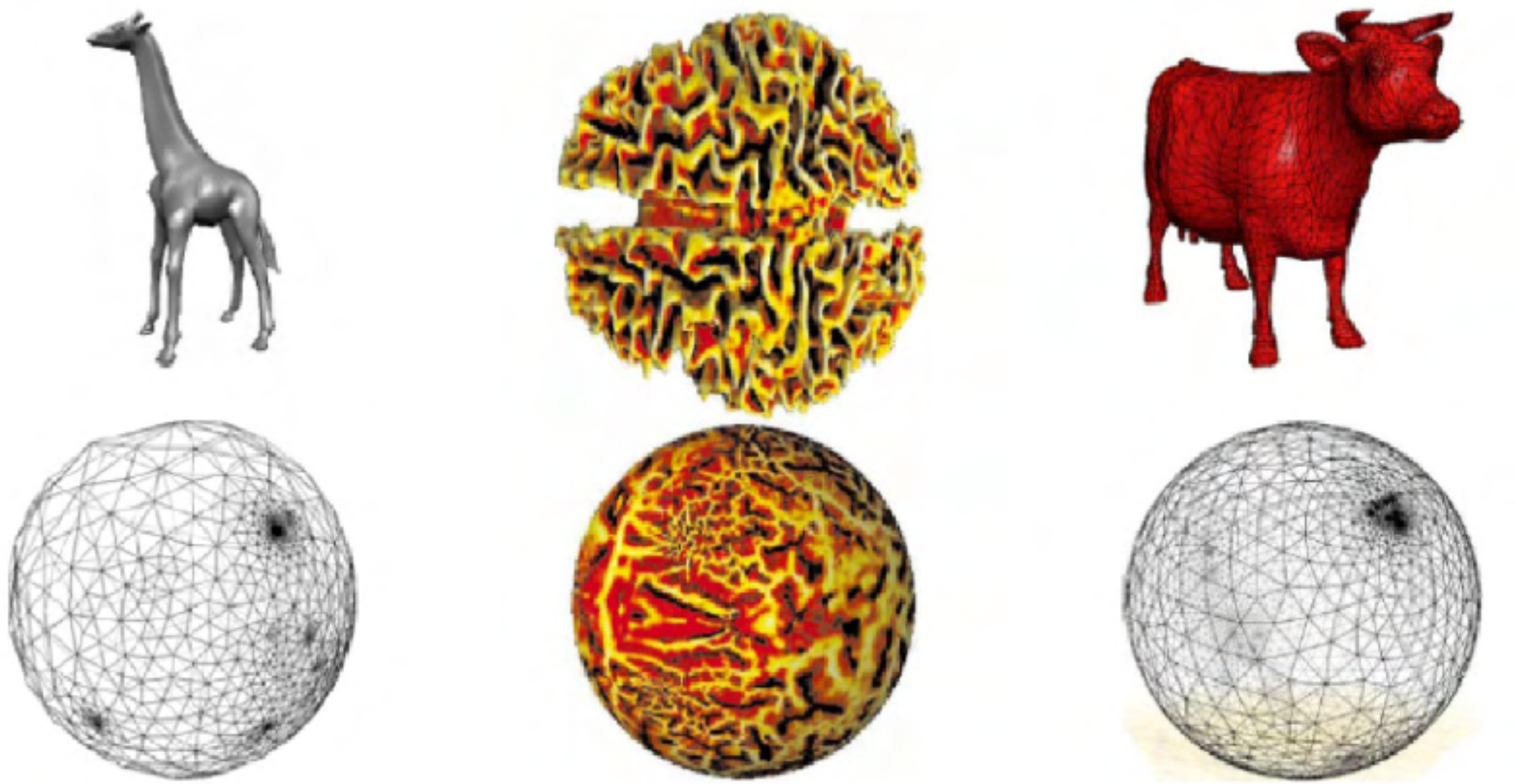


Levy, Petitjean, Ray, Maillot: *Least Squares Conformal Maps for Automatic Texture Atlas Generation*, SIGGRAPH, 2002

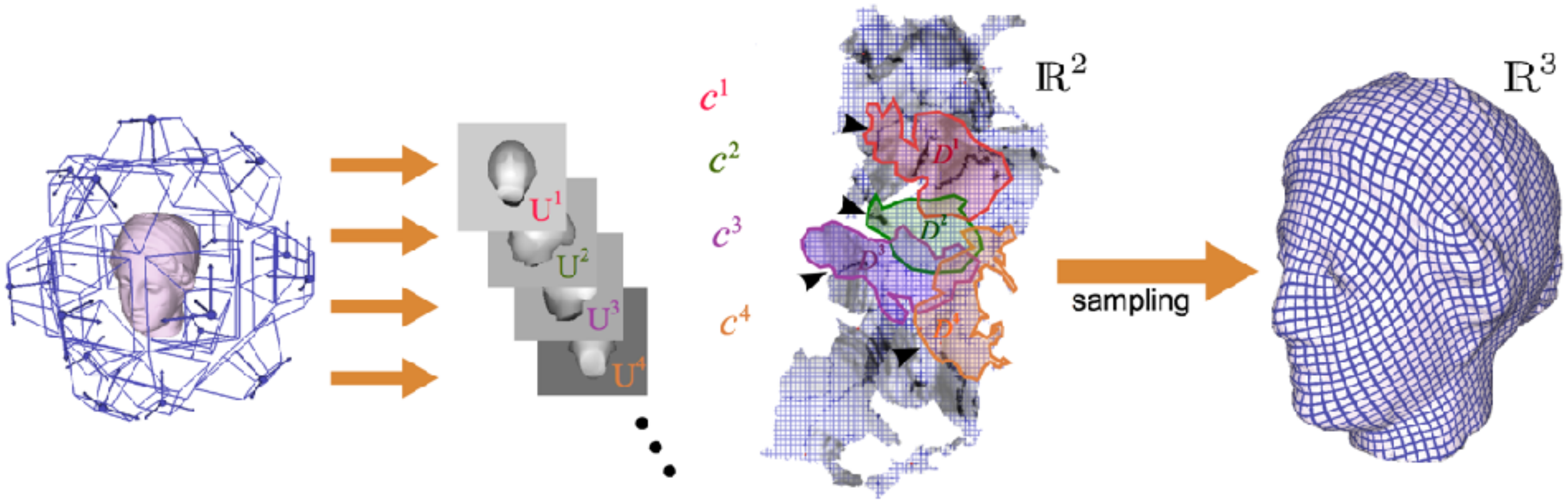
Non-Planar Domains



(a) [Alexa 2000] (b) [Haker et al., 2000] (c) [Isenburg et al., 2001]

seamless, continuous parameterization of genus-0 surfaces

Global Parameterization – Range Images



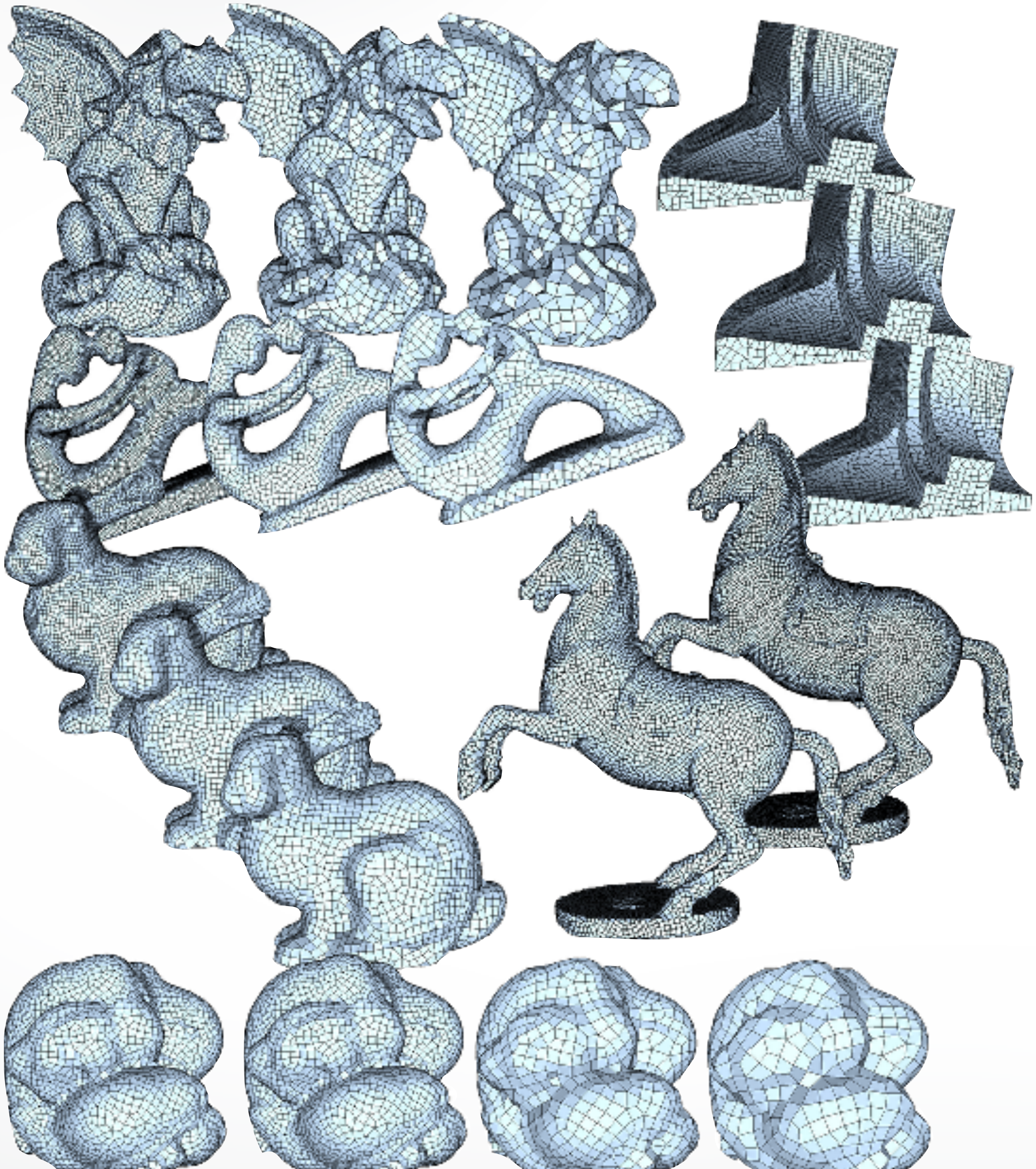
Constrained Parameterizations

Levy: *Constraint Texture Mapping*, SIGGRAPH 2001.

Literature

- Book, Chapter 5
- Hormann et al.: Mesh Parameterization, Theory and Practice, Siggraph 2007 Course Notes
- Floater and Hormann: Surface Parameterization: a tutorial and survey, advances in multiresolution for geometric modeling, Springer 2005
- Hormann, Polthier, and Sheffer, Mesh Parameterization: Theory and Practice, SIGGRAPH Asia 2008 Course Notes

Next Time



Decimation

http://cs621.hao-li.com

Thanks!