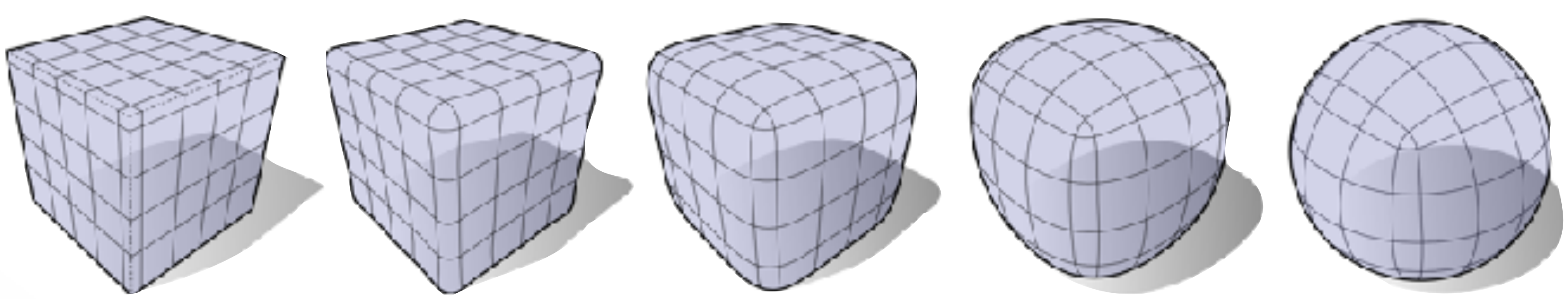
4.1 Classic Differential Geometry 2



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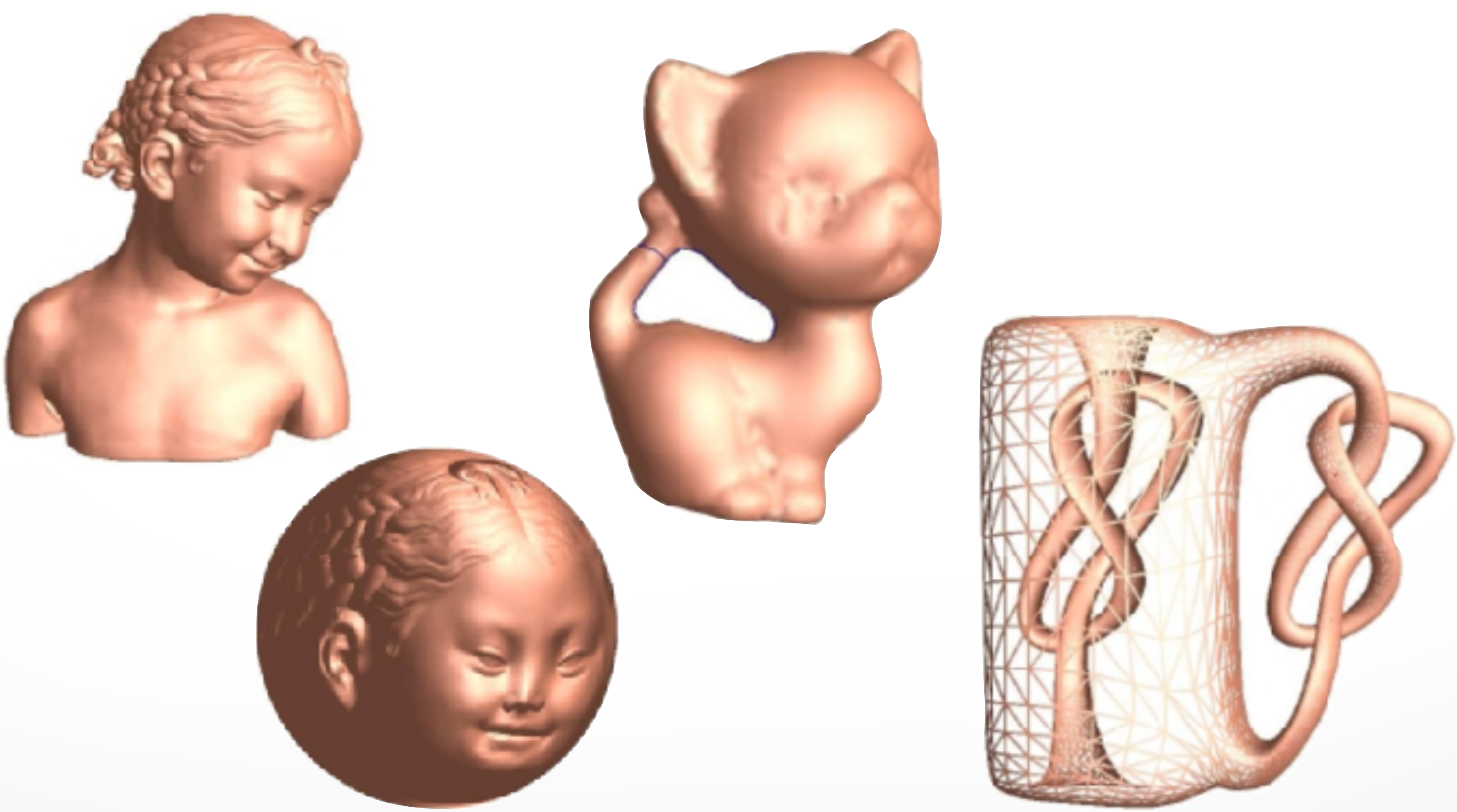
Outline

- Parametric Curves
- **Parametric Surfaces**

Surfaces

What characterizes shape?

- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions



Metric on Surfaces

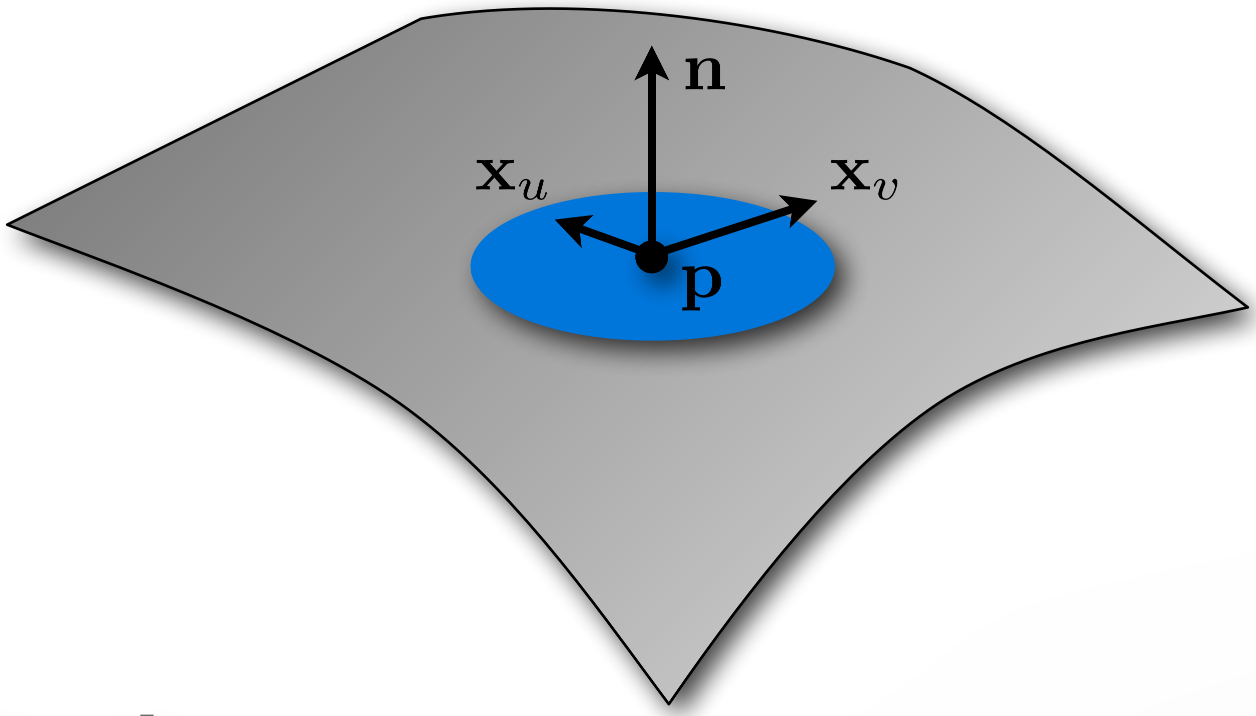
Measure Stuff

- angle, length, area
  - requires an inner product
- we have:
  - Euclidean inner product in domain
- we want to turn this into:
  - inner product on surface

Parametric Surfaces

Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$ normal exists

Angles on Surface

Curve $[u(t), v(t)]$ in uv-plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

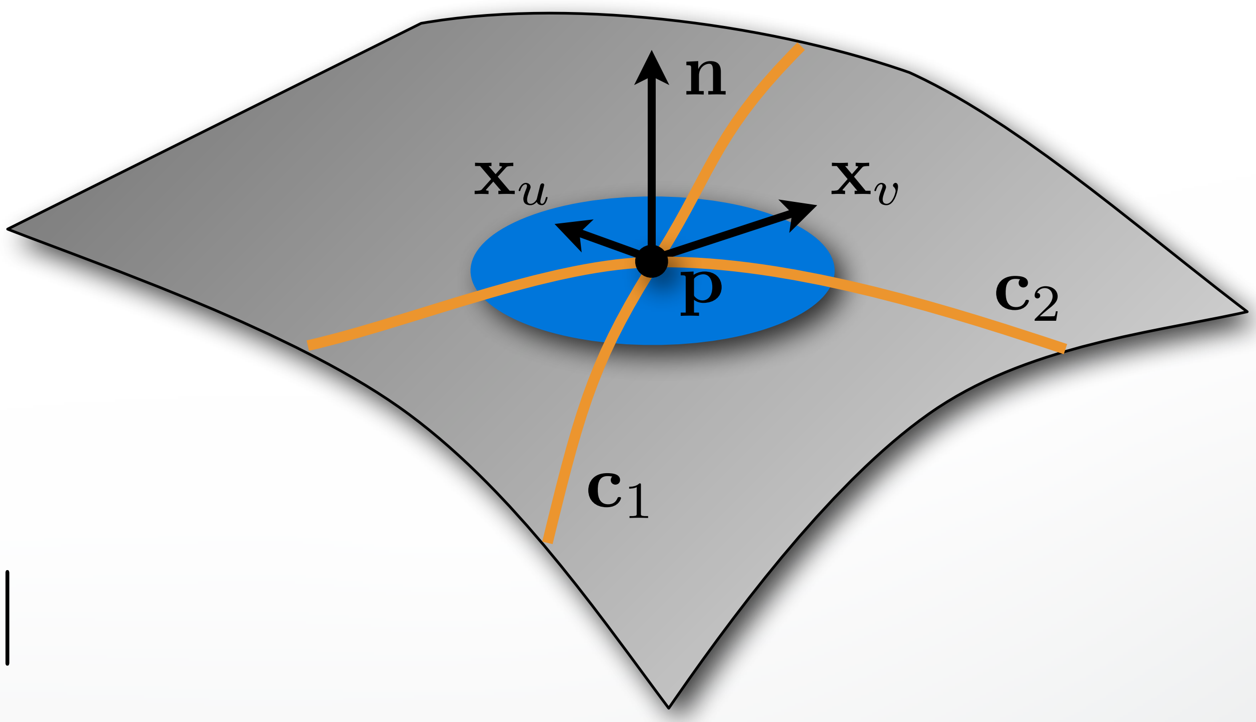
Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

- angle of intersection?
- two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



Angles on Surface

Curve $[u(t), v(t)]$ **in uv-plane defines curve on the surface** $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t)\,, v(t))$$

Two curves $\mathbf{c}_1$ **and** $\mathbf{c}_2$ **intersecting at** $\mathbf{p}$

$$\begin{aligned}
\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\
&= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1)\, \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\
&= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}
\end{aligned}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure (w.r.t. surface metric)

Angles $\quad \mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

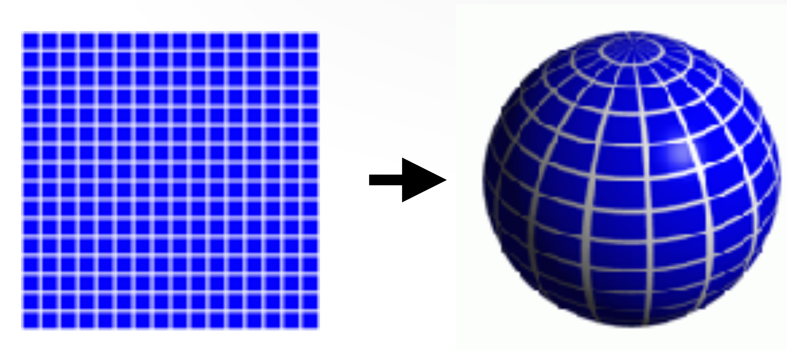
squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

Sphere Example



Spherical parameterization

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Tangent vectors

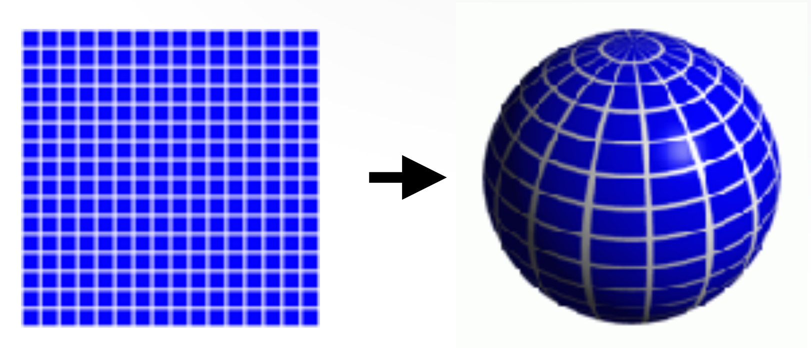
$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \qquad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Sphere Example

Length of equator $\mathbf{x}(t, \pi/2)$



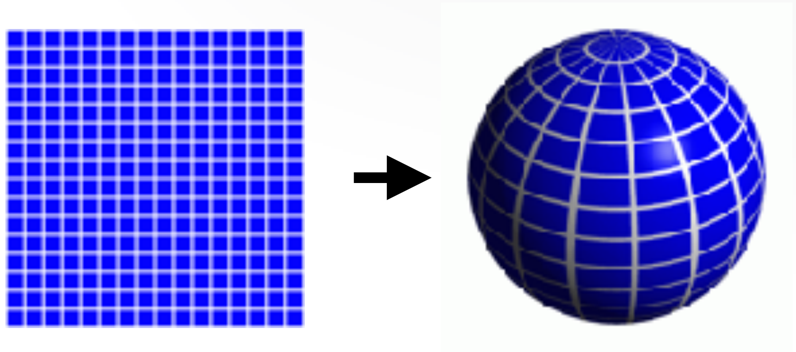
$$\int_0^{2\pi} 1 \, \mathrm{d}s \;=\; \int_0^{2\pi} \sqrt{E\left(u_t\right)^2 + 2F u_t v_t + G\left(v_t\right)^2} \, \mathrm{d}t$$

$$= \int_0^{2\pi} \sin v \, \mathrm{d}t$$

$$= \; 2\pi \sin v \;=\; 2\pi$$

Sphere Example

Area of a sphere



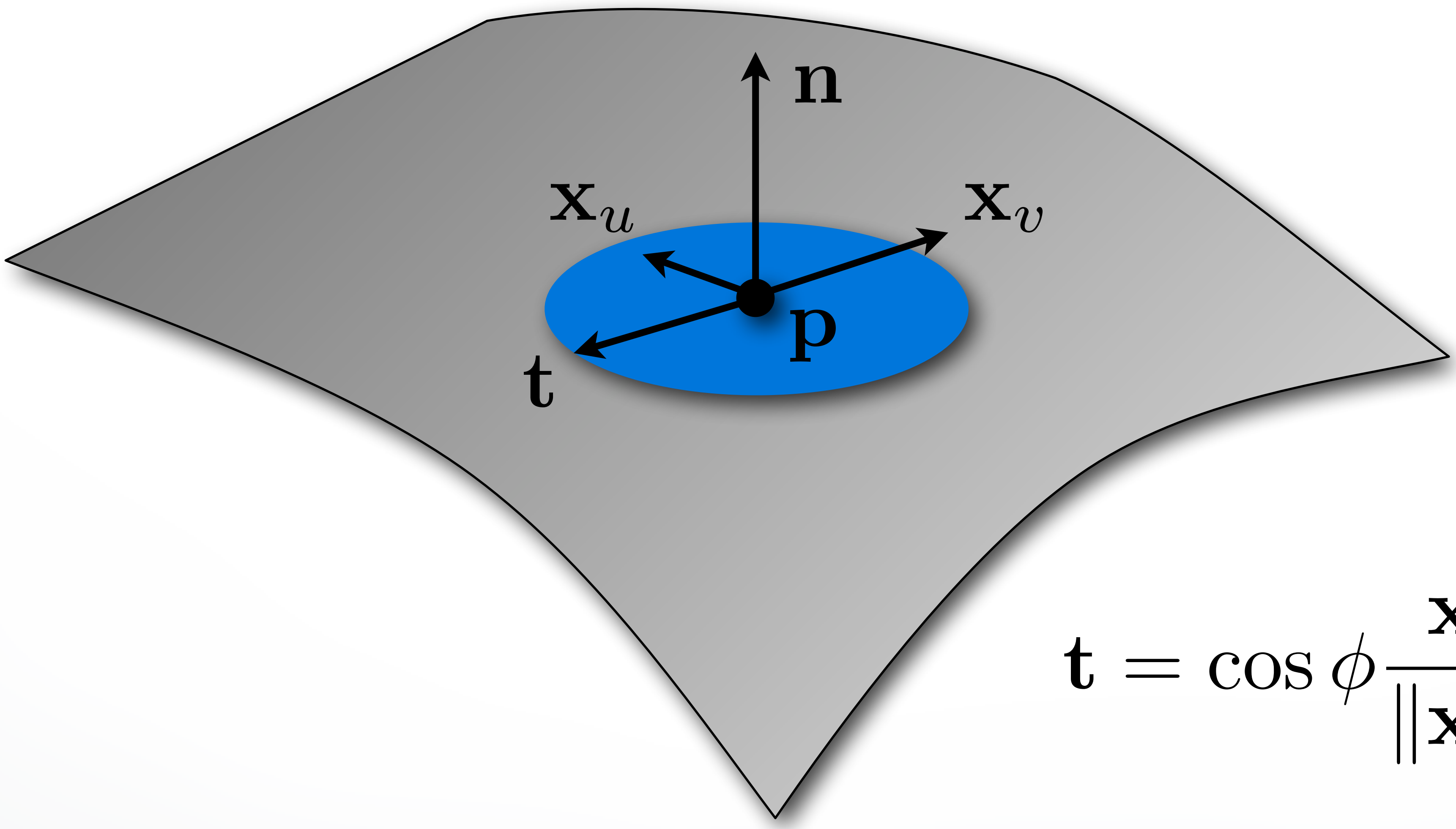
$$\int_0^{\pi}\int_0^{2\pi} 1\,\mathrm{d}A = \int_0^{\pi}\int_0^{2\pi} \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin v\,\mathrm{d}u\,\mathrm{d}v$$

$$= 4\pi$$

Normal Curvature

Tangent vector $\mathbf{t}$ ...



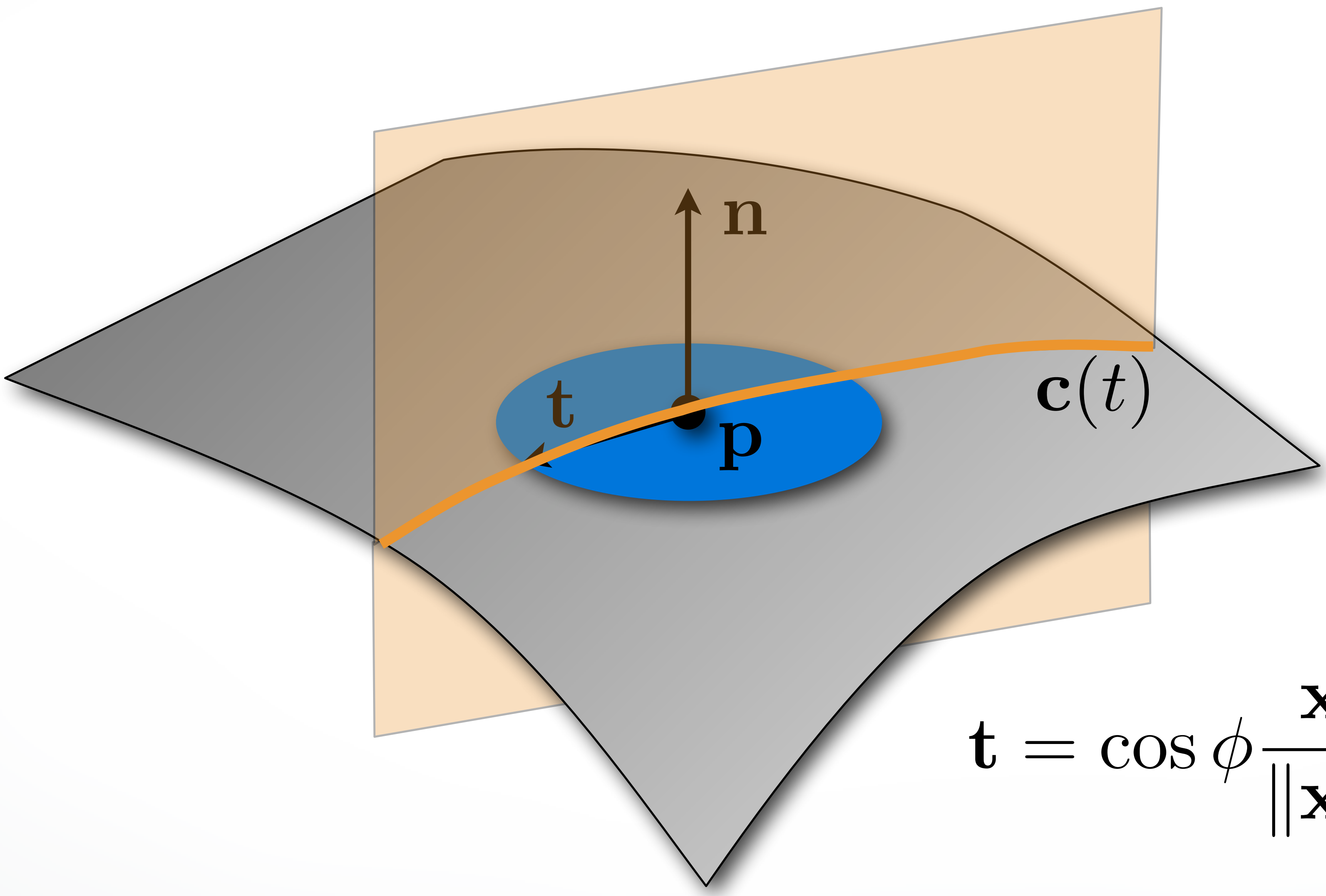
$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

unit vector

Normal Curvature

… defines intersection plane, yielding curve $\mathbf{c}(t)$

normal curve

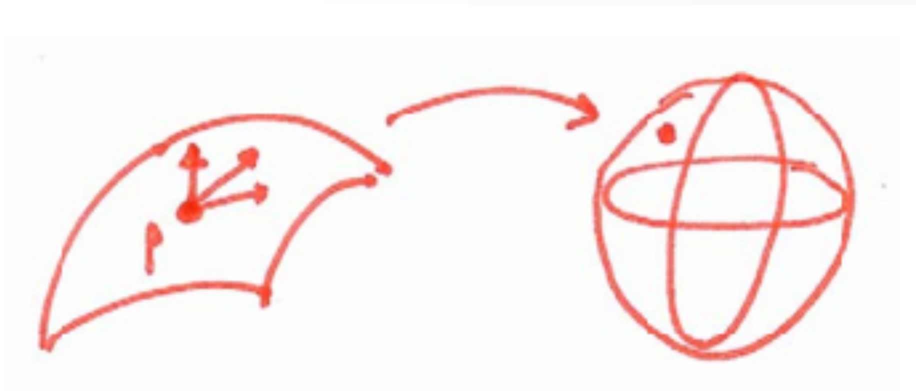


$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Geometry of the Normal

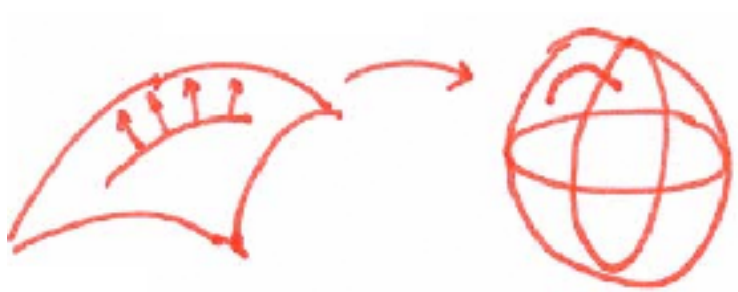
Gauss map

- normal at point



$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \qquad N : S \to \mathbb{S}^2$$

- consider curve in surface again
  - study its curvature at p
  - normal "tilts" along curve



Normal Curvature

Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p}(t) = \mathbf{x}(u, v)$

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

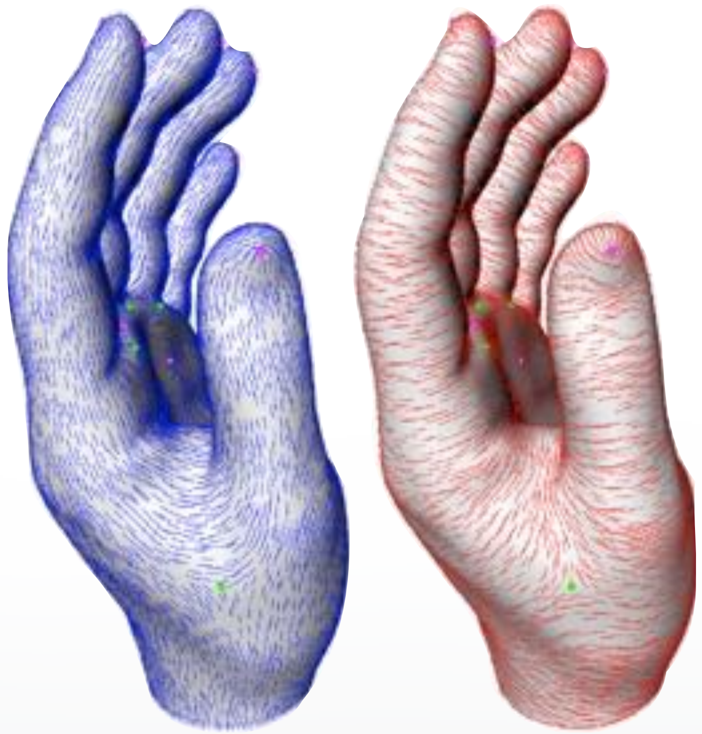
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \, \mathbf{II} \, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \, \mathbf{I} \, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

$$\begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler theorem $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Corresponding principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal



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Special curvatures

- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$ **extrinsic**
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ **intrinsic (only first FF)**

Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$
$$\operatorname{tr} dN_p = \kappa_1 + \kappa_2 = H$$

- eigenvalues and orthovectors

$$dN_p(e_1) = \kappa_1 e_1 \qquad dN_p(e_2) = \kappa_2 e_2$$

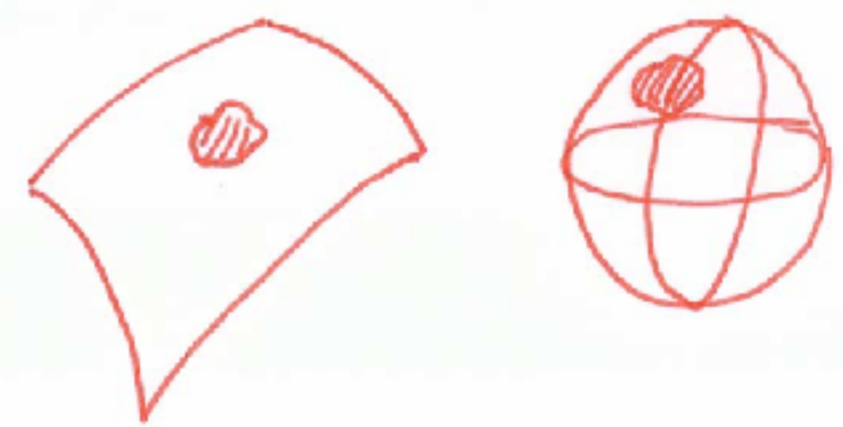
$$II_p|_{\mathbb{S} \subset T_p S} \quad \begin{cases} \max \to \kappa_1 \\ \min \to \kappa_2 \end{cases}$$

Mean and Gaussian Curvature

Integral representations

$$H_p/2 = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta$$

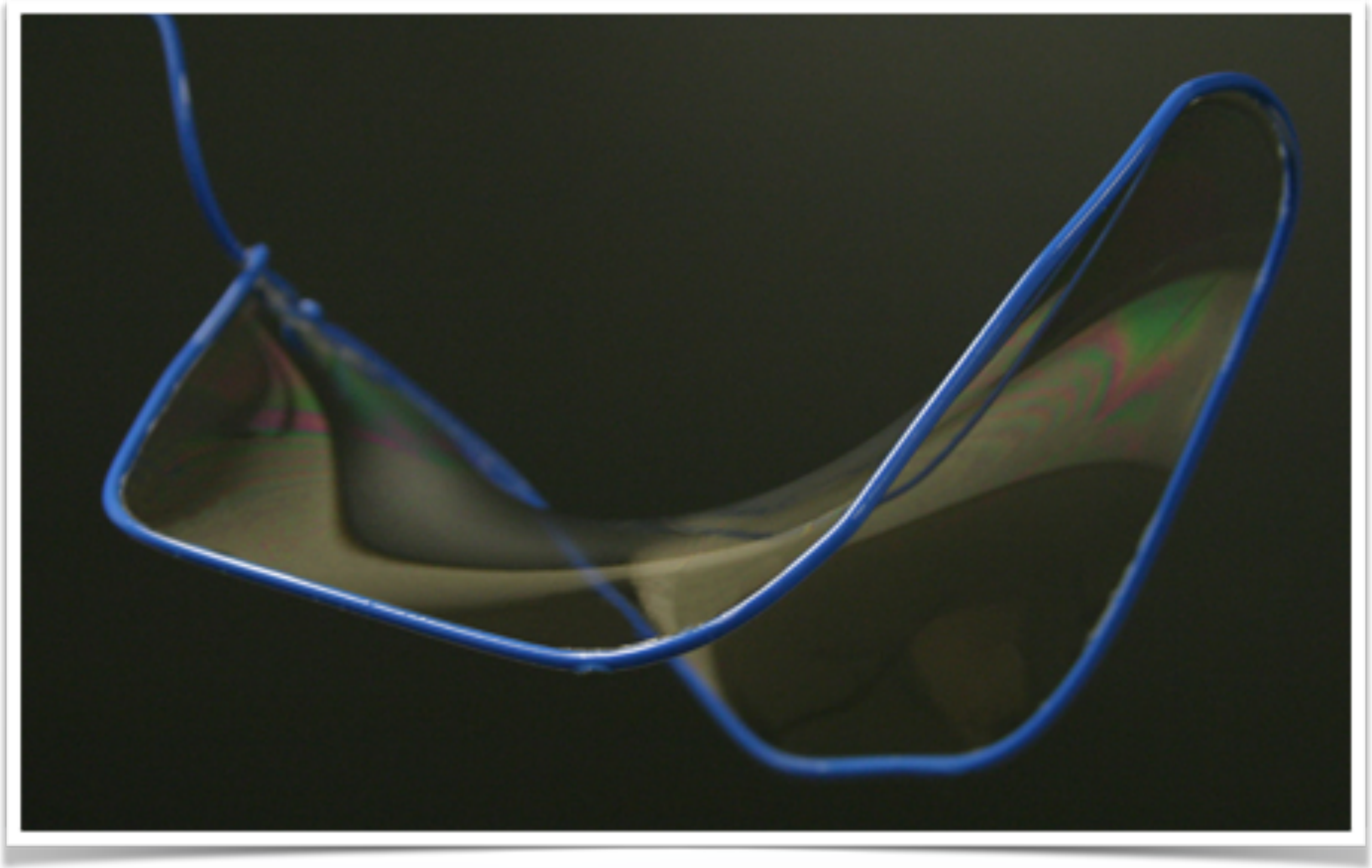
$$K_p = \lim_{A \to 0} \frac{A_G}{A}$$



Curvature of Surfaces

Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere $\rightarrow$ minimal surface



soap film

Curvature of Surfaces

Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

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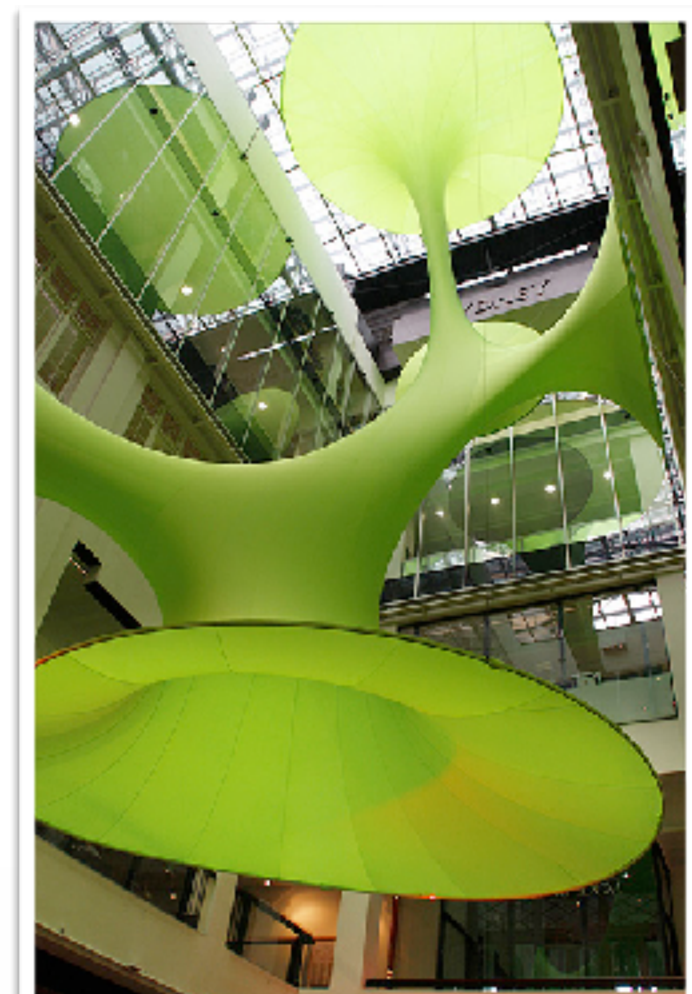


Green Void, Sydney
Architects: Lava

Curvature of Surfaces

Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

- $K = 0$ everywhere $\rightarrow$ developable surface

surface that can be flattened to a plane without distortion (stretching or compression)

Disney, Concert Hall, L.A.
Architects: Gehry Partners

Timber Fabric
IBOIS, EPFL

Shape Operator

Derivative of Gauss map

- second fundamental form

$$II_p(v) = \langle dN_p(v), v \rangle$$

- local coordinates

$$II_p = -\begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Intrinsic Geometry

Properties of the surface that only depend on the first fundamental form

- length
- angles
- Gaussian curvature (Theorema Egregium)

remarkable theorem (Gauss)

$$K = \lim_{r \to 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

Gaussian curvature of a surface is invariant under local isometry

Classification

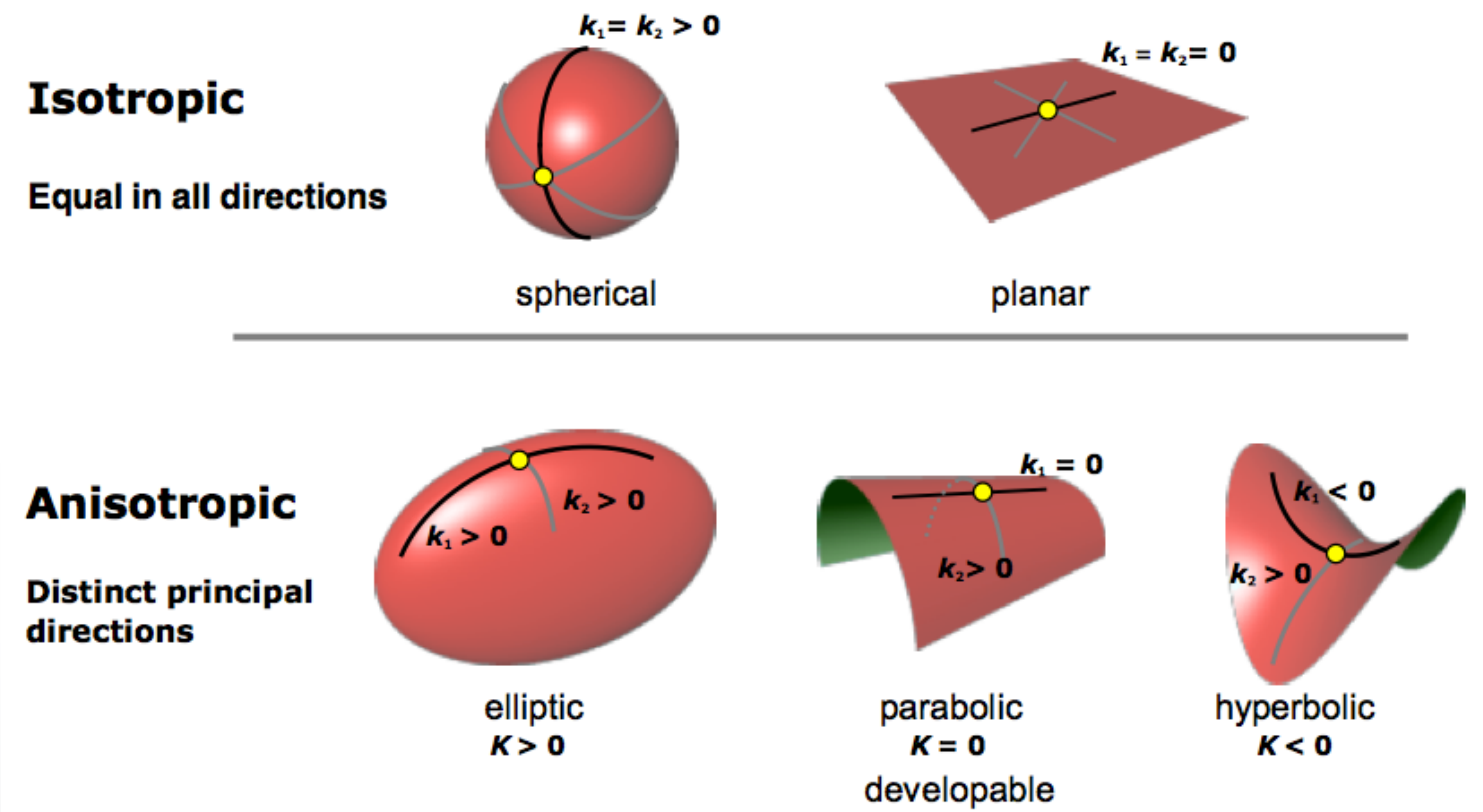
Point $\mathbf{x}$ on the surface is called

- elliptic, if $K > 0$
- hyperbolic, if $K < 0$
- parabolic, if $K = 0$
- umbilic, if $\kappa_1 = \kappa_2$ or isotropic

Gaussian curvature K

Classification

Point $\mathbf{x}$ on the surface is called



Gauss-Bonnet Theorem

For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

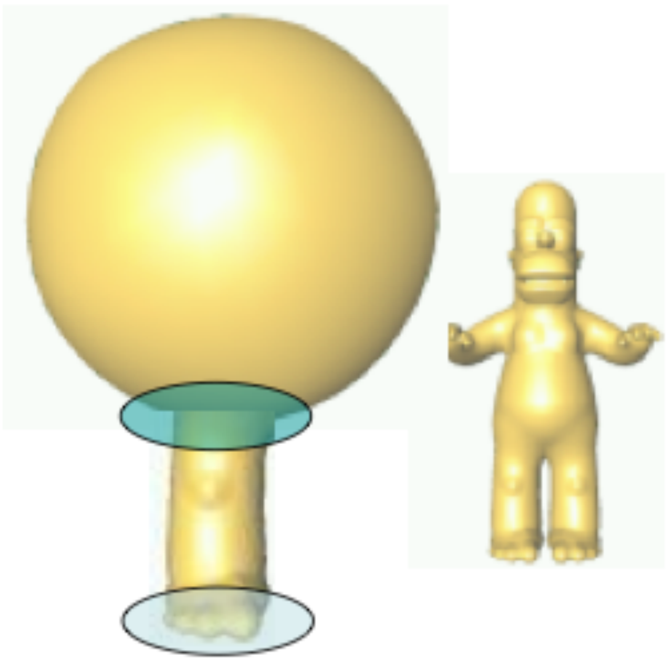
Gauss-Bonnet Theorem

Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



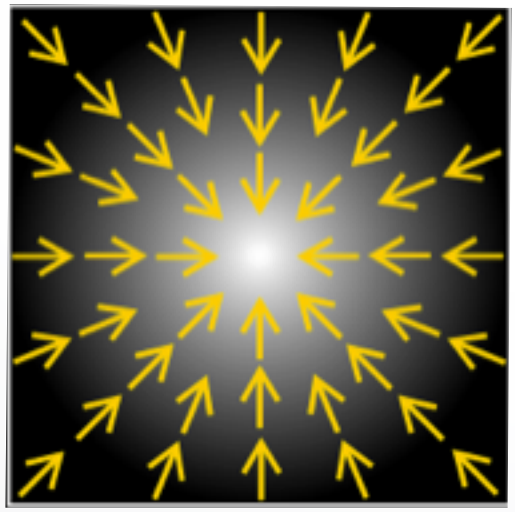
when sphere is deformed, new
positive and negative curvature cancel out

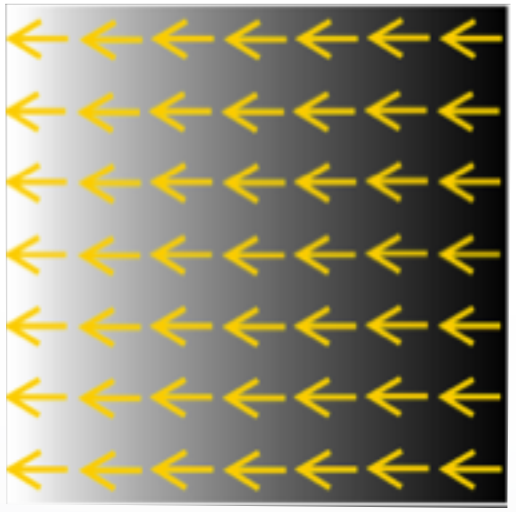
Differential Operators

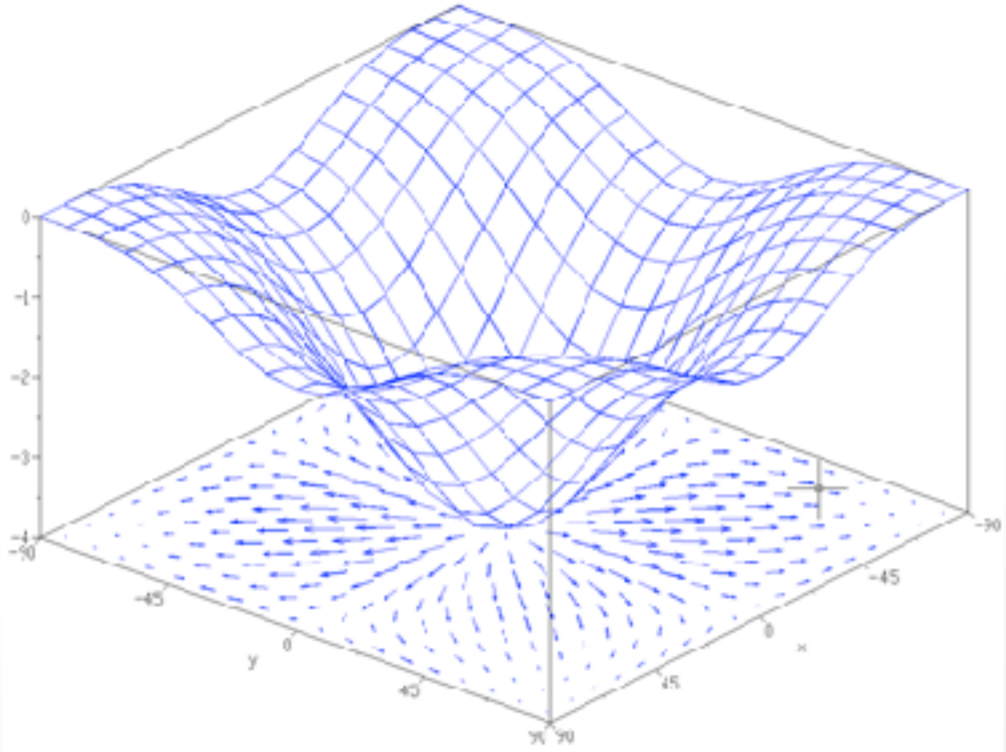
Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend







Differential Operators

Divergence

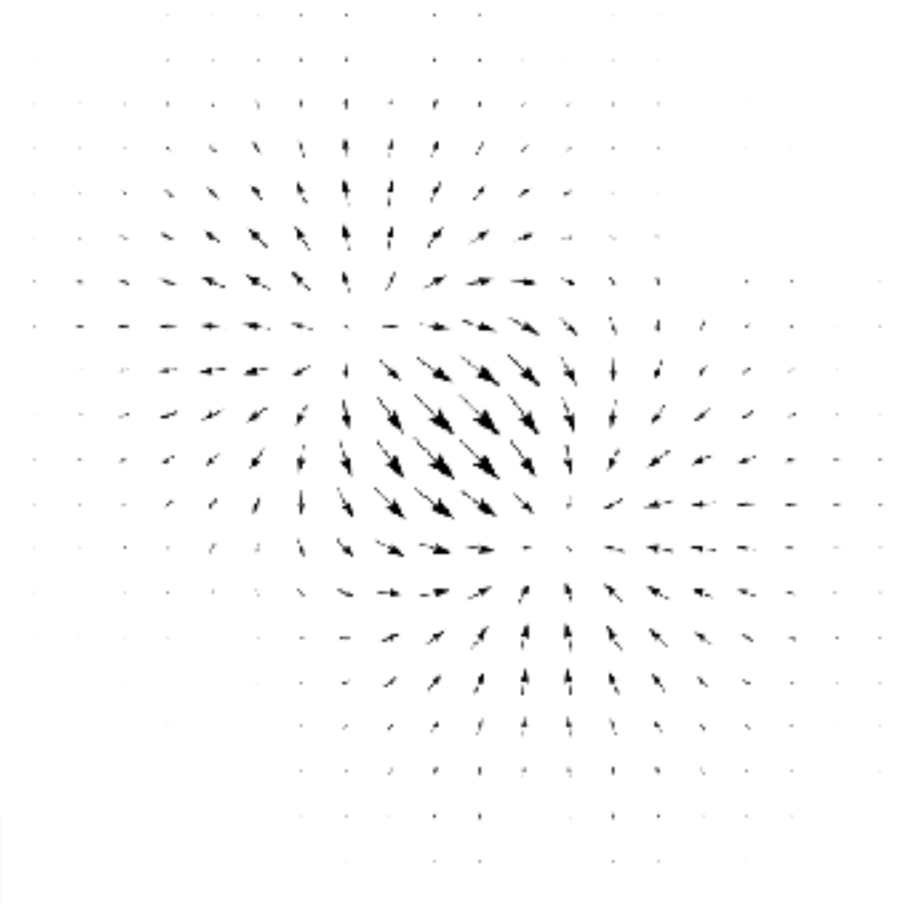
$$\text{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: incompressible fluid
  - velocity field is divergence-free

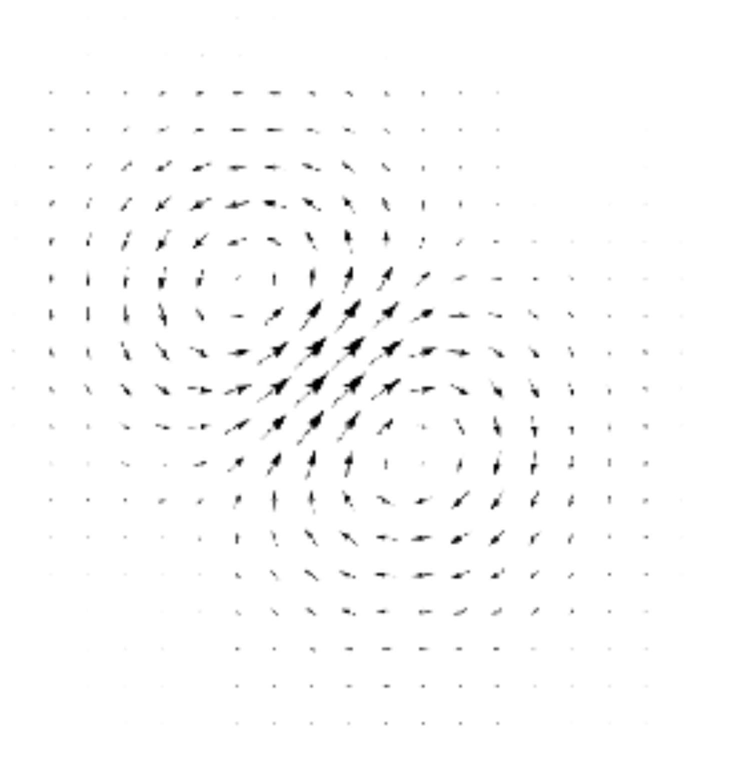
Differential Operators

Divergence

$$\text{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

Laplace Operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

Extension of Laplace of functions on manifolds

Laplace-Beltrami

gradient operator

...of the surface

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}}\, \nabla_{\mathcal{S}} f$$

function on manifold S

divergence operator

Laplace on the surface

Laplace-Beltrami Operator

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

function on manifold S

divergence operator

surface normal

Literature

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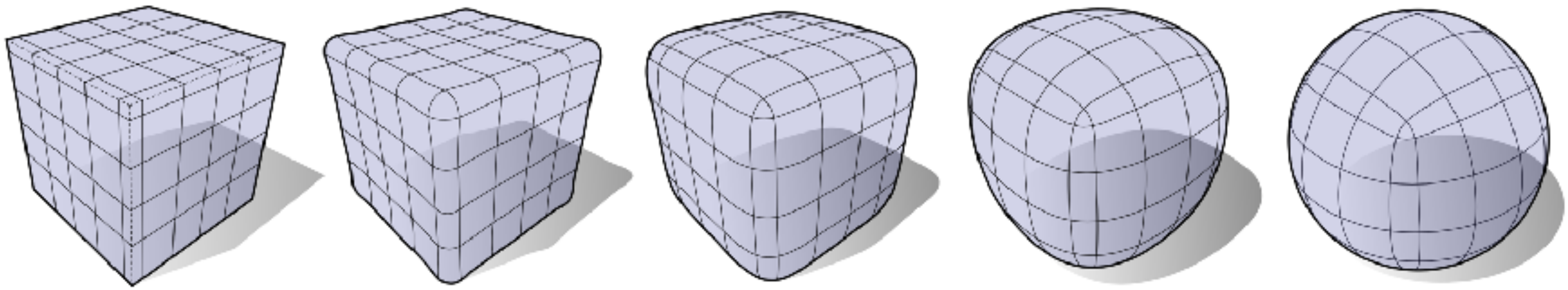
Next Time



Discrete Differential Geometry

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Thanks!