4.1 Classic Differential Geometry 2

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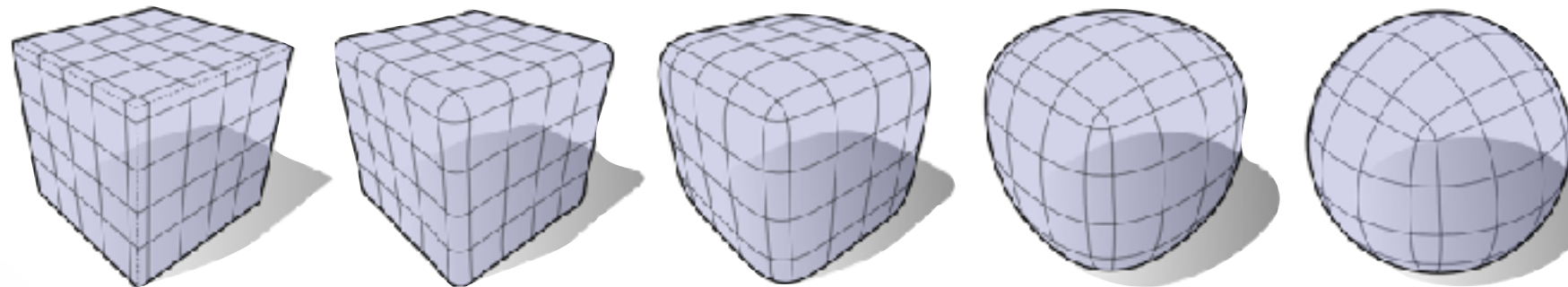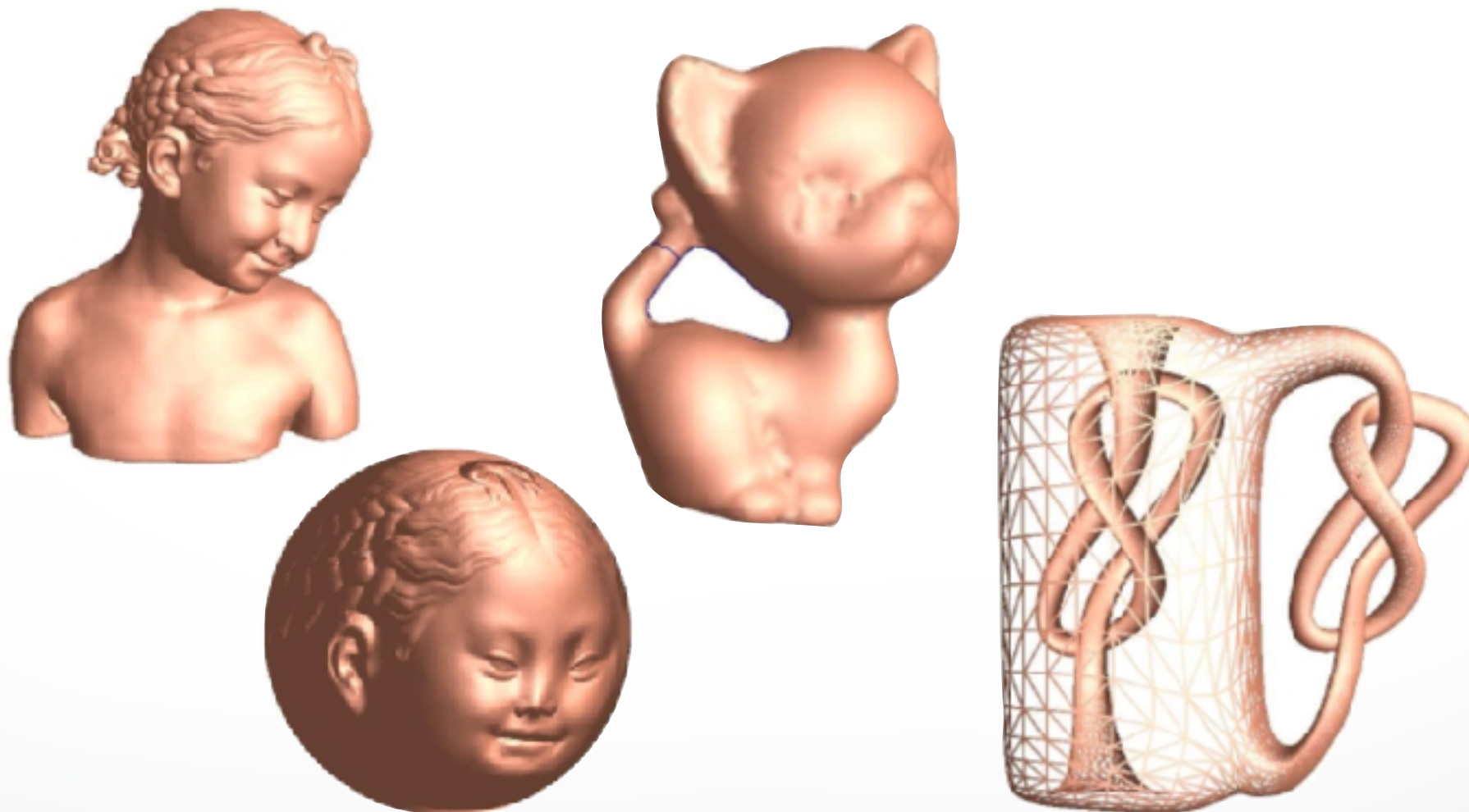
Outline

- Parametric Curves

- **Parametric Surfaces**

Surfaces

What characterizes shape?

- shape does not depend on Euclidean motions

  - metric and curvatures

- smooth continuous notions to discrete notions

Metric on Surfaces

Measure Stuff

- angle, length, area
  - requires an inner product
- we have:
  - Euclidean inner product in domain
- we want to turn this into:
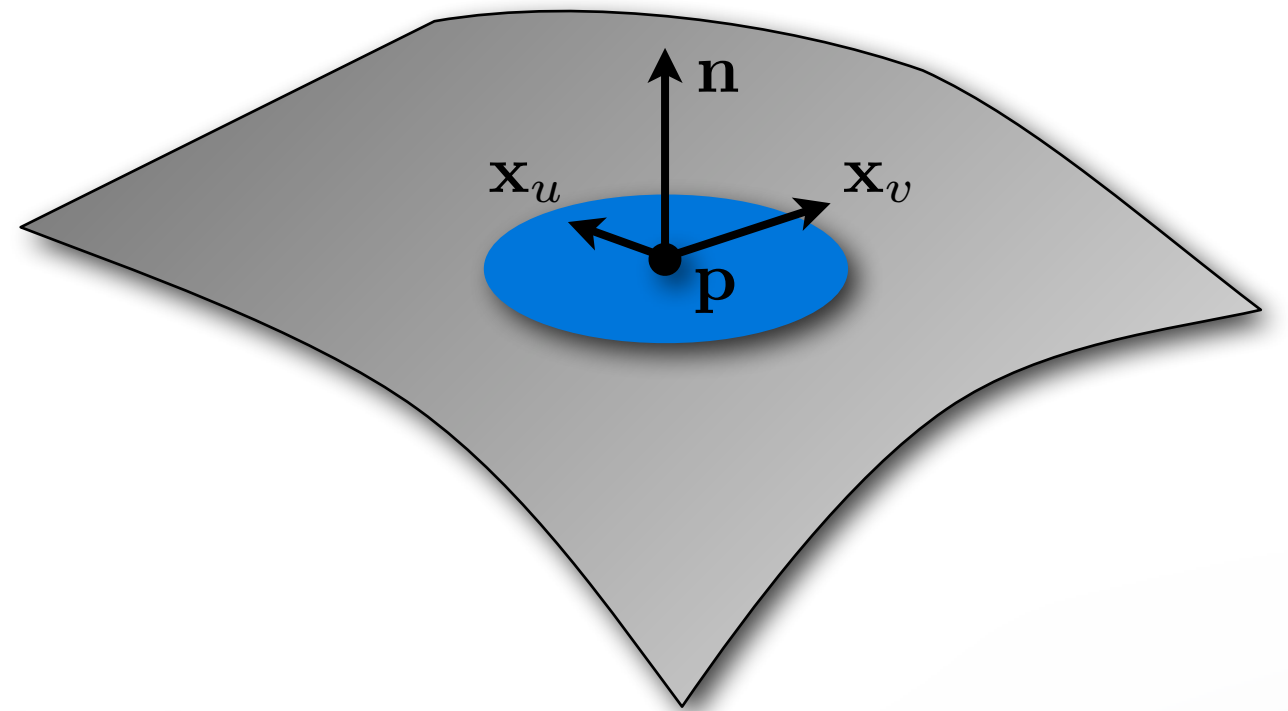 - inner product on surface

Parametric Surfaces

Continuous surface

$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$ **normal exists**

Curve $[u(t), v(t)]$ in uv-plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$
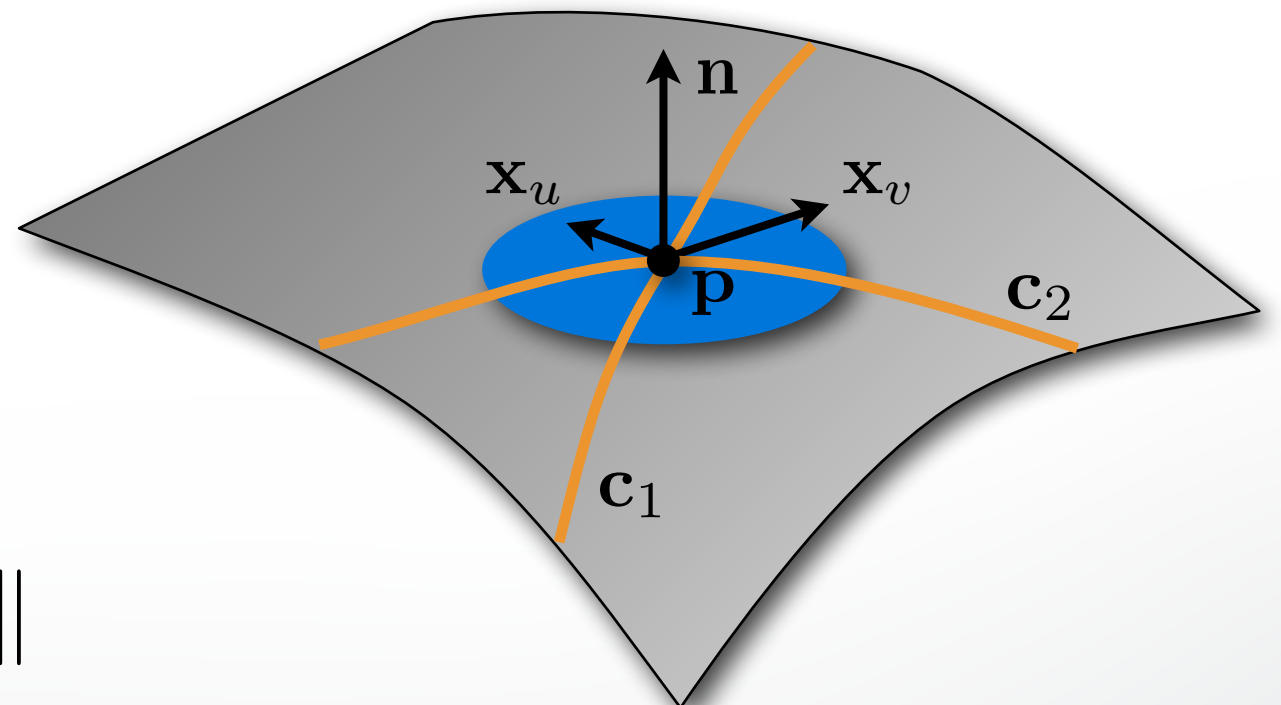
Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

- angle of intersection?
- two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$

Curve $[u(t), v(t)]$ **in uv-plane defines curve on the surface** $\mathbf{x}(u, v)$

$$\mathbf{c}(t) \ = \ \mathbf{x}(u(t)\,, v(t))$$

Two curves $\mathbf{c}_1$ **and** $\mathbf{c}_2$ **intersecting at** $\mathbf{p}$

$$\mathbf{t}_1^T \mathbf{t}_2 \ = \ (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v)$$

$$= \ \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1)\, \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v$$

$$= \ (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} \;=\; \begin{pmatrix} E & F \\ F & G \end{pmatrix} \;:=\; \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle \;:=\; \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure (w.r.t. surface metric)

Angles

$$\mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$$

Length

$$\begin{aligned}
\mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\
&= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2
\end{aligned}$$

squared infinitesimal length

Area

$$\begin{aligned}
\mathrm{d}A &= \| \mathbf{x}_u \times \mathbf{x}_v \| \, \mathrm{d}u\, \mathrm{d}v \\
&= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} \, \mathrm{d}u\, \mathrm{d}v \\
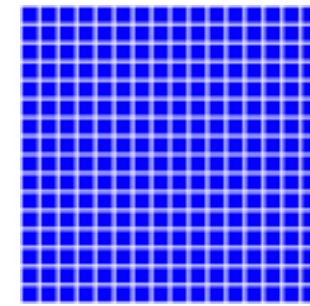&= \sqrt{EG - F^2} \mathrm{d}u\, \mathrm{d}v
\end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

Sphere Example

Spherical parameterization

$$\mathbf{x}(u,v) \;=\; \left(\begin{array}{c} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{array} \right), \qquad (u,v) \in [0, 2\pi) \times [0, \pi)$$
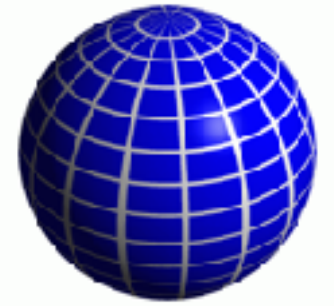
Tangent vectors

$$\mathbf{x}_u(u,v) \;=\; \left(\begin{array}{c} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{array} \right) \qquad \mathbf{x}_v(u,v) \;=\; \left(\begin{array}{c} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{array} \right)$$

First fundamental Form

$$\mathbf{I} \;=\; \left(\begin{array}{cc} \sin^2 v & 0 \\ 0 & 1 \end{array} \right)$$

Sphere Example

Length of equator $\mathbf{x}(t, \pi/2)$
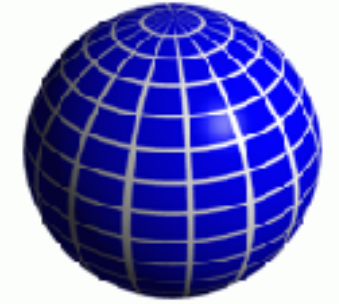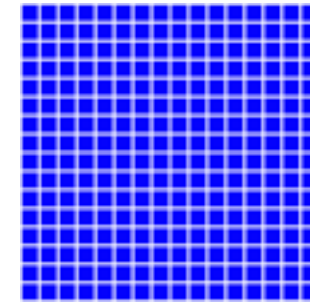
$$\int_0^{2\pi} 1 \, \mathrm{d}s = \int_0^{2\pi} \sqrt{E\left(u_t\right)^2 + 2F u_t v_t + G\left(v_t\right)^2} \, \mathrm{d}t$$

$$= \int_0^{2\pi} \sin v \, \mathrm{d}t$$

$$= 2\pi \sin v = 2\pi$$

Sphere Example

Area of a sphere
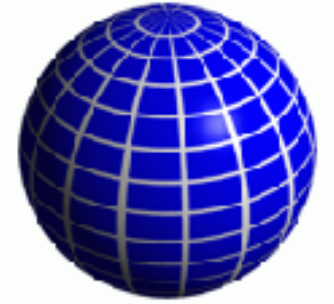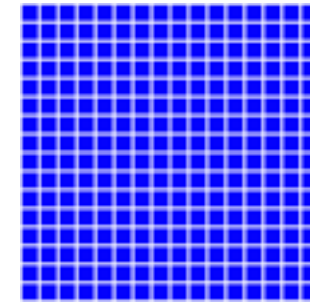
$$\int_0^\pi \int_0^{2\pi} 1\, \mathrm{d}A \;=\; \int_0^\pi \int_0^{2\pi} \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v$$

$$=\; \int_0^\pi \int_0^{2\pi} \sin v\, \mathrm{d}u\, \mathrm{d}v$$

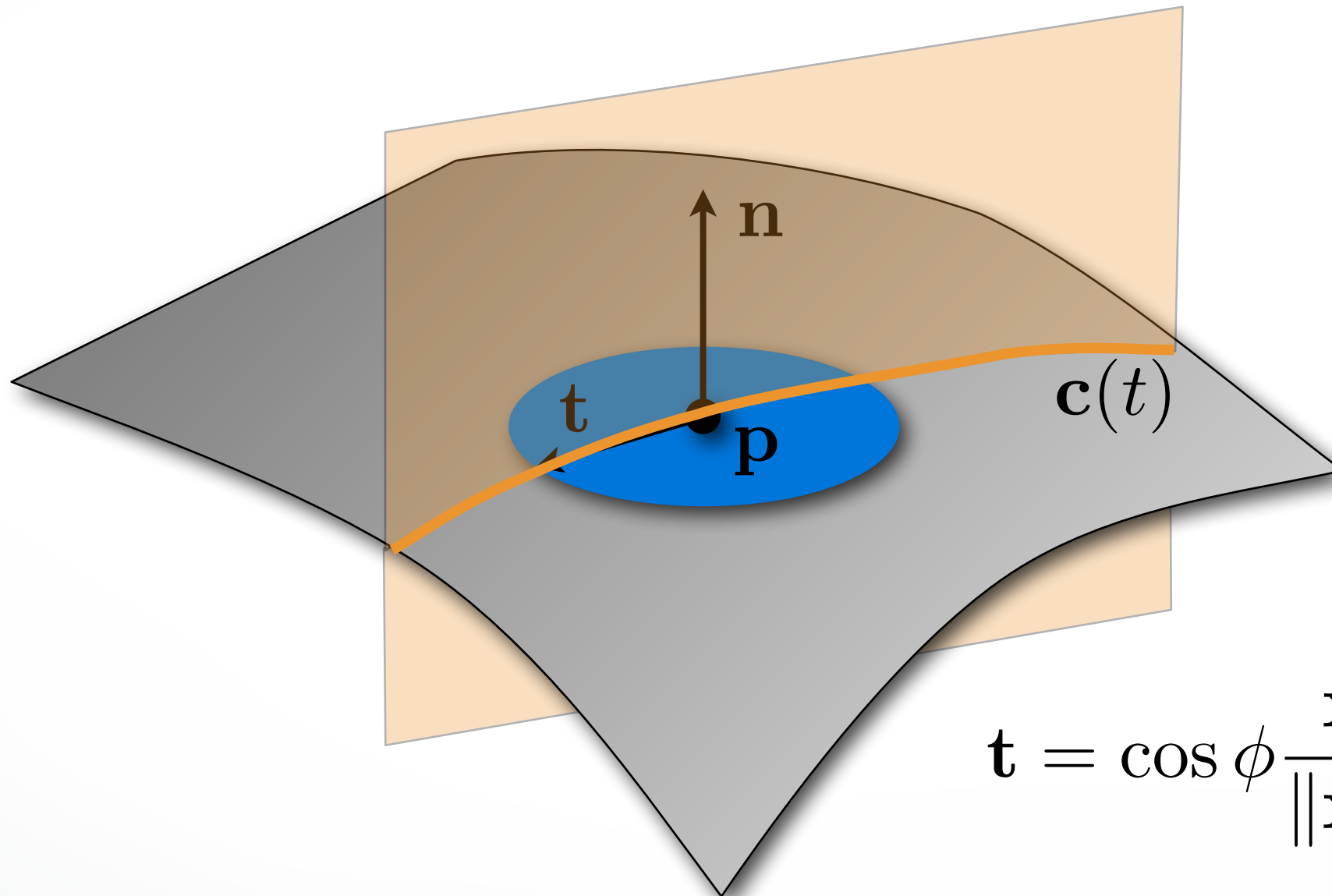$$=\; 4\pi$$

Normal Curvature

Tangent vector t …
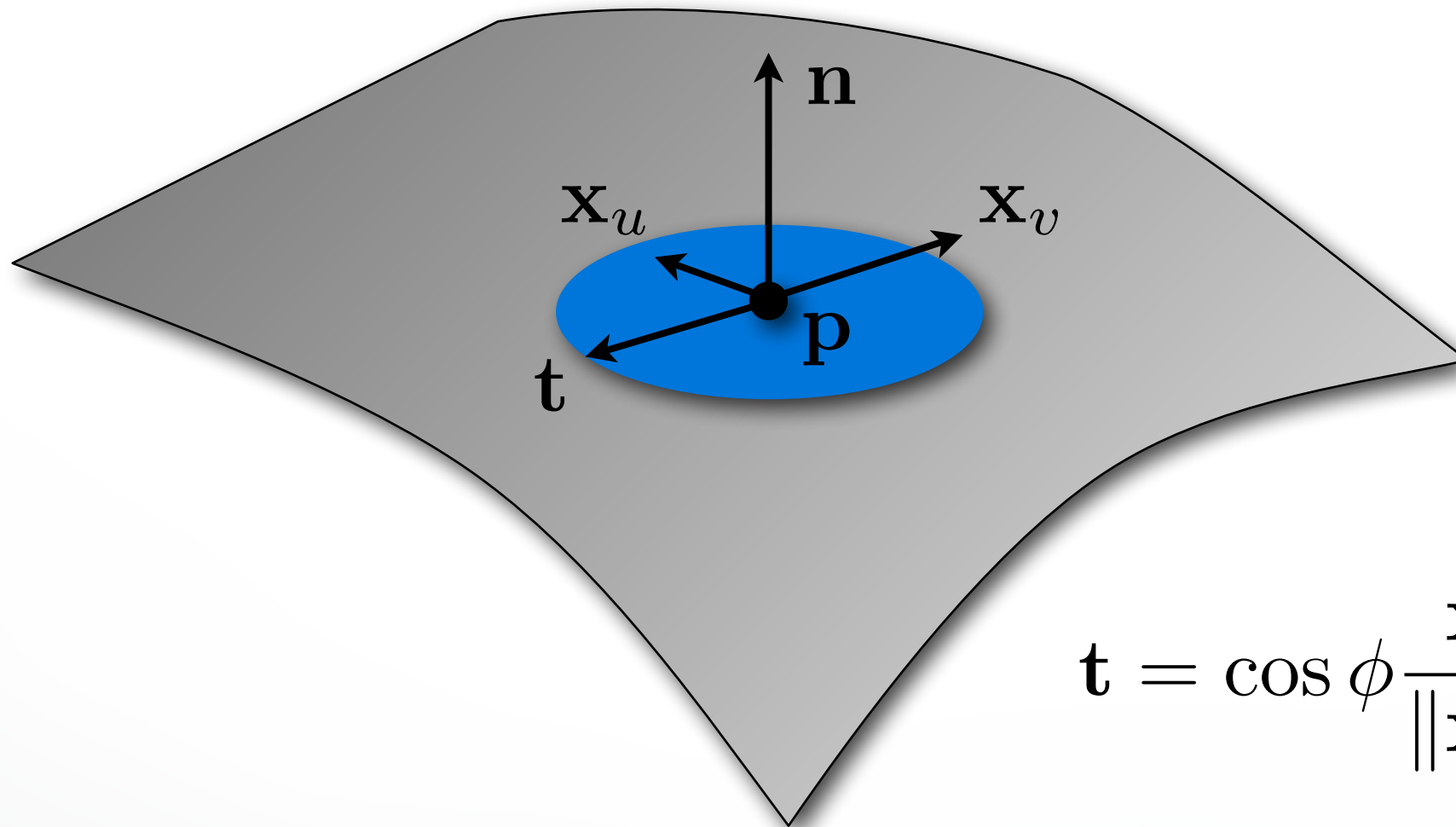
$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

unit vector

Normal Curvature

... defines intersection plane, yielding curve $\mathbf{c}(t)$

normal curve

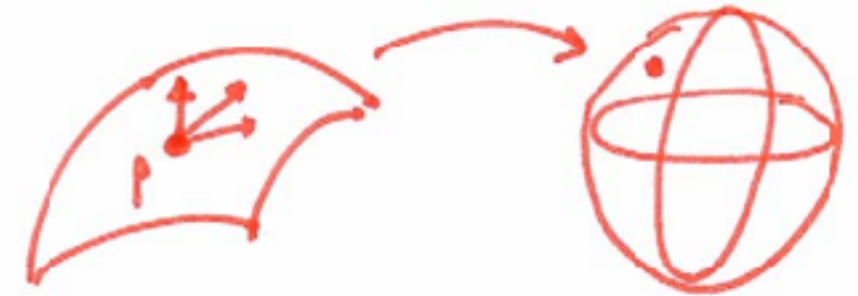$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Gauss map

- normal at point

$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \qquad N : S \to \mathbb{S}^2$$

- consider curve in surface again

  - study its curvature at p

  - normal "tilts" along curve

Normal Curvature

Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p}(t) = \mathbf{x}(u,v)$

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$
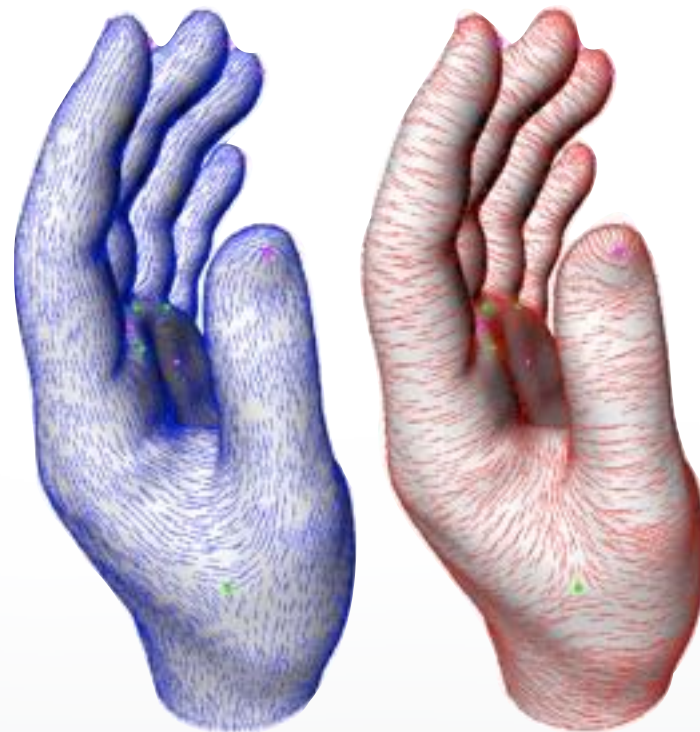
normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I}\, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \qquad \begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a,b) \end{aligned}$$

Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max\limits_{\phi} \kappa_n(\phi)$

- Minimum curvature $\kappa_2 = \min\limits_{\phi} \kappa_n(\phi)$

- Euler theorem $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$

- Corresponding principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal

Surface Curvature(s)

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Special curvatures

- Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$ **extrinsic**
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ **intrinsic (only first FF)**

Invariants

Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$
$$\operatorname{tr} dN_p = \kappa_1 + \kappa_2 = H$$

- eigenvalues and orthovectors

$$dN_p(e_1) = \kappa_1 e_1 \qquad dN_p(e_2) = \kappa_2 e_2$$

$$II_p|_{\mathbb{S} \subset T_p S} \begin{cases} \max \to \kappa_1 \\ \min \to \kappa_2 \end{cases}$$

Mean Curvature

Integral representations

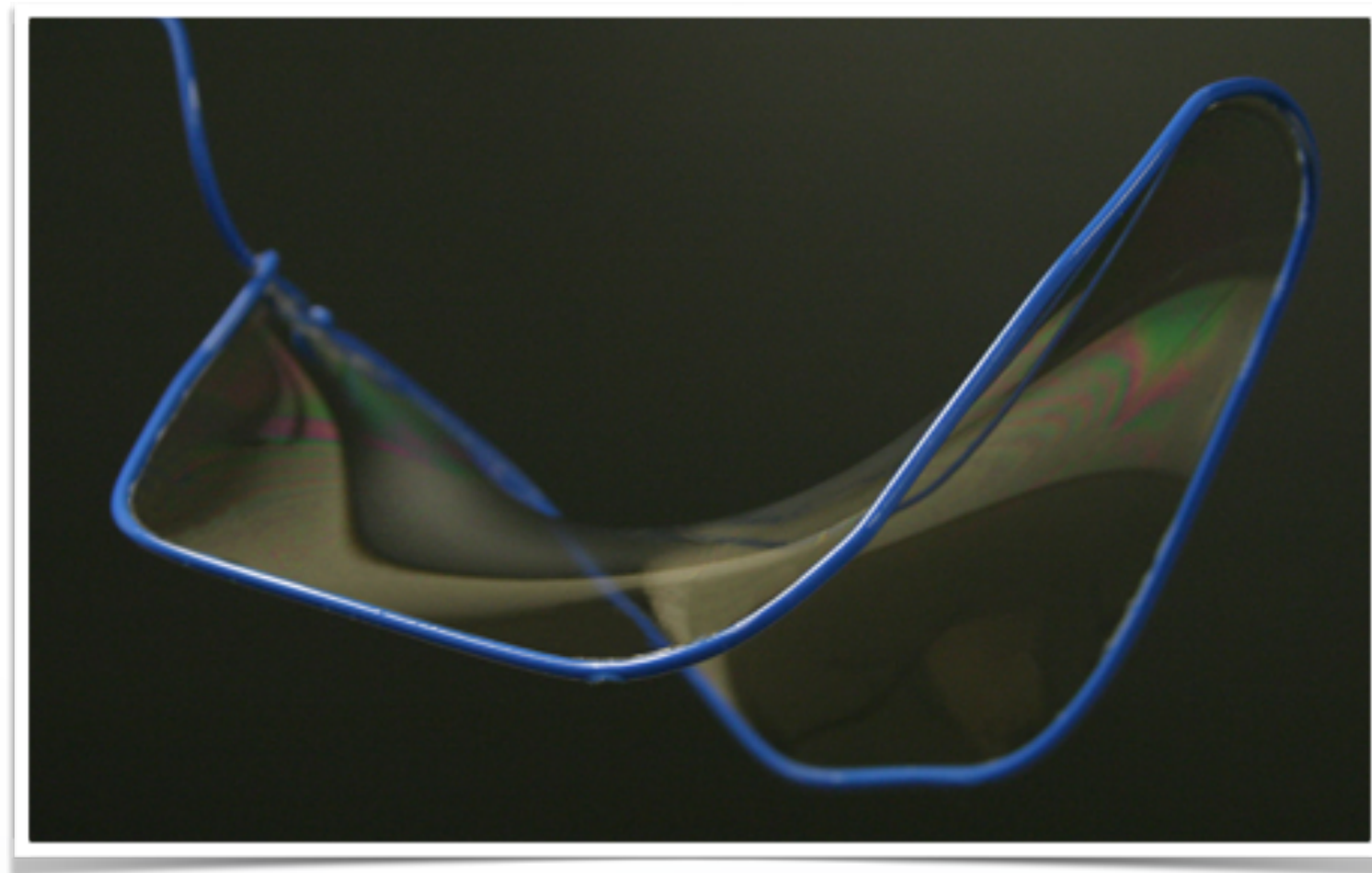$$H_p/2 = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\,d\theta$$

$$K_p = \lim_{A \to 0} \frac{A_G}{A}$$
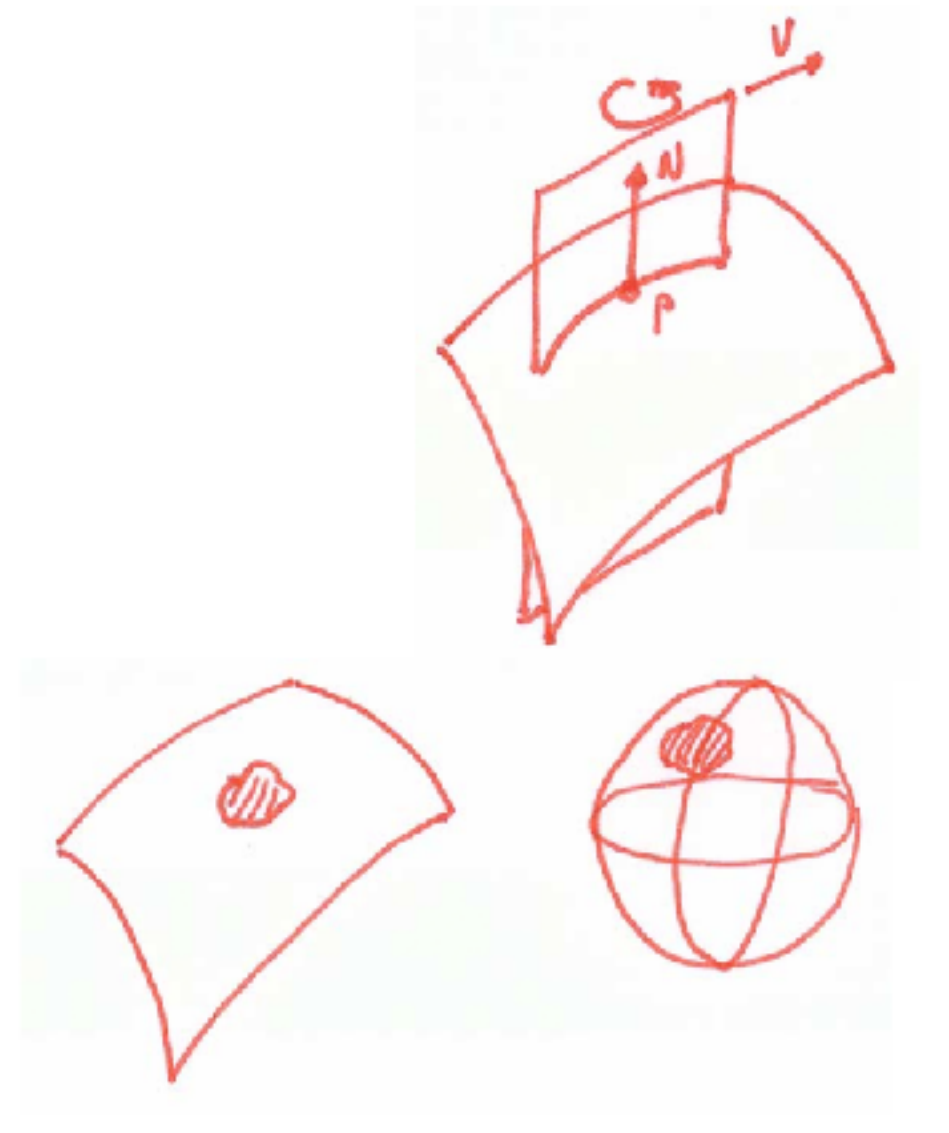
Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere $\rightarrow$ minimal surface

soap film

Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere $\rightarrow$ minimal surface

Green Void, Sydney
Architects: Lava
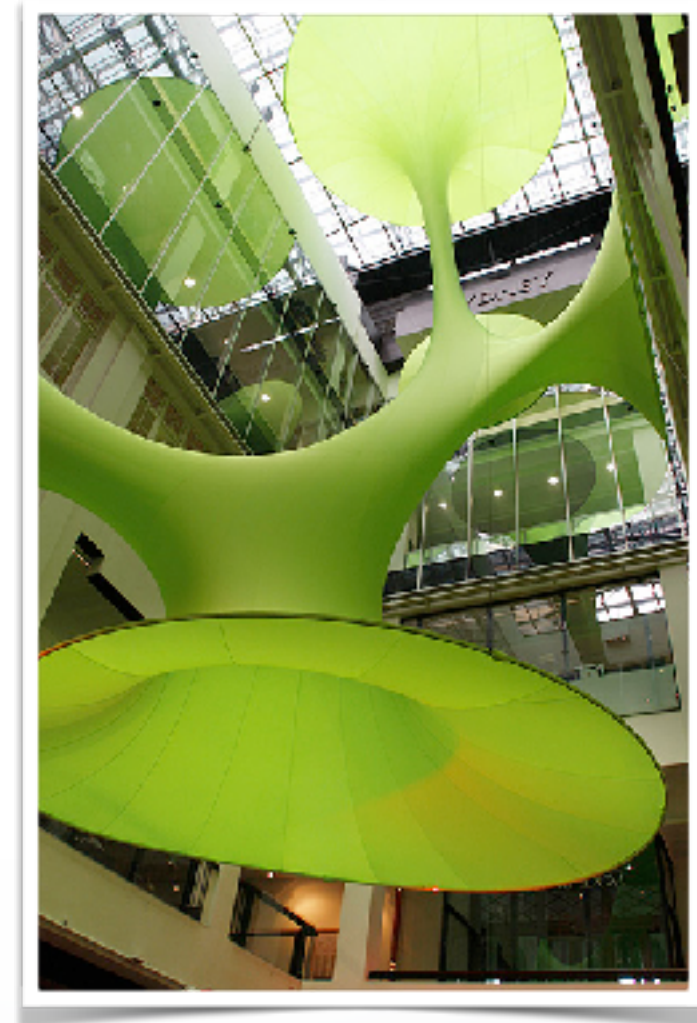
Curvature of Surfaces

Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

- $K = 0$ everywhere $\rightarrow$ developable surface

surface that can be flattened to a plane without distortion (stretching or compression)

Disney, Concert Hall, L.A.
Architects: Gehry Partners

Timber Fabric
IBOIS, EPFL

Derivative of Gauss map

- second fundamental form

$$II_p(v) = \langle dN_p(v), v \rangle$$

- local coordinates

$$II_p = - \begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Intrinsic Geometry

Properties of the surface that only depend on the first fundamental form

- length

- angles

- Gaussian curvature (Theorema Egregium)
  remarkable theorem (Gauss)

$$K = \lim_{r \to 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

Gaussian curvature of a surface is invariant under local isometry

Classification

Point $\mathbf{x}$ on the surface is called

- elliptic, if $K > 0$

- hyperbolic, if $K < 0$

  Gaussian curvature K

- parabolic, if $K = 0$

- umbilic, if $\kappa_1 = \kappa_2$ **or isotropic**

Classification

Point x on the surface is called

Isotropic

Equal in all directions

$k_1 = k_2 > 0$

spherical

$k_1 = k_2 = 0$
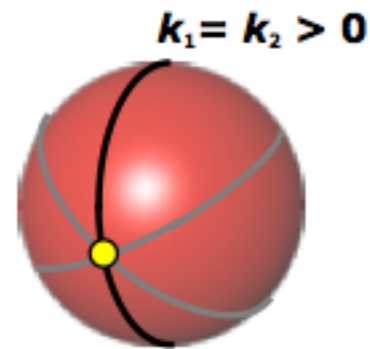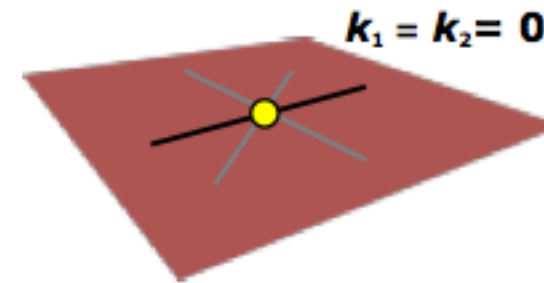
planar

Anisotropic

Distinct principal directions

$k_2 > 0$
$k_1 > 0$
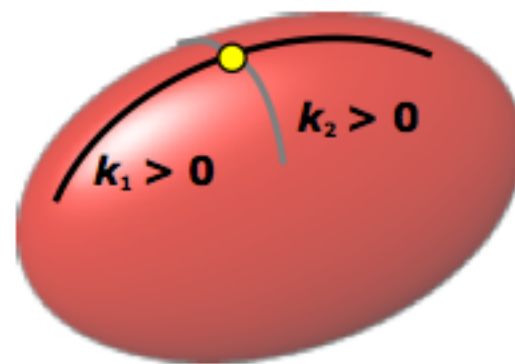
elliptic
$K > 0$

$k_1 = 0$
$k_2 > 0$
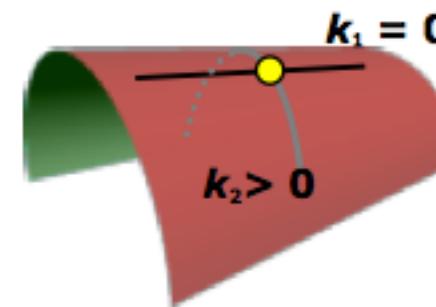
parabolic
$K = 0$
developable

$k_1 < 0$
$k_2 > 0$
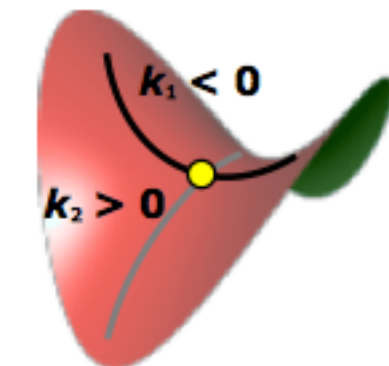
hyperbolic
$K < 0$

For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

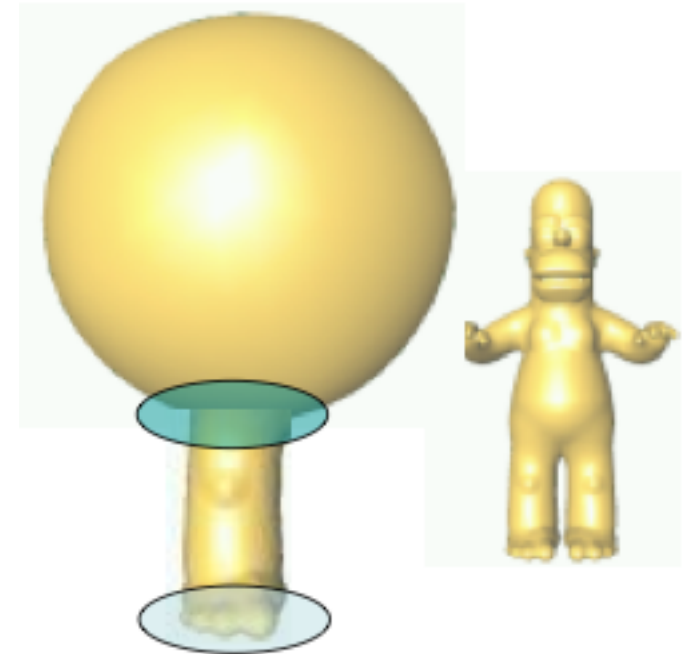$$\int K = 2\pi\chi$$

$$\int K(\ \ ) = \int K(\ \ ) = \int K(\ \ ) = 4\pi$$

Gauss-Bonnet Theorem

Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1\kappa_2 = 1/r^2$$

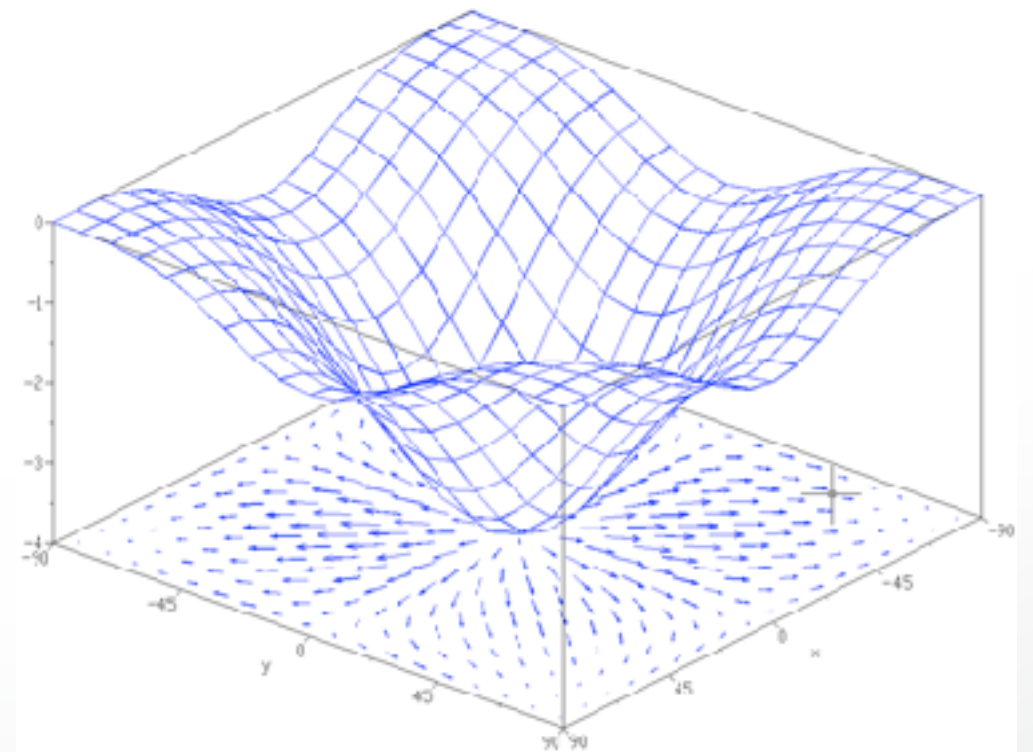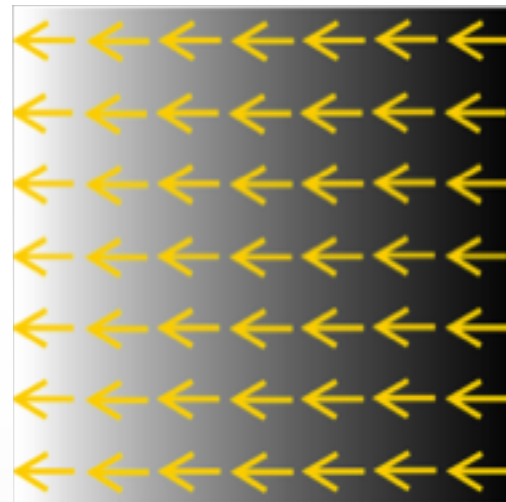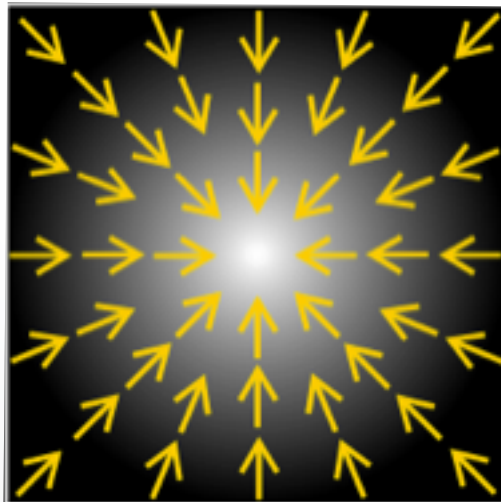$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

when sphere is deformed, new
positive and negative curvature cancel out

Differential Operators

Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend
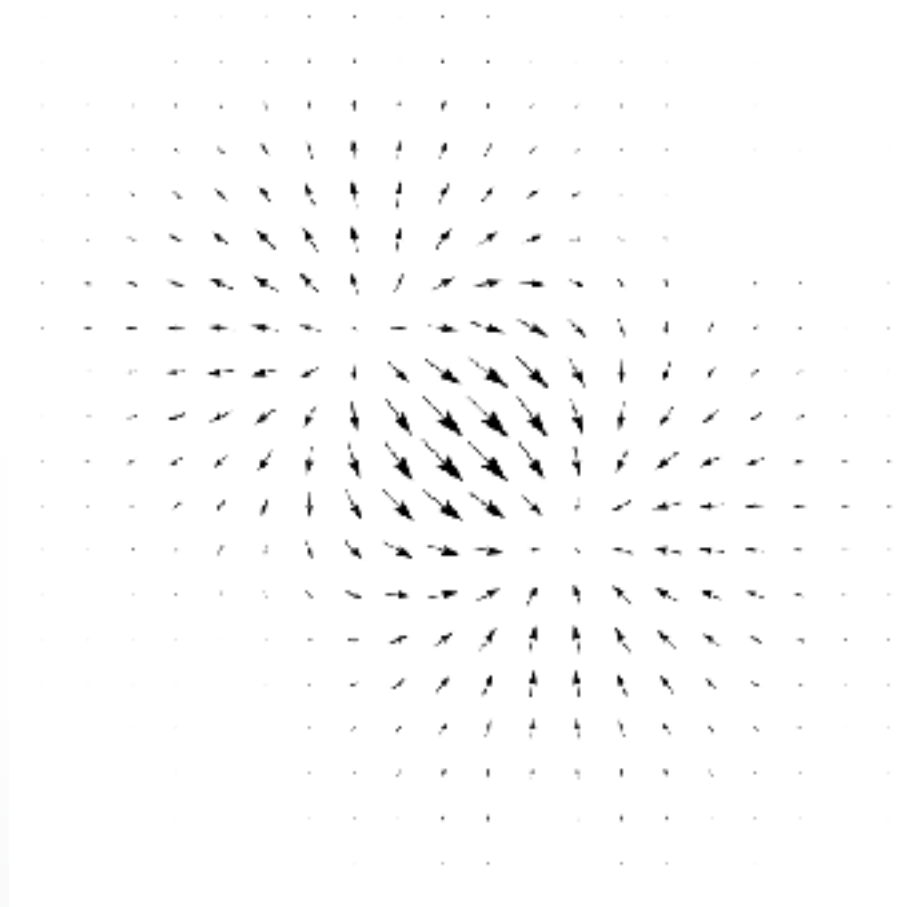
Differential Operators

Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

- volume density of outward flux of vector field

- magnitude of source or sink at given point

- Example: incompressible fluid
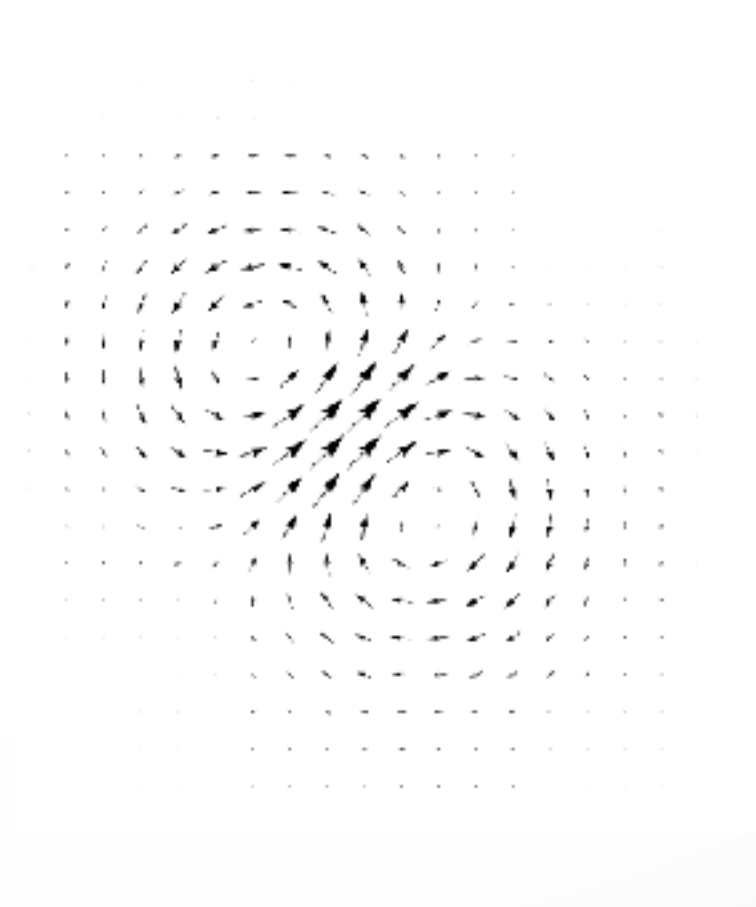 - velocity field is divergence-free

Differential Operators

Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

high divergence

low divergence

Laplace Operator

Laplace
operator

gradient
operator

2nd partial
derivatives

$$\Delta f = \mathrm{div}\nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in
Euclidean space

divergence
operator

Cartesian
coordinates

Laplace-Beltrami Operator

Extension of Laplace of functions on manifolds

Laplace-
Beltrami

gradient
operator

…of the surface

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} f$$

function on
manifold S

divergence
operator

Laplace on the surface

Laplace-Beltrami Operator

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}}\mathbf{x} = \mathrm{div}_{\mathcal{S}} \ \nabla_{\mathcal{S}}\mathbf{x} = -2H\mathbf{n}$$

function on manifold S

divergence operator

surface normal

Literature

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- G. Farin: **Curves and Surfaces for CAGD**, Morgan Kaufmann, 2001

- W. Boehm, H. Prautzsch: **Geometric Concepts for Geometric Design**, AK Peters 1994

- H. Prautzsch, W. Boehm, M. Paluszny: **Bézier and B-Spline Techniques**, Springer 2002

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- http://graphics.stanford.edu/courses/cs468-13-spring/schedule.html
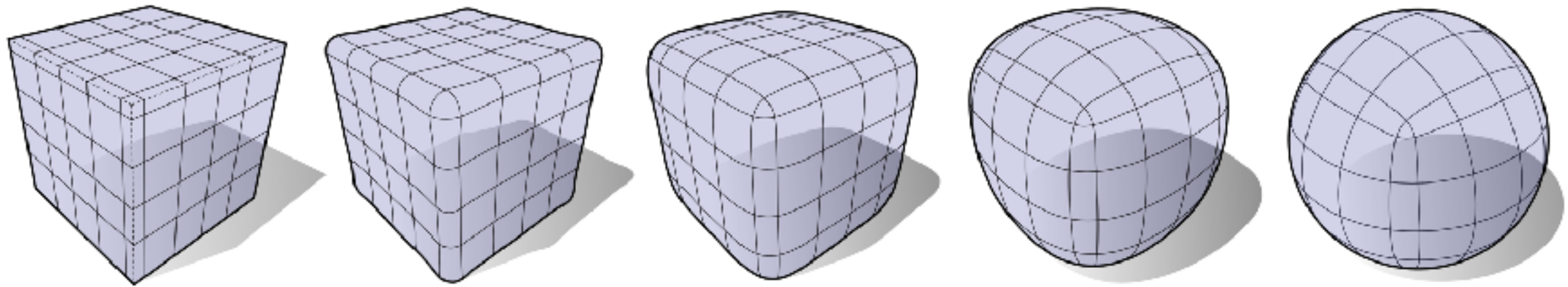
Discrete Differential Geometry

Thanks!