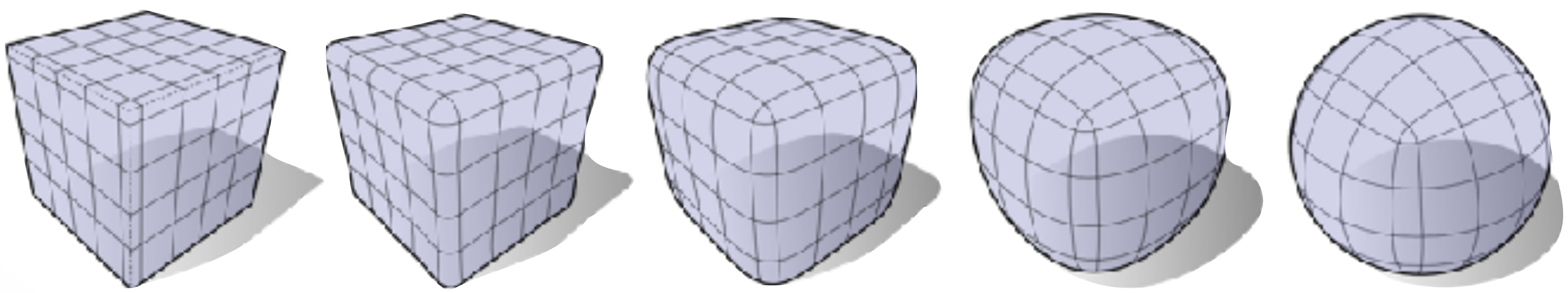
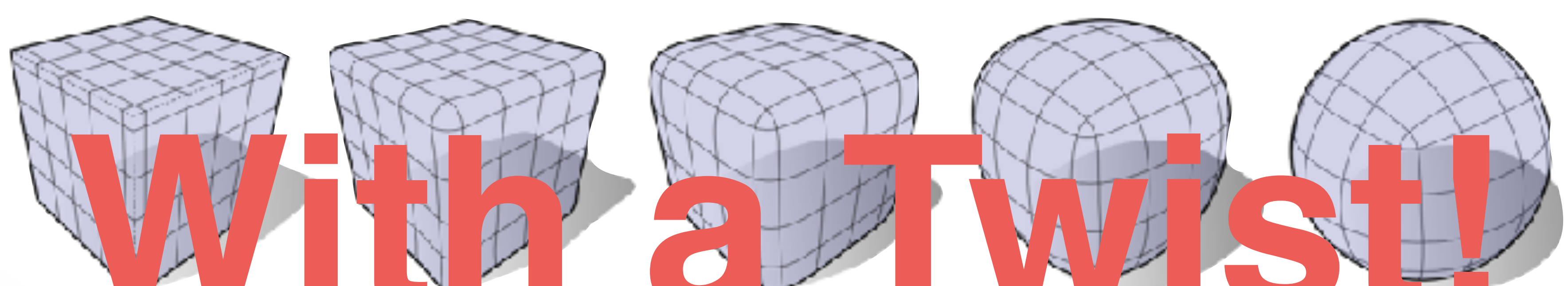
3.2 Classic Differential Geometry 1



Hao Li

http://cs621.hao-li.com

3.2 Classic Differential Geometry 1



With a Twist!

Hao Li

http://cs621.hao-li.com

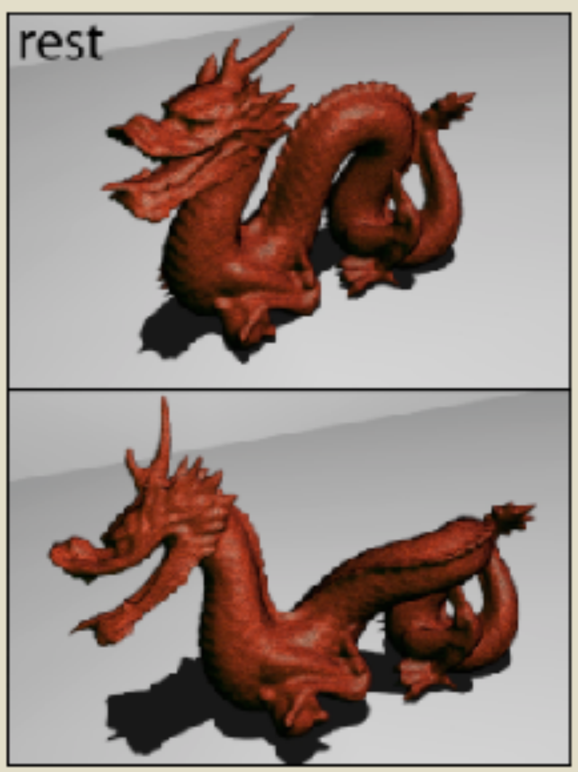
Some Updates: **run.usc.edu/vega**

Another awesome free library with half-edge data-structure

By Prof. Jernej Barbic

Vega FEM

MAIN DOWNLOAD/FAQ SCREENSHOTS ABOUT



JURIJ VEGA (1754-1802)

VEGA FEM LIBRARY

USC Viterbi School of Engineering

NEW: Vega FEM 2.0 released on Oct 8, 2013. New features described below.

Vega is a computationally efficient and stable C/C++ physics library for three-dimensional deformable object simulation. It is designed to model large deformations, including geometric and material nonlinearities, and can also efficiently simulate linear systems. Vega is open-source and free. It is released under the 3-clause BSD license, which means that it can be used freely both in academic research and in commercial applications.

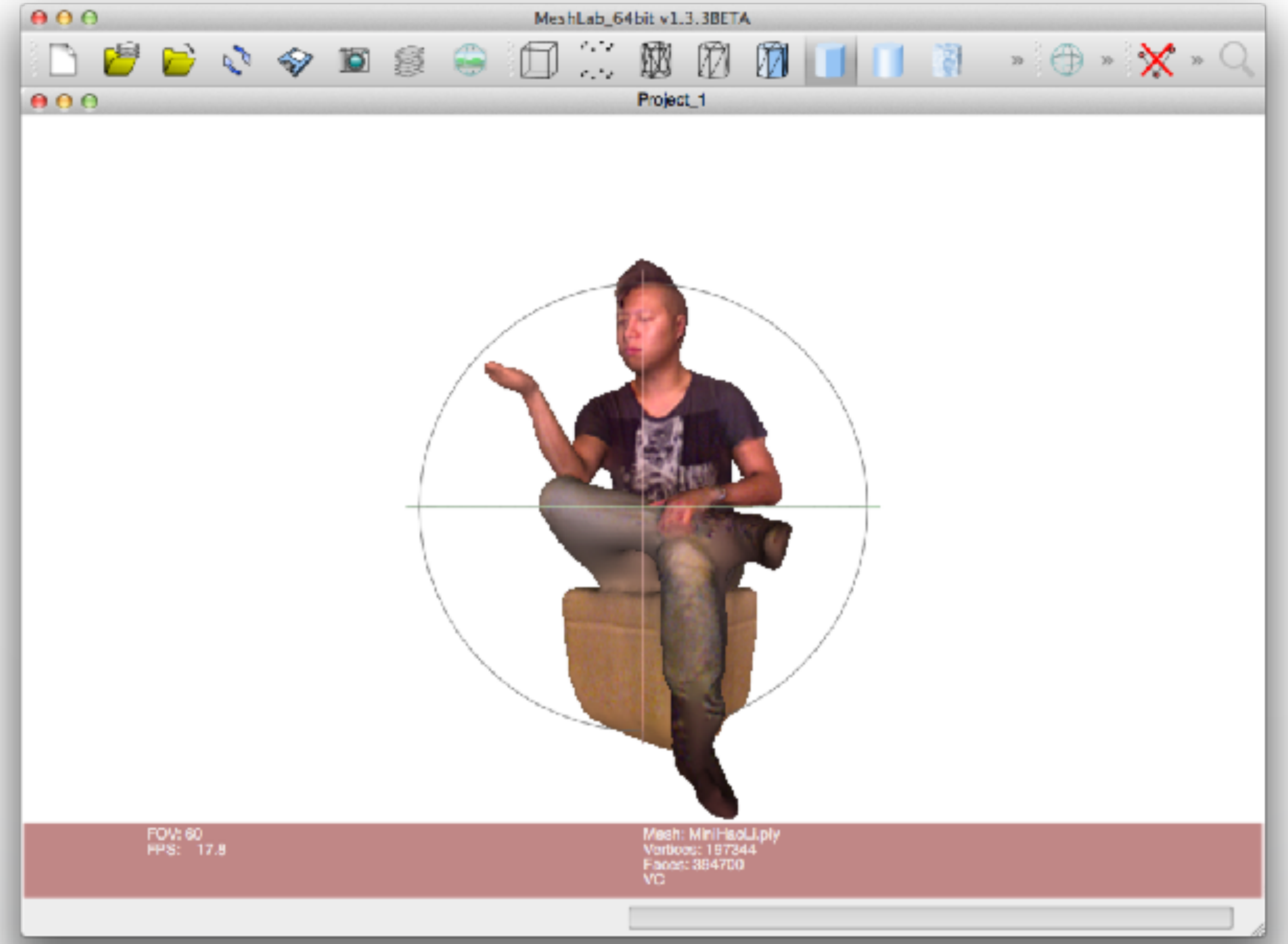
Vega implements several widely used methods for simulation of large deformations of 3D solid deformable objects:

- co-rotational linear FEM elasticity [MG04]; it can also compute the exact tangent stiffness matrix [Bar12] (similar to [CPSS10]),
- linear FEM elasticity [Sha90],
- invertible isotropic nonlinear FEM models [ITF04, TSIF05],

FYI

MeshLab

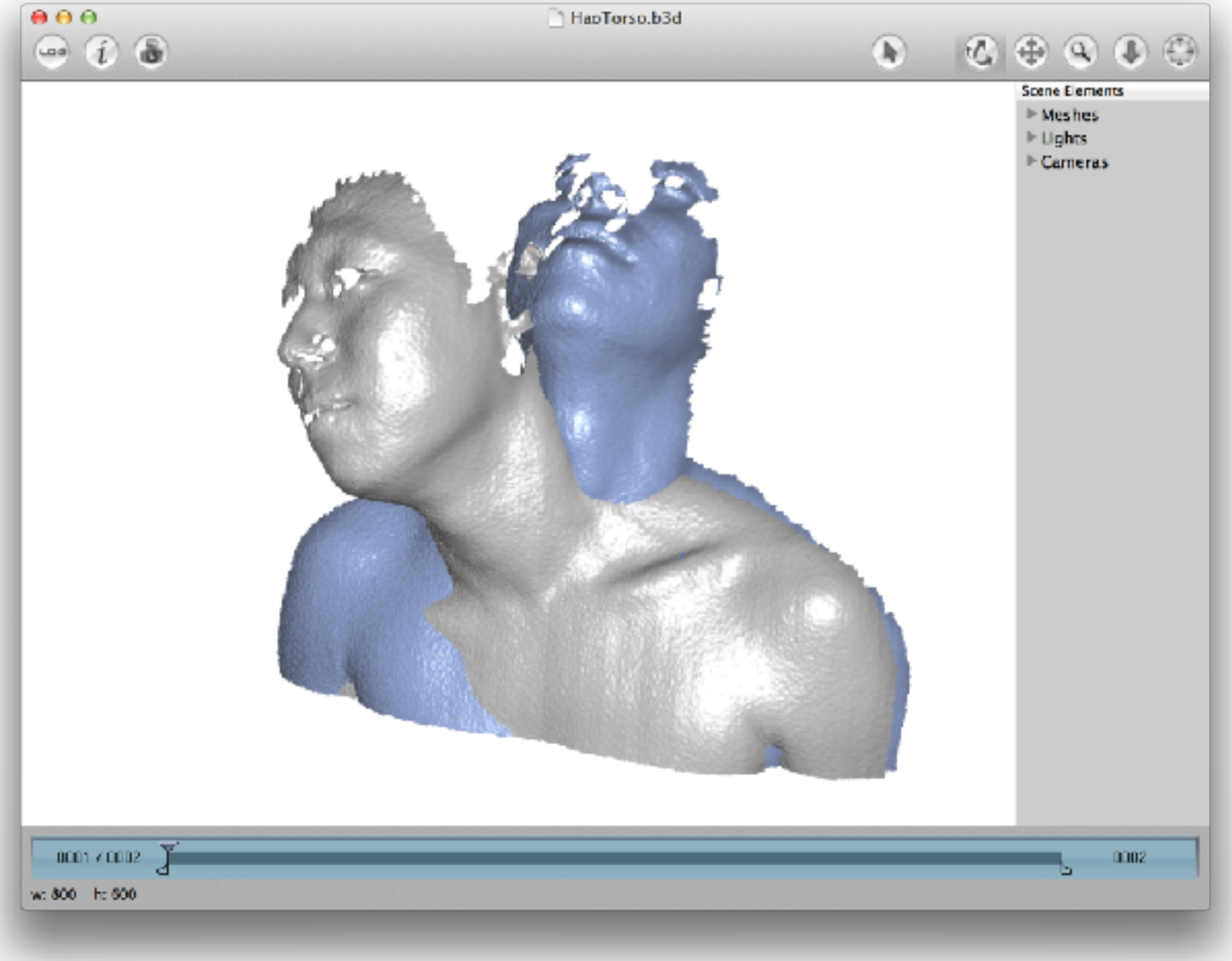
Popular Mesh Processing Software (meshlab.sourceforge.net)



FYI

BeNTO3D

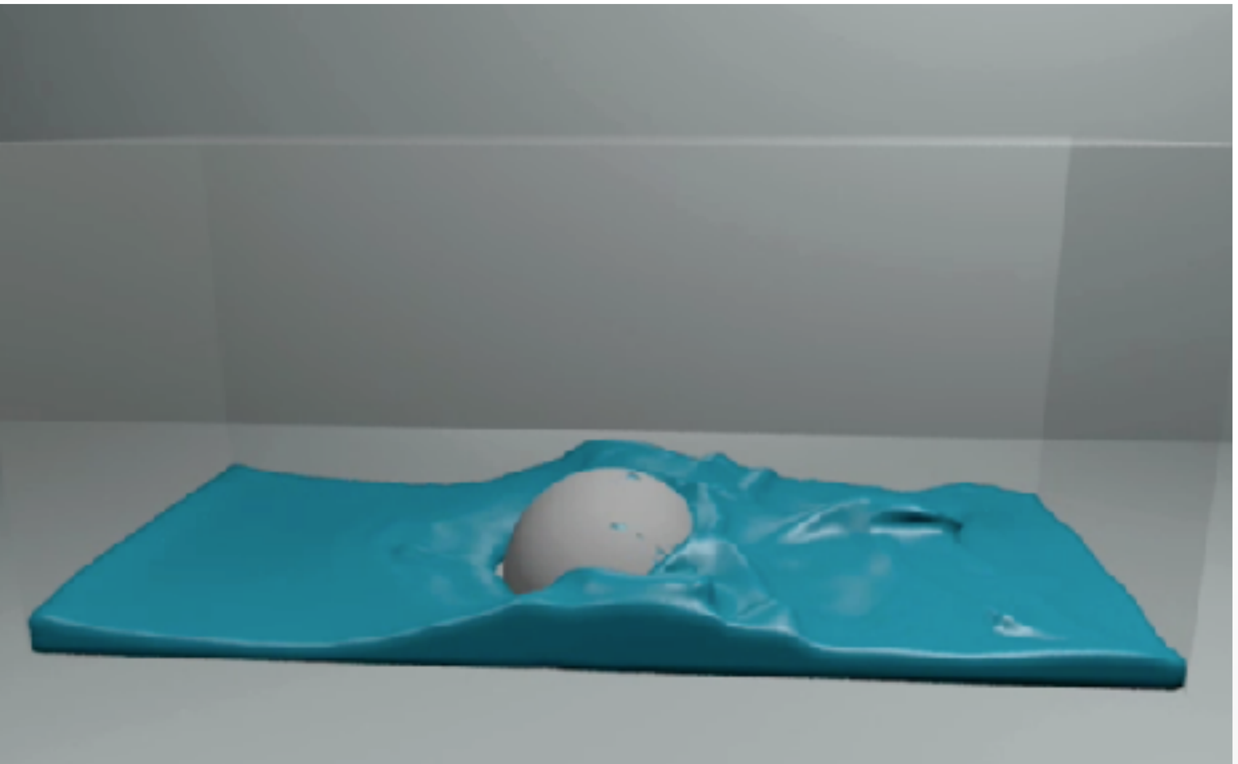
Mesh Processing Framework for Mac (www.bento3d.com)

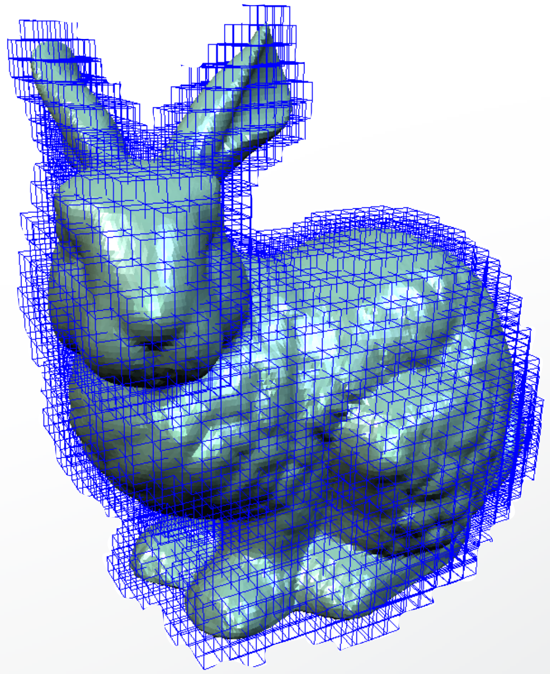


Last Time

Discrete Representations

- Explicit (parametric, polygonal meshes) **Geometry**
- Implicit Surfaces (SDF, grid representation) **Topology**
- Conversions
  - E→I: Closest Point, SDF, Fast Marching
  - I→E: Marching Cubes Algorithm

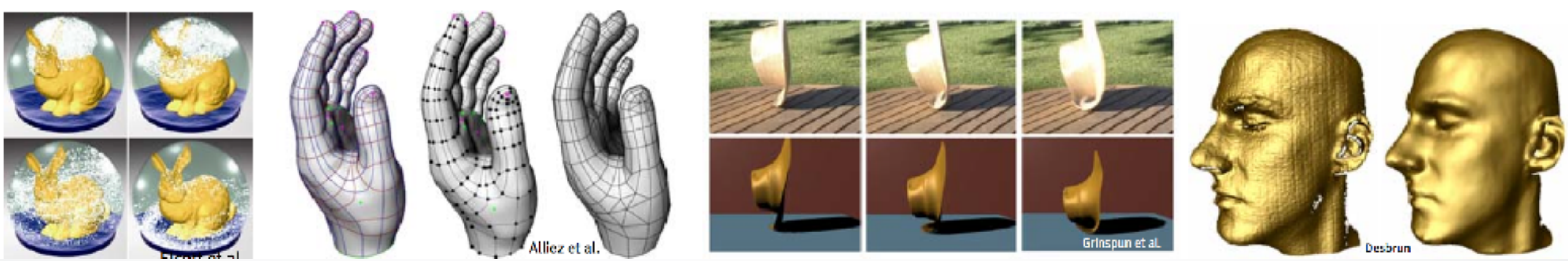




Differential Geometry

Why do we care?

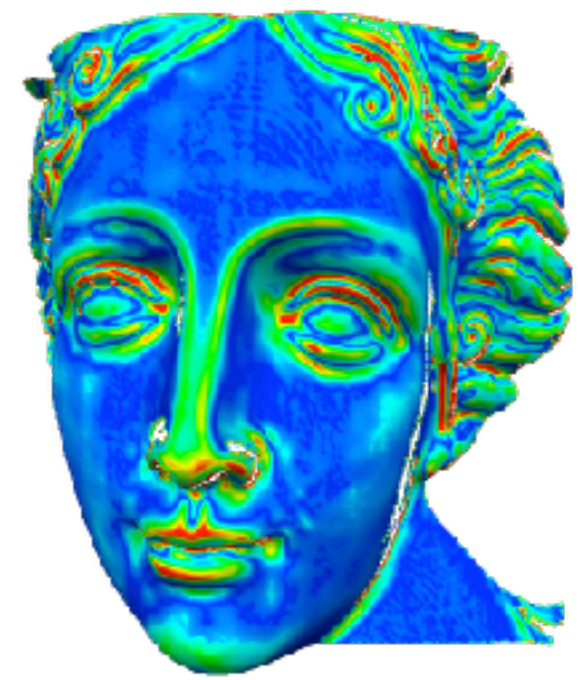
- Geometry of surfaces
- Mothertongue of physical theories
- Computation: processing / simulation

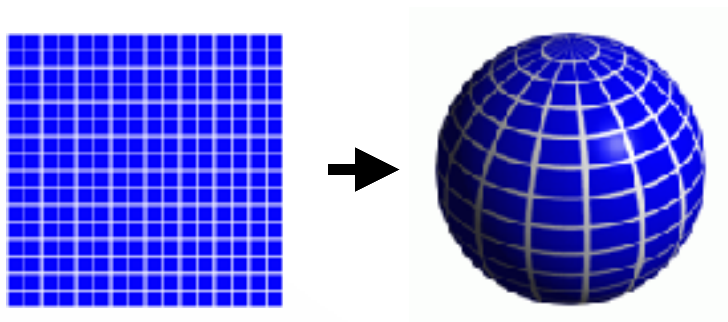


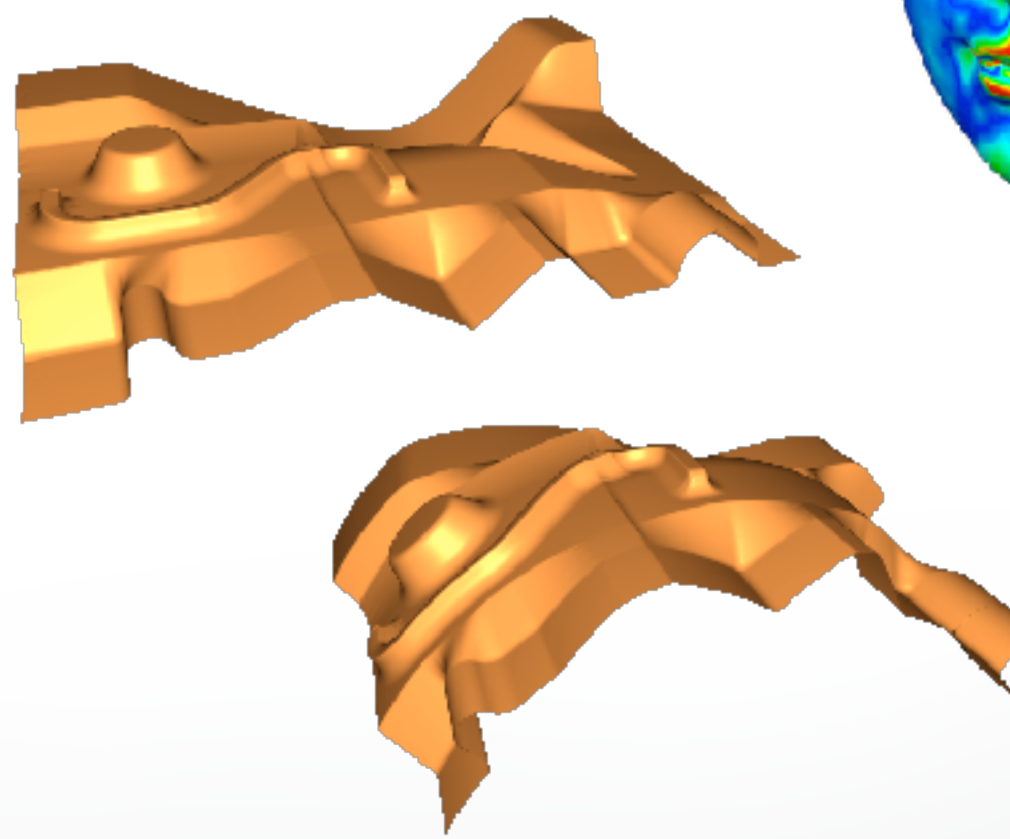
Motivation

We need differential geometry to compute

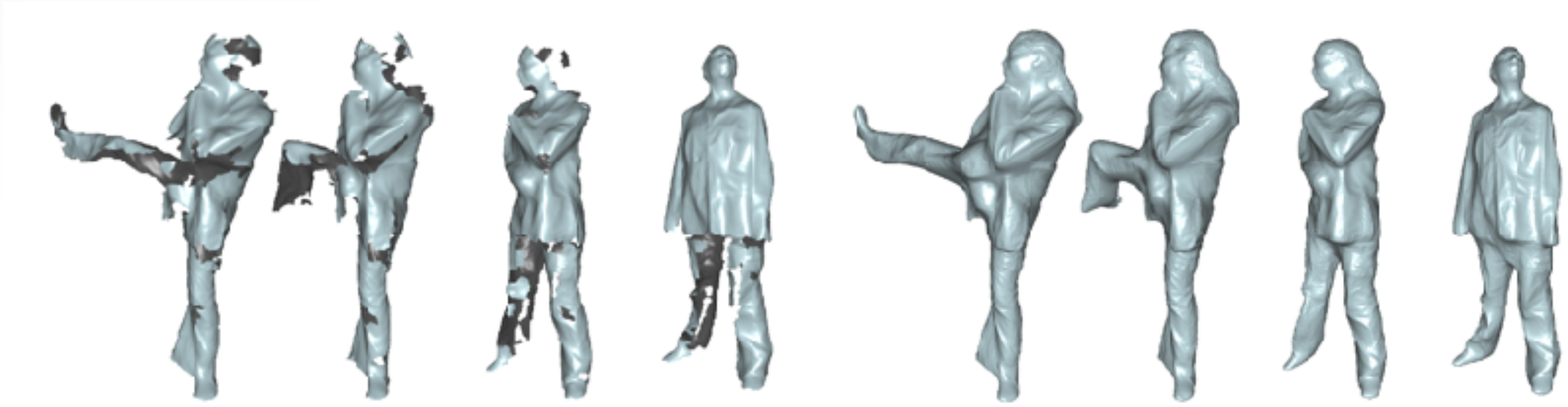
- surface curvature
- paramaterization distortion
- deformation energies

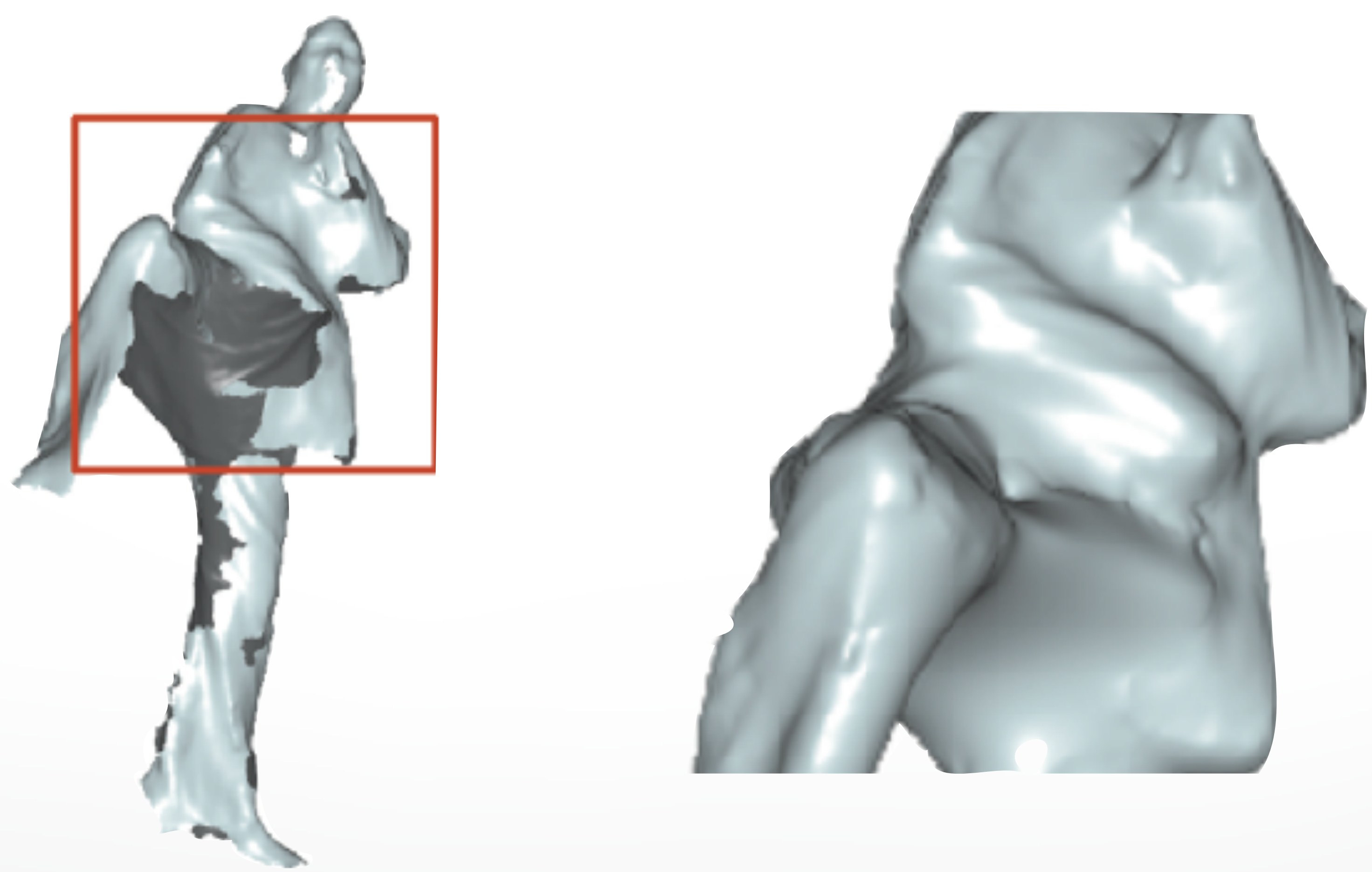




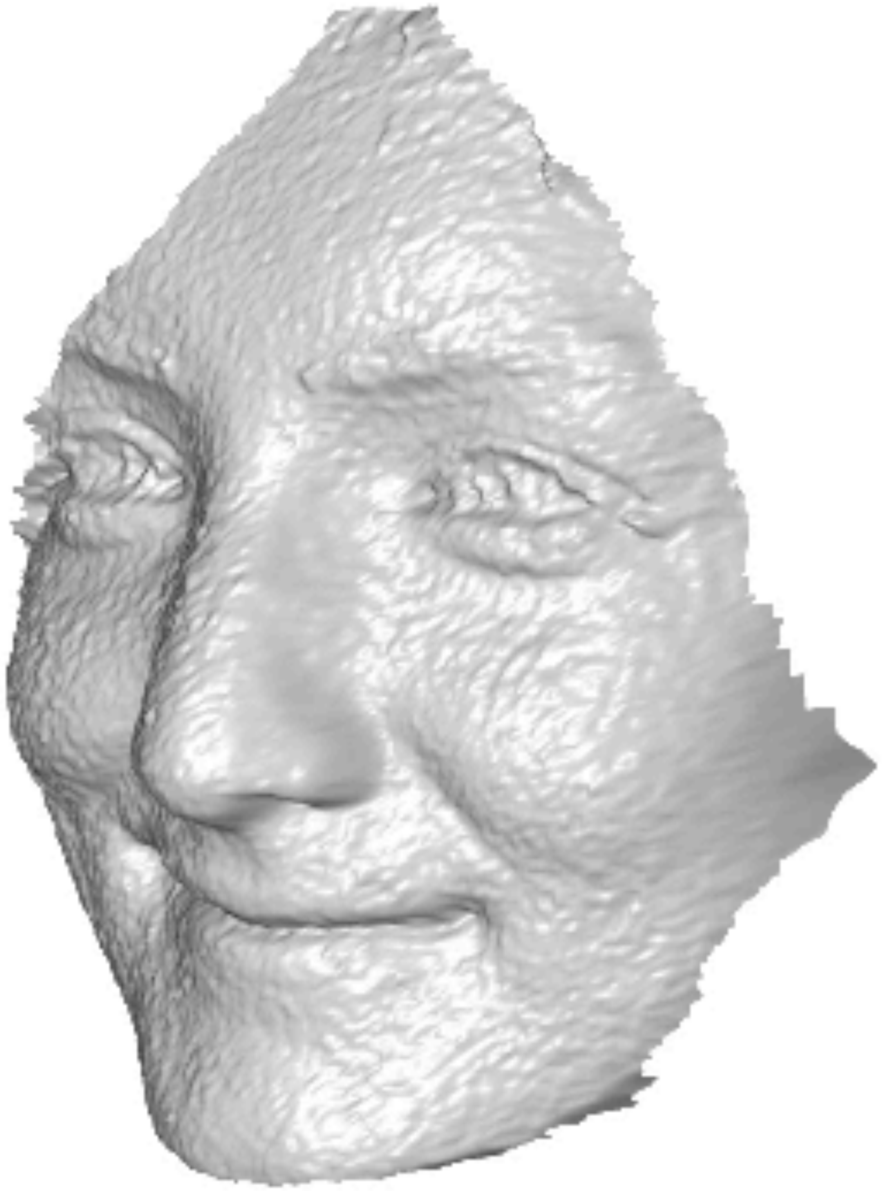


Applications: 3D Reconstruction





Applications: Head Modeling

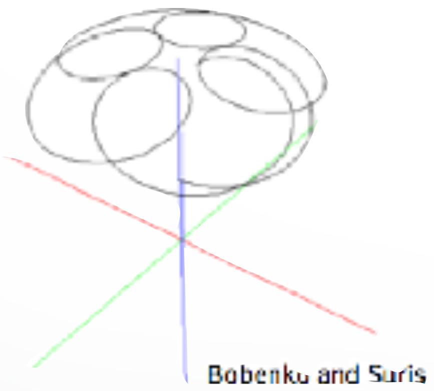


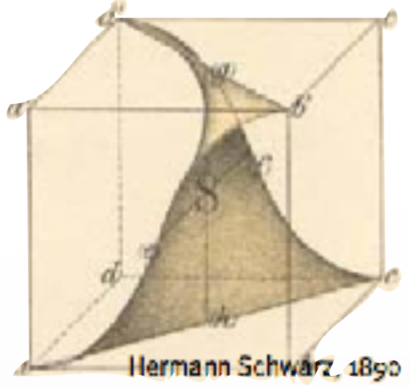
Applications: Facial Animation

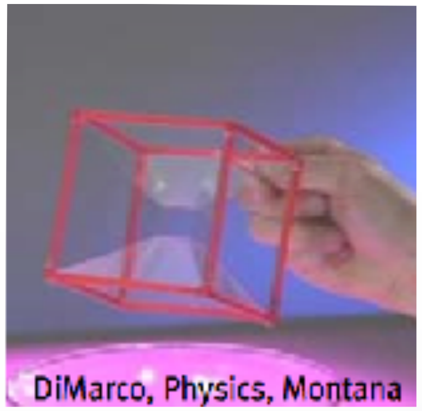
Motivation

Geometry is the key

- studied for centuries (Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether…)
- mostly differential geometry
    - differential and integral calculus
- invariants and symmetries







Getting Started

How to apply DiffGeo ideas?

- surfaces as a collection of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - **discrete** vs. **discretized**

Let's look at that first

Getting Started

What characterizes structure(s)?

- What is shape?
  - Euclidean Invariance

- What is physics?
  - Conservation/Balance Laws

- What can we measure?
  - area, curvature, mass, flux, circulation

Getting Started

Invariant descriptors

- quantities invariant under a set of transformations

Intrinsic descriptor

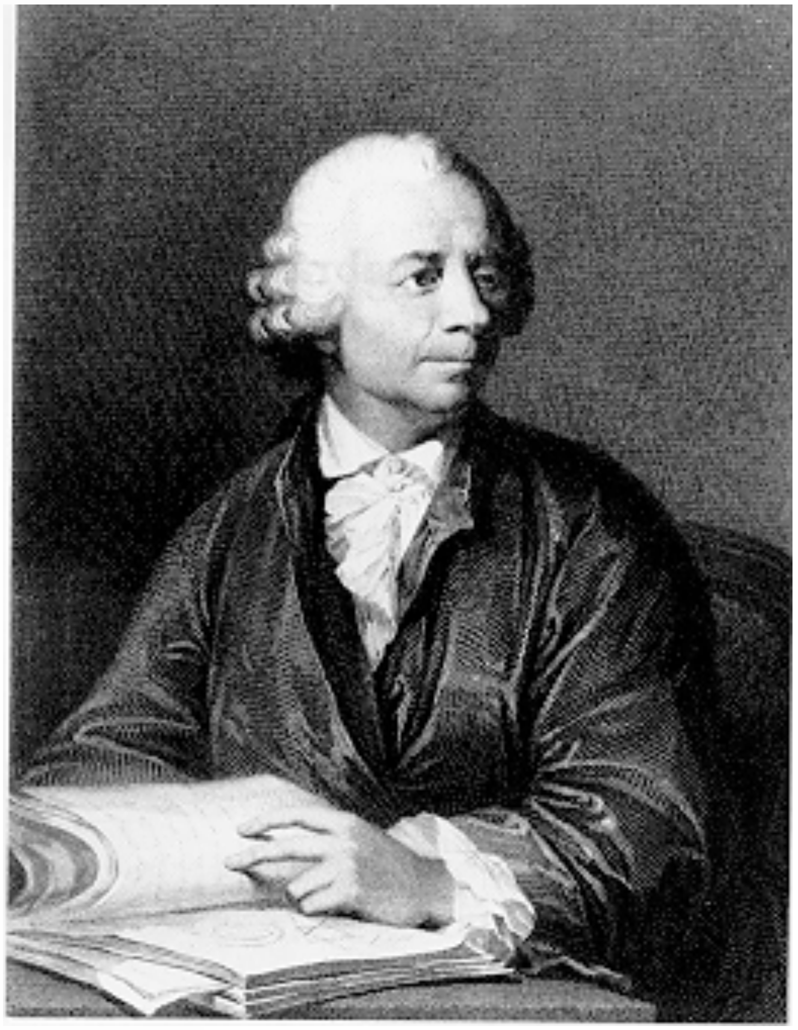
- quantities which do not depend on a coordinate frame

Outline

- **Parametric Curves**
- Parametric Surfaces

Formalism & Intuition

Differential Geometry

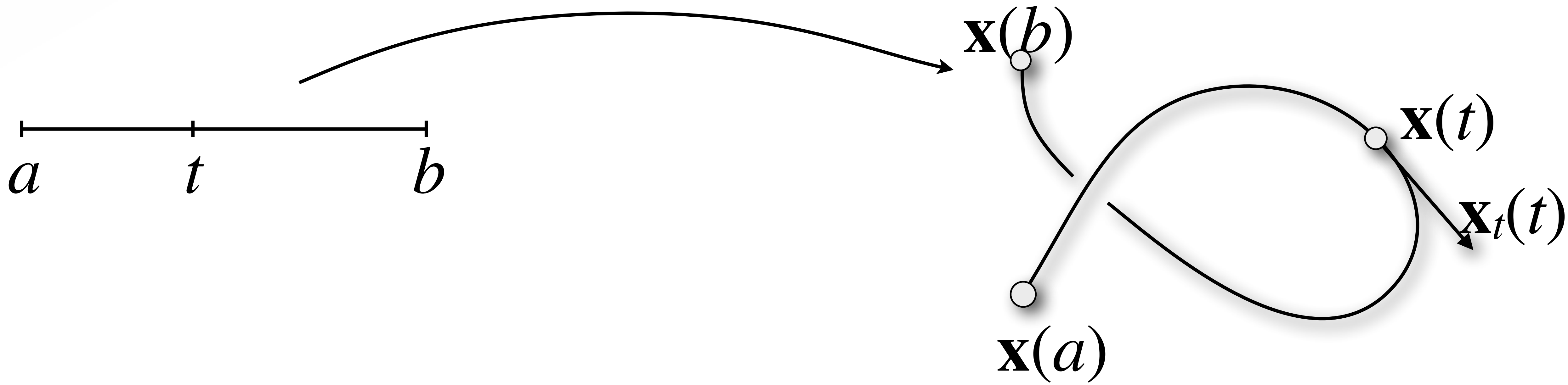


Leonard Euler (1707-1783)

Carl Friedrich Gauss (1777-1855)

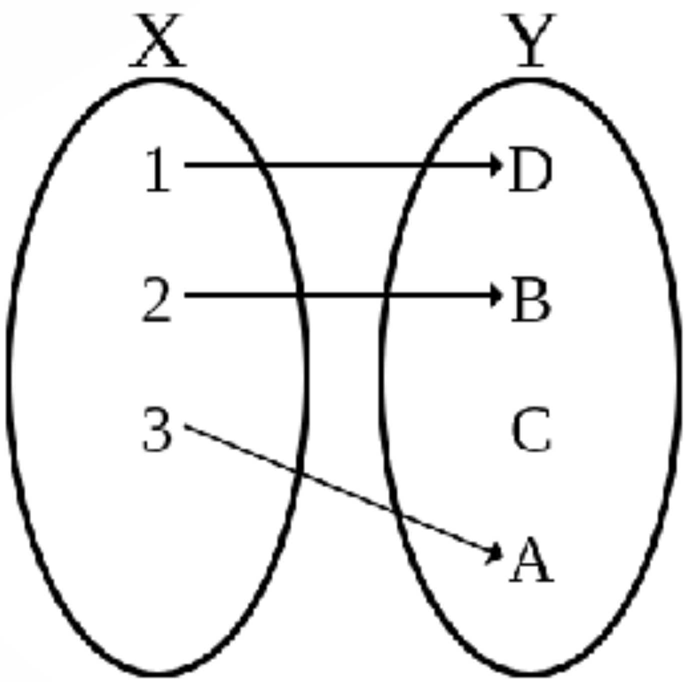
Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$

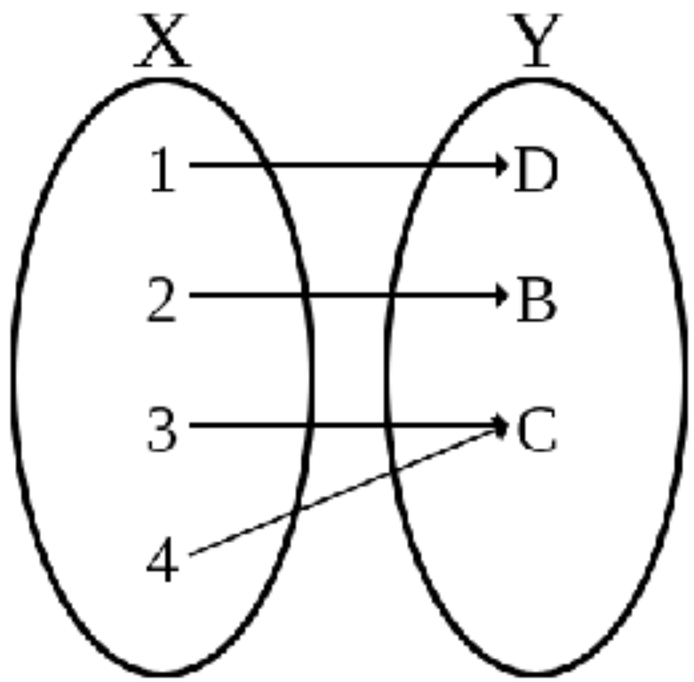


$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)/\mathrm{d}t \\ \mathrm{d}y(t)/\mathrm{d}t \\ \mathrm{d}z(t)/\mathrm{d}t \end{pmatrix}$$

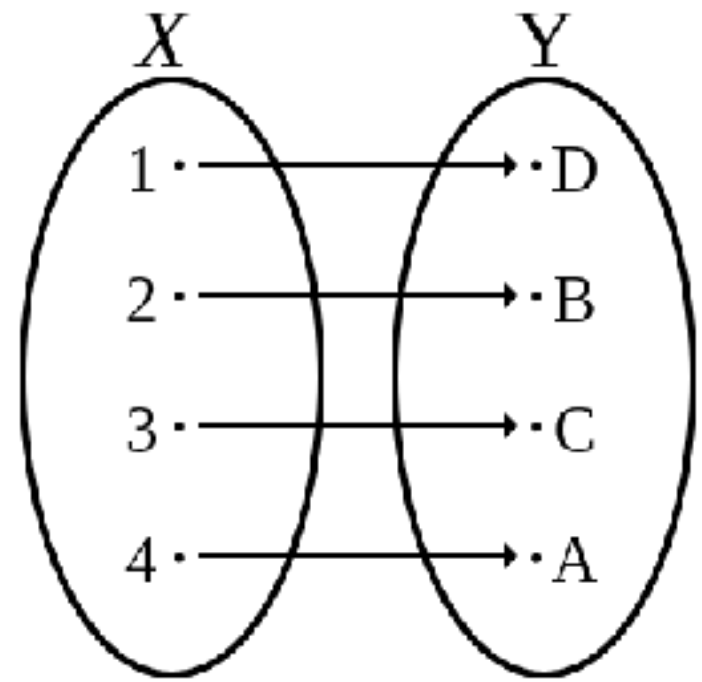
Recall: **Mappings**



Injective



Surjective



Bijective

NO SELF-INTERSECTIONS

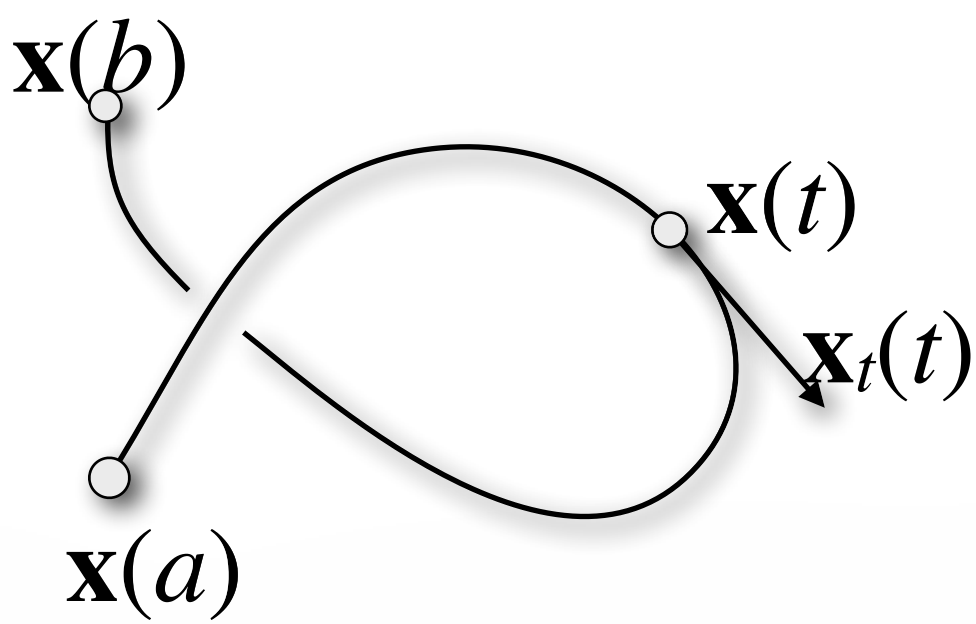
SELF-INTERSECTIONS

AMBIGUOUS PARAMETERIZATION

Parametric Curves

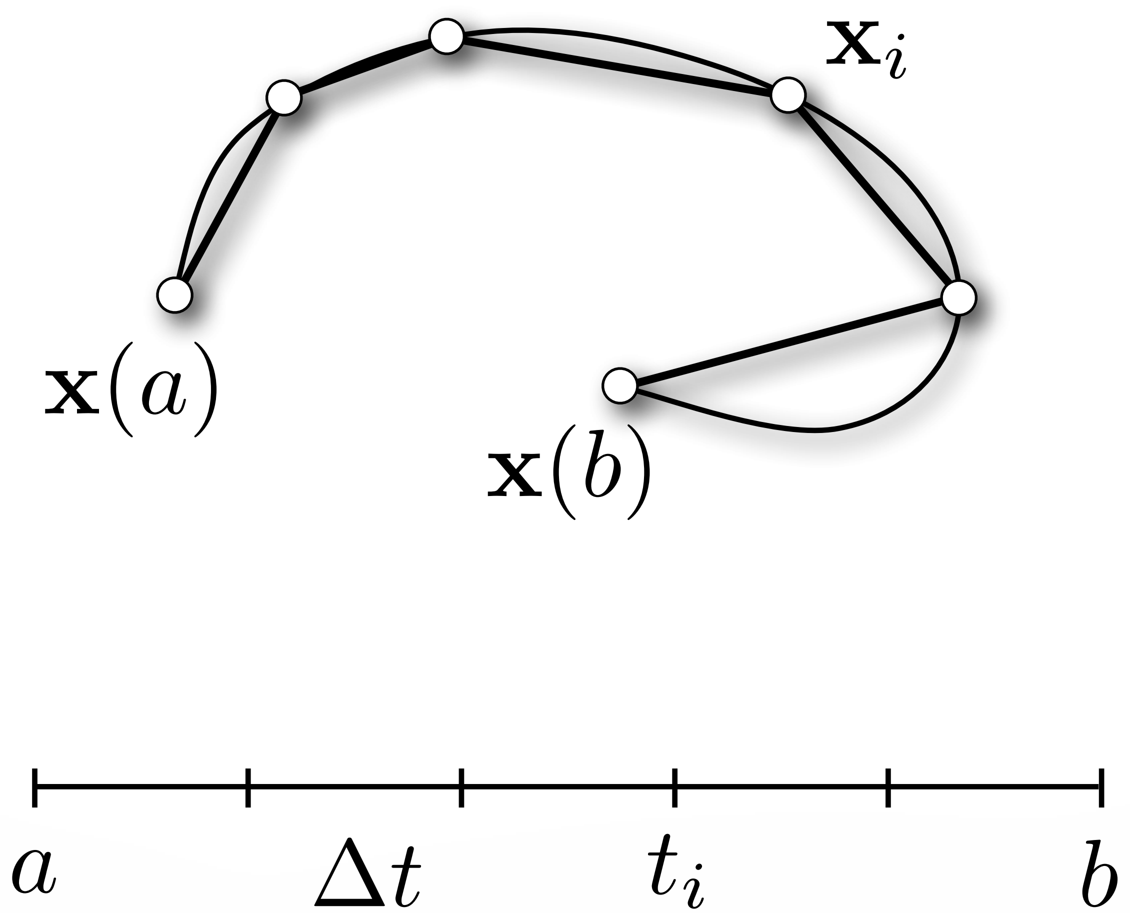
A parametric curve $\mathbf{x}(t)$ is

- simple: $\mathbf{x}(t)$ is injective (no self-intersections)
- differentiable: $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
- regular: $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$



Length of a Curve

Let $t_i = a + i\Delta t$ **and** $\mathbf{x}_i = \mathbf{x}(t_i)$



Length of a Curve

Polyline chord length

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t\,, \qquad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

norm change

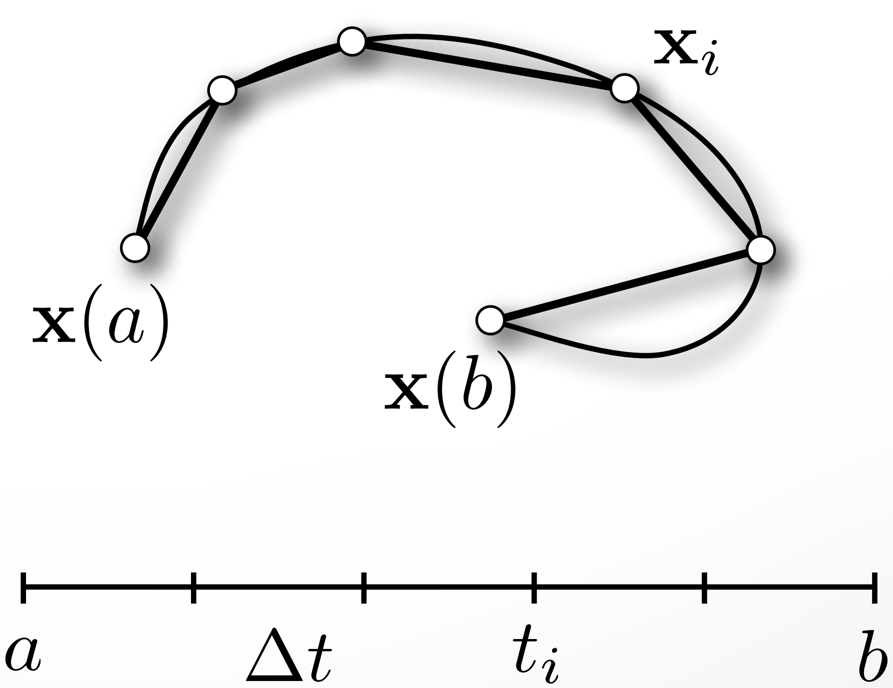
Curve arc length ($\Delta t \to 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\|\, \mathrm{d}t$$

length =

integration of infinitesimal change

× norm of speed



Re-Parameterization

Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

Re-parameterization w.r.t. $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3\,, \quad t \mapsto \mathbf{x}(u(t))$$

Derivative (chain rule)

$$\frac{\mathrm{d}\mathbf{x}(u(t))}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{x}_u(u(t))\; u_t(t)$$

Re-Parameterization

Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \to \mathbb{R}^2 \quad , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \to [0, 1] \quad , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \to \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

$$\Rightarrow \quad \mathbf{g}(\phi(t)) \; = \; \mathbf{f}(t)$$

Arc Length Parameterization

Mapping of parameter domain:

$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\| \,\mathrm{d}t$$

Parameter s for $\mathbf{x}(s)$ equals length from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\mathbf{x}(s) \;=\; \mathbf{x}(s(t)) \qquad\qquad \mathrm{d}s \;=\; \|\mathbf{x}_t\| \,\mathrm{d}t$$

same infinitesimal change

Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1\,, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

defines orthonormal frame

The Frenet Frame

Taylor expansion

$$\mathbf{x}(t+h) \;=\; \mathbf{x}(t) \;+\; \mathbf{x}_t(t)\,h \;+\; \frac{1}{2}\mathbf{x}_{tt}(t)\,h^2 \;+\; \frac{1}{6}\mathbf{x}_{ttt}(t)\,h^3 \;+\; \ldots$$

for convergence analysis and approximations

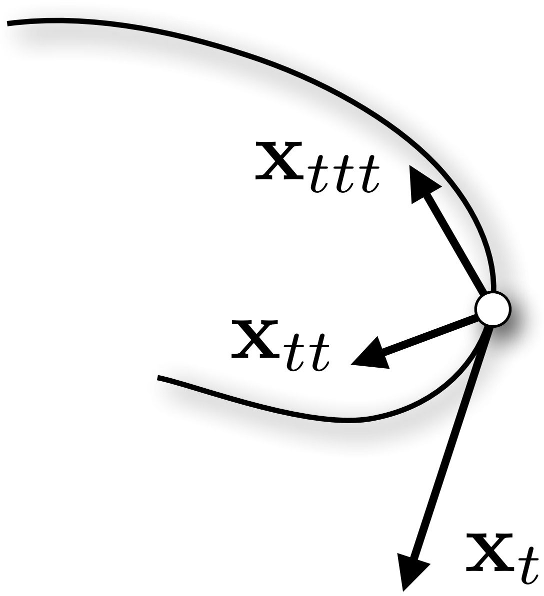
Define local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ (Frenet frame)

$$\mathbf{t} \;=\; \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|} \qquad \mathbf{n} \;=\; \mathbf{b} \times \mathbf{t} \qquad \mathbf{b} \;=\; \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

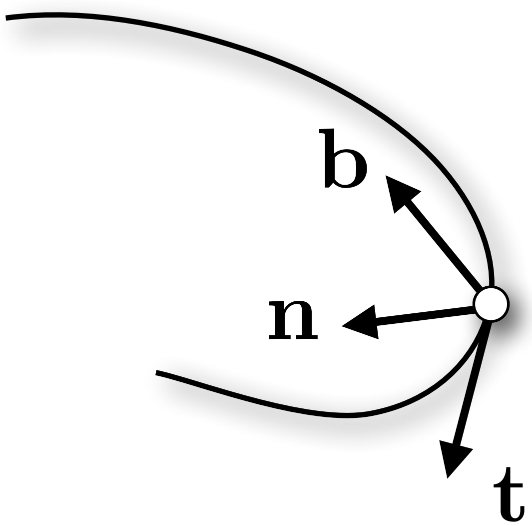
tangent — main normal — binormal

The Frenet Frame

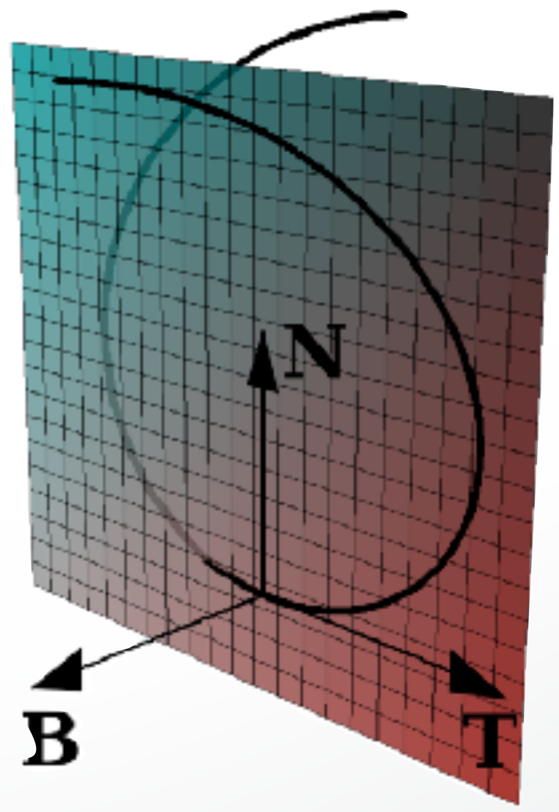
Orthonormalization of local frame



local affine frame



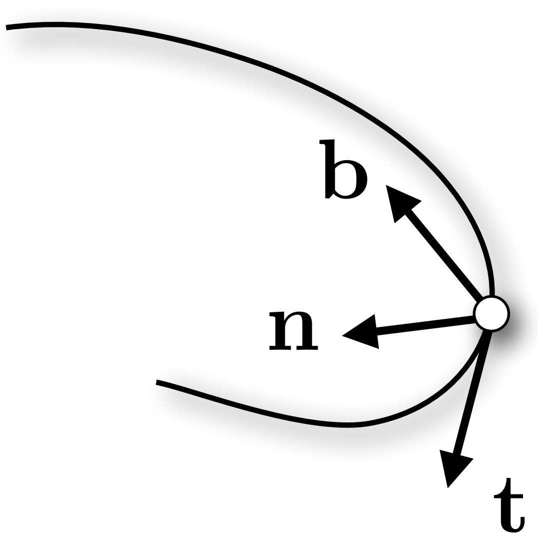
Frenet frame



The Frenet Frame

Frenet-Serret: Derivatives w.r.t. arc length s

$$
\begin{aligned}
\mathbf{t}_s &= \quad\quad\quad +\kappa\mathbf{n} \\
\mathbf{n}_s &= -\kappa\mathbf{t} \quad\quad\quad +\tau\mathbf{b} \\
\mathbf{b}_s &= \quad\quad\quad -\tau\mathbf{n}
\end{aligned}
$$



Curvature (deviation from straight line)

$$\kappa = \|\mathbf{x}_{ss}\|$$

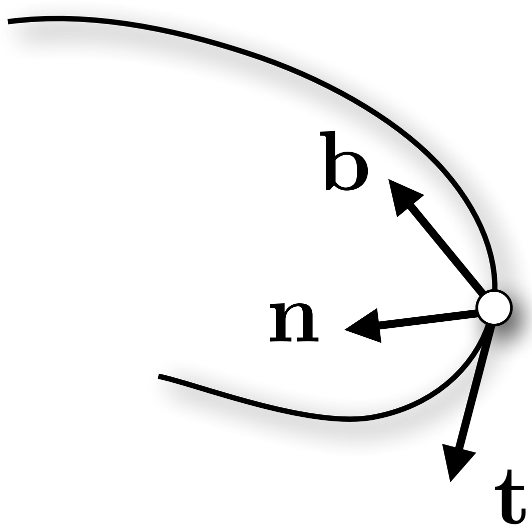
Torsion (deviation from planarity)

$$\tau = \frac{1}{\kappa^2}\det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature and Torsion

Planes defined by $\mathbf{x}$ and two vectors:

- osculating plane: vectors $\mathbf{t}$ and $\mathbf{n}$
- normal plane: vectors $\mathbf{n}$ and $\mathbf{b}$
- rectifying plane: vectors $\mathbf{t}$ and $\mathbf{b}$



Osculating circle

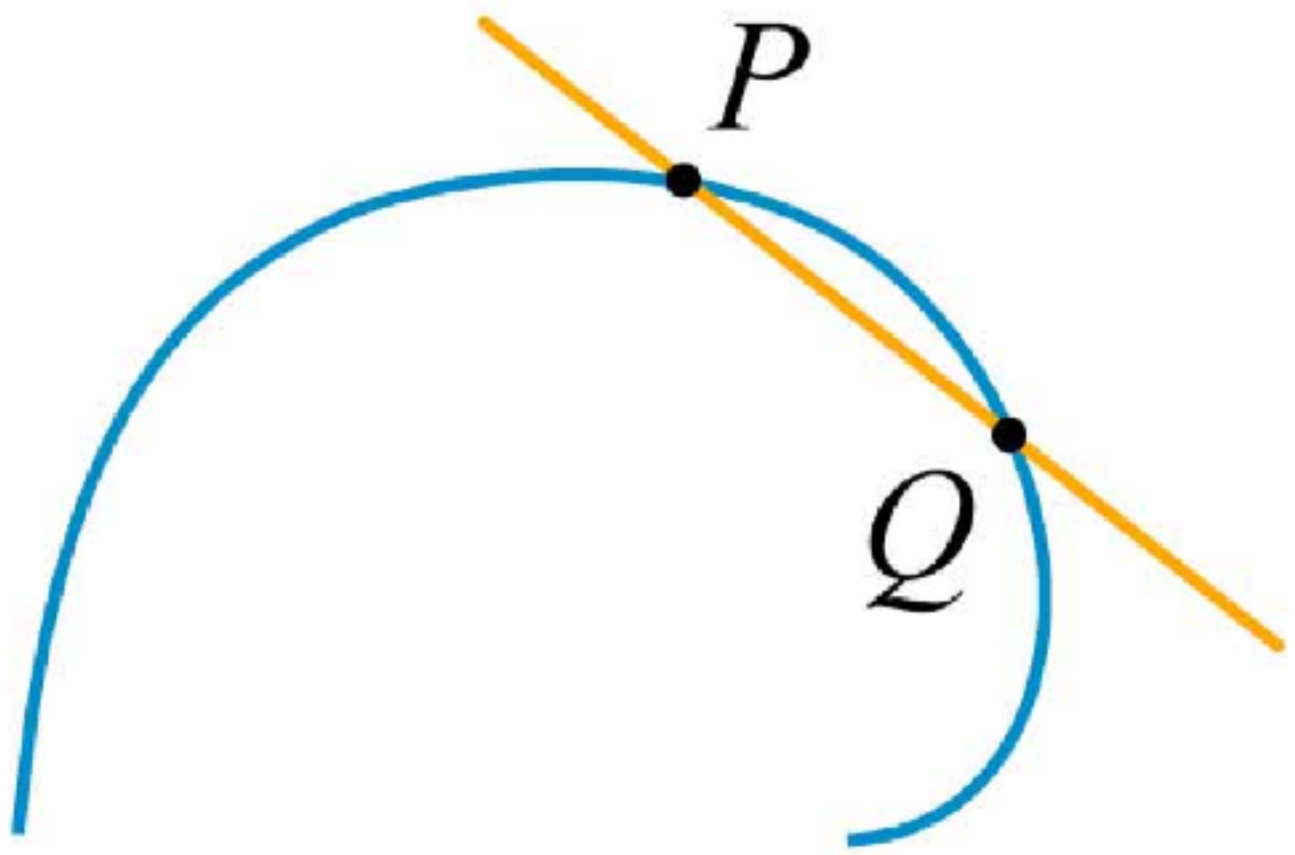
- second order contact with curve
- center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
- radius $1/\kappa$

Curvature and Torsion

- **Curvature**: Deviation from straight line
- **Torsion**: Deviation from planarity
- Independent of parameterization
  - **intrinsic** properties of the curve
- Euclidean invariants
  - **invariant** under rigid motion
- Define curve **uniquely** up to a rigid motion

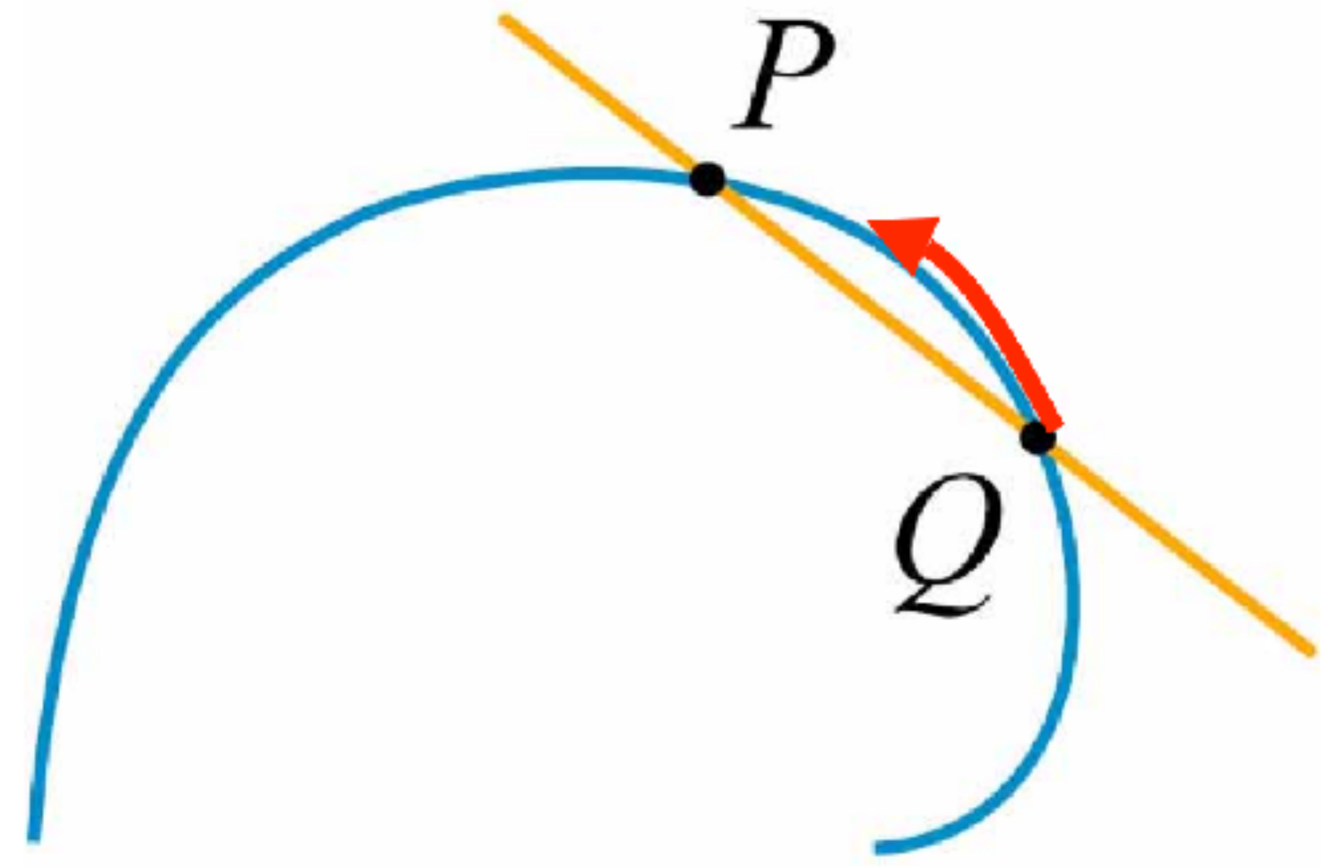
Curvature: Some Intuition

A line through two points on the curve (Secant)



Curvature: Some Intuition

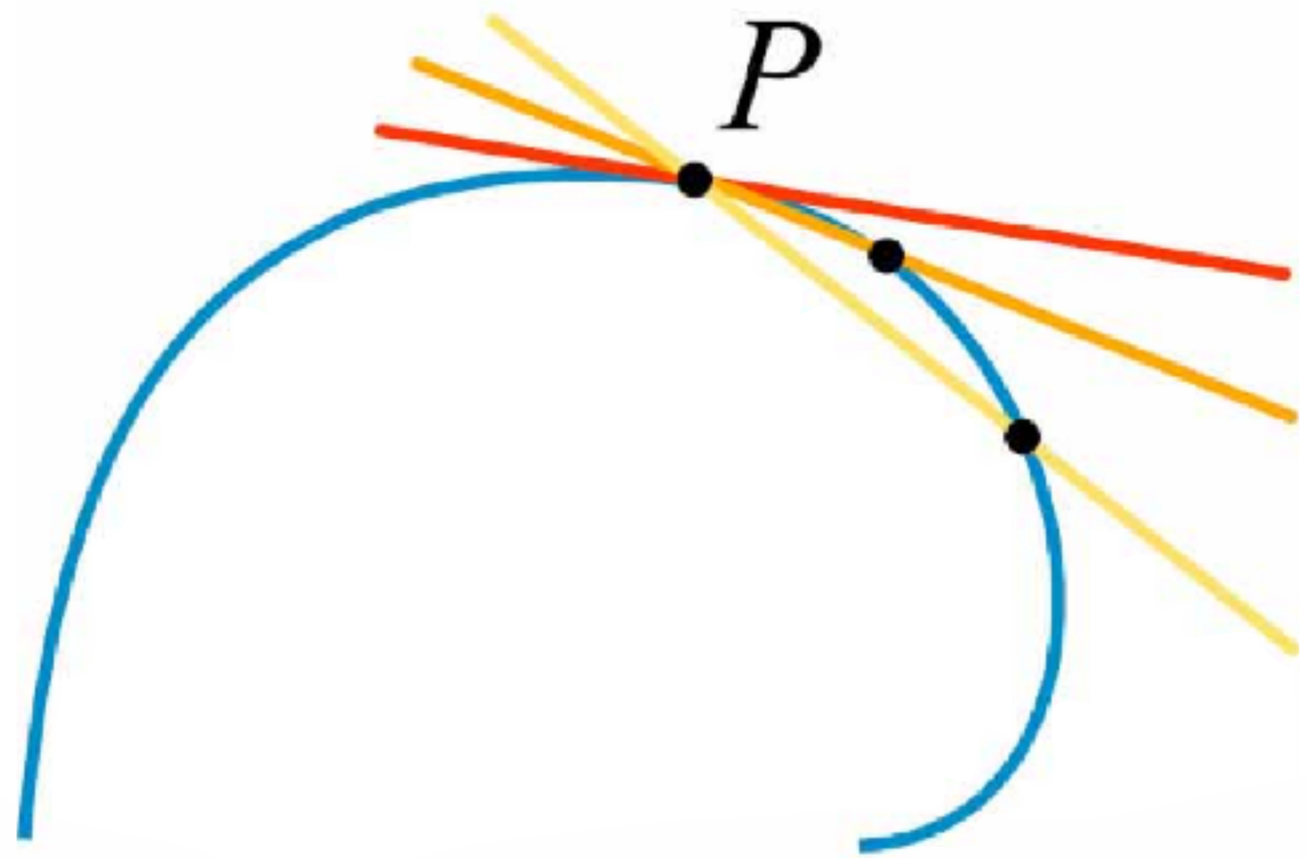
A line through two points on the curve (Secant)



Curvature: Some Intuition

Tangent, the first approximation

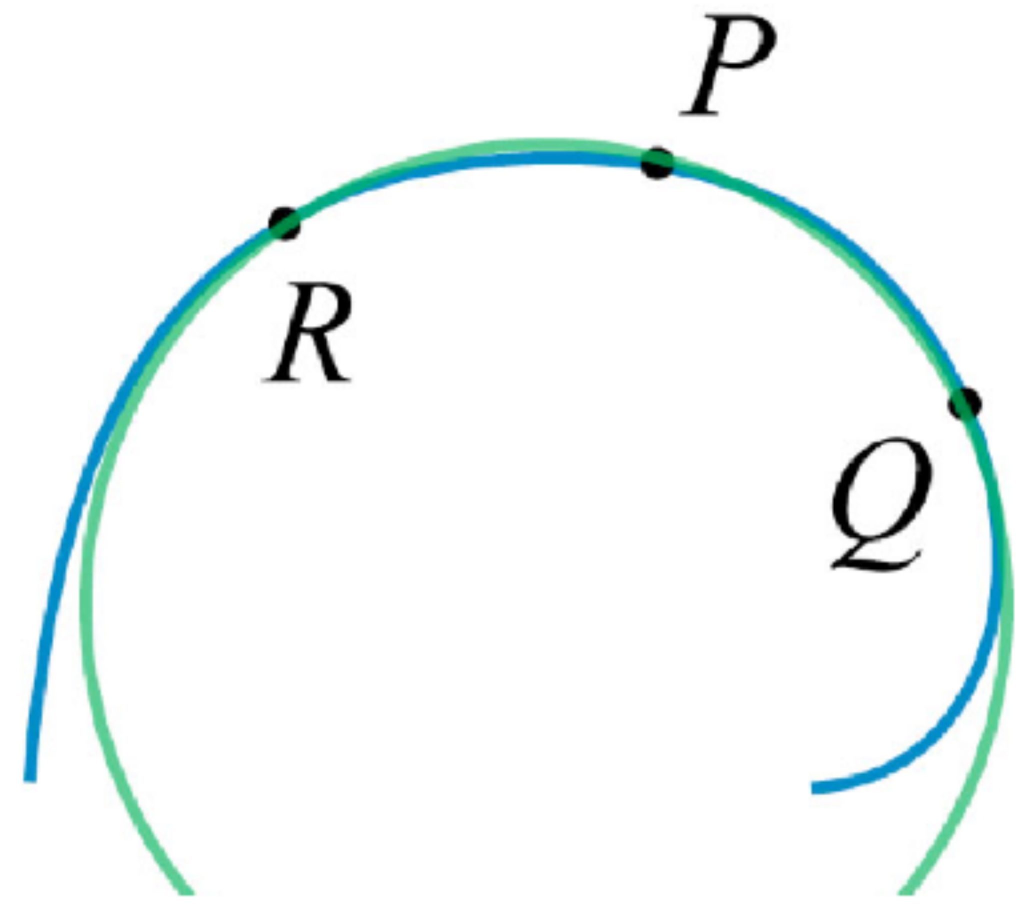
limiting secant as the two points come together



Curvature: Some Intuition

Circle of curvature

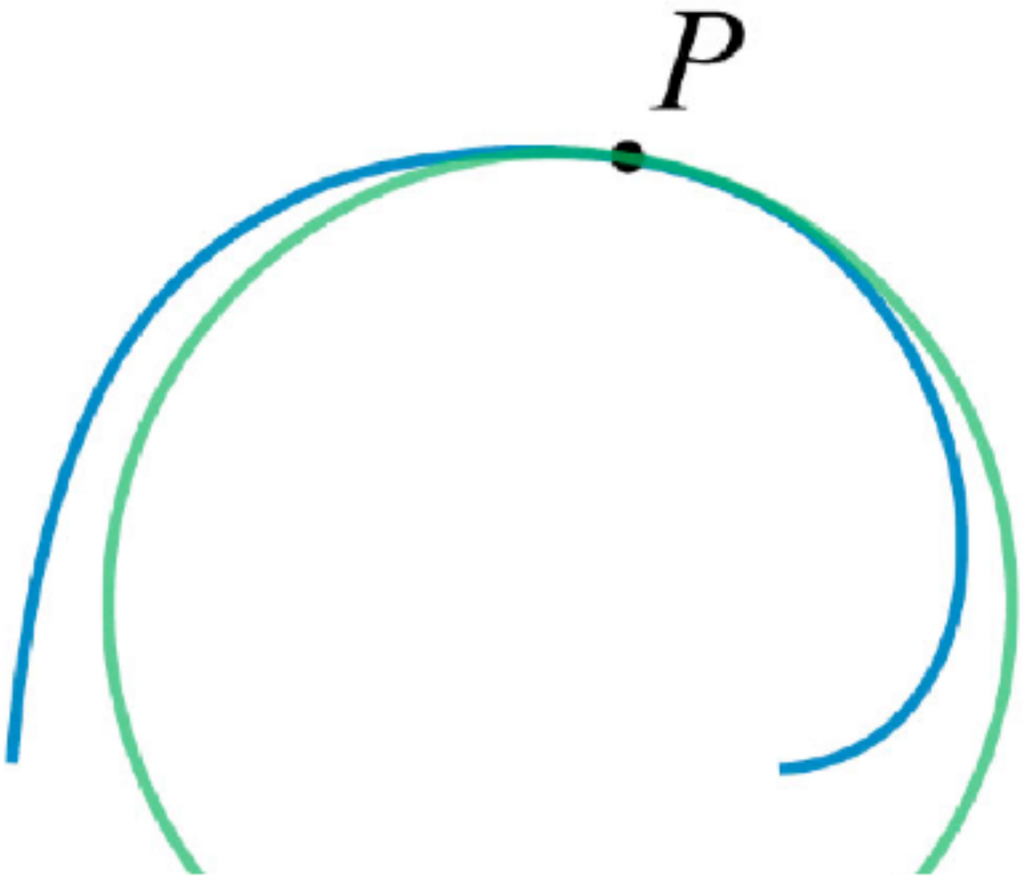
Consider the circle passing through 3 pints of the curve



Curvature: Some Intuition

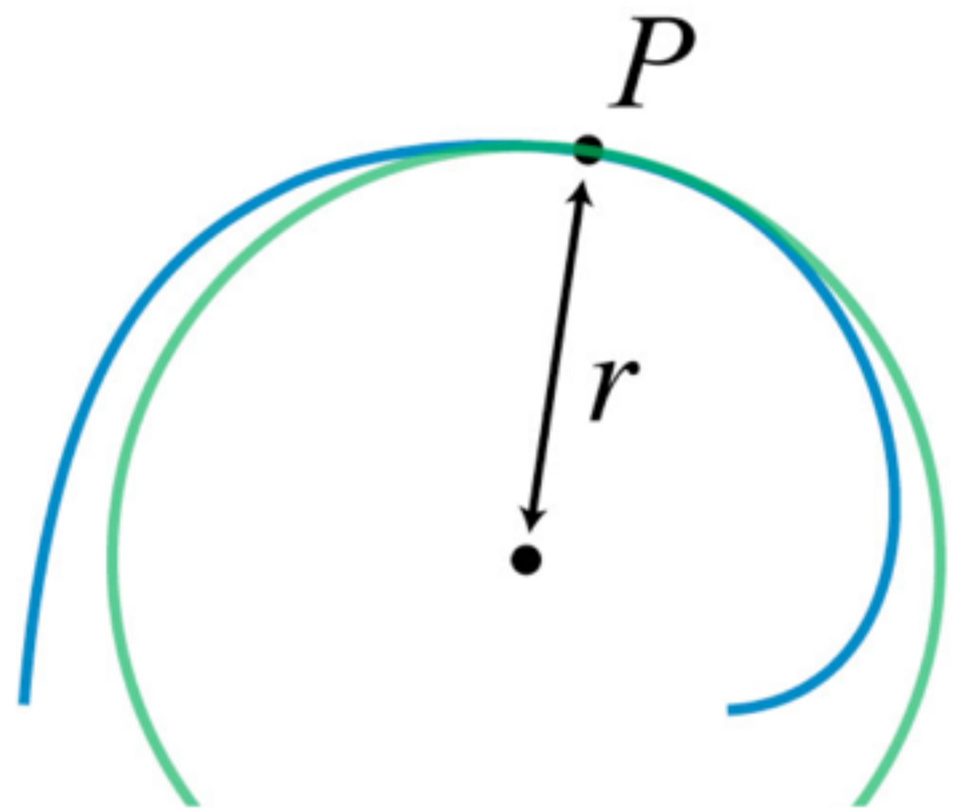
Circle of curvature

The limiting circle as three points come together



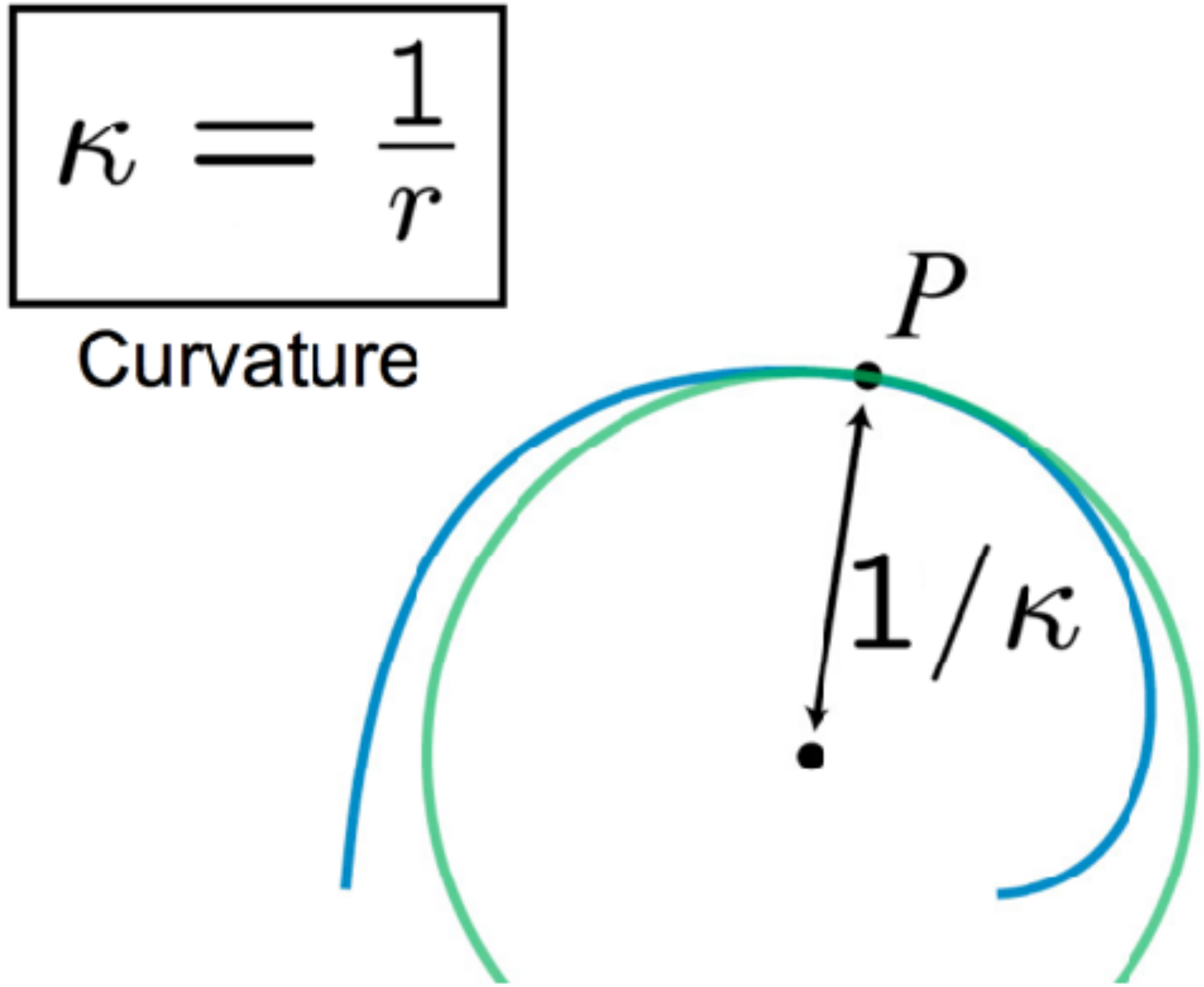
Curvature: Some Intuition

Radius of curvature r



Curvature: Some Intuition

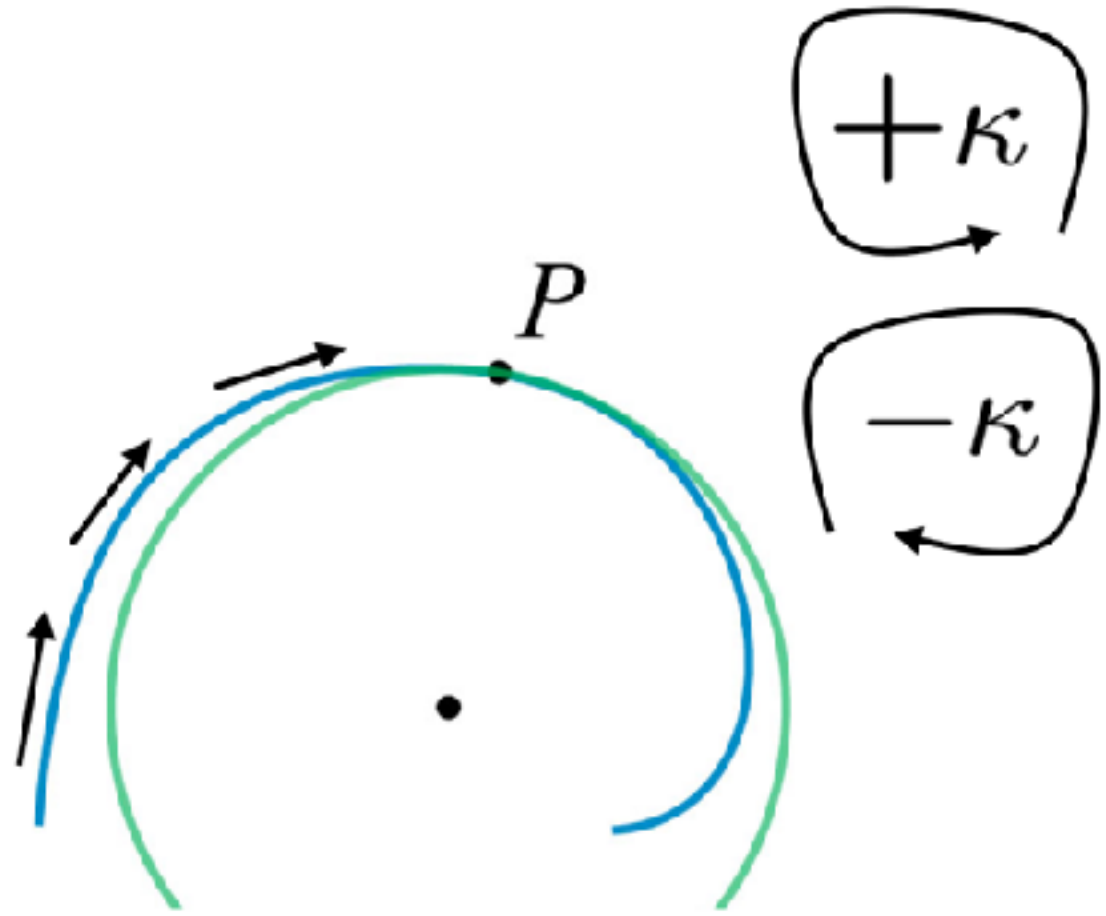
Radius of curvature *r*



Curvature: Some Intuition

Signed curvature

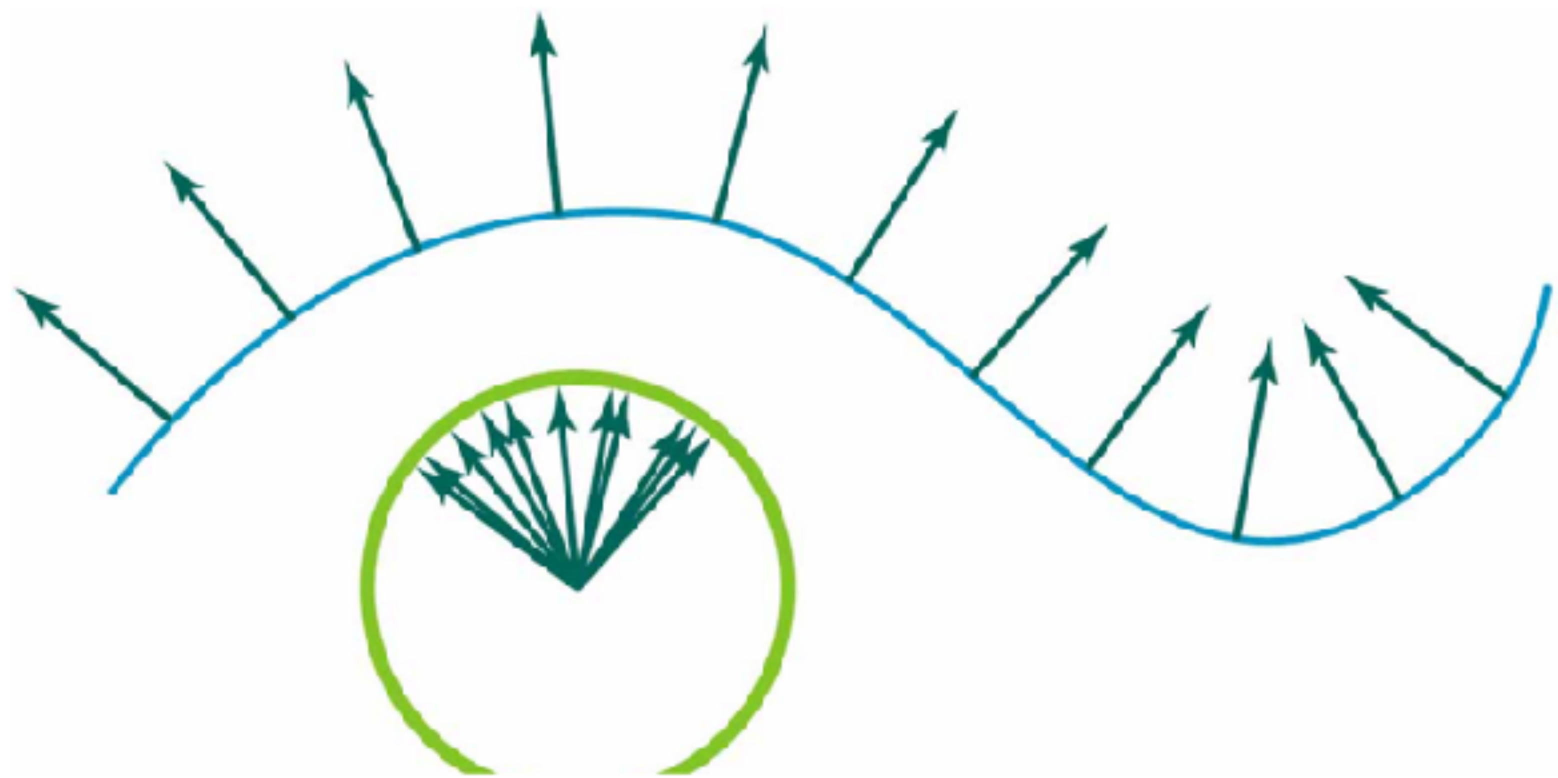
Sense of traversal along curve



Curvature: Some Intuition

Gauß map $\hat{n}(x)$

Point on curve maps to point on unit circle



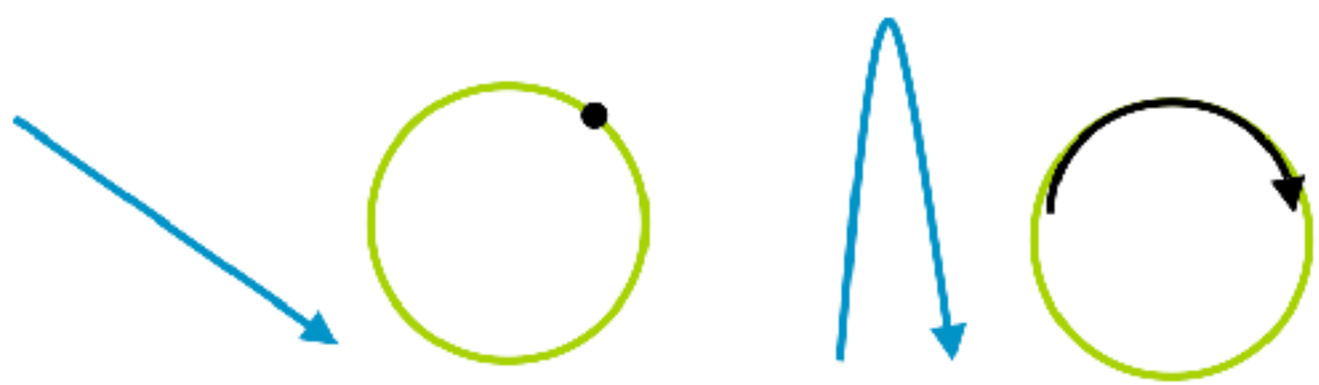
Curvature: Some Intuition

Shape operator (Weingarten map)

Change in normal as we slide along curve

negative directional derivative D of Gauß map

$$\boxed{\mathbf{S}(\mathbf{v}) = -D_{\mathbf{v}}\hat{\mathbf{n}}}$$



describes directional curvature

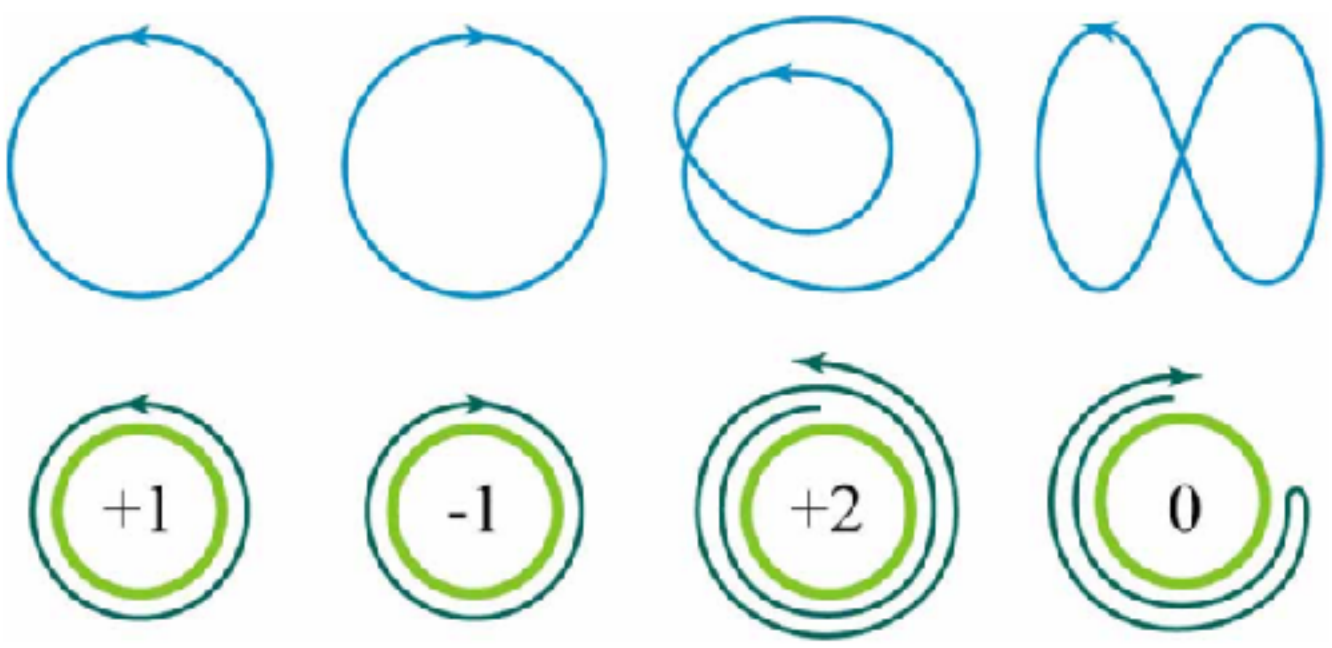
using normals as degrees of freedom

→ accuracy/convergence/implementation (discretization)

Curvature: Some Intuition

Turning number, *k*

Number of orbits in Gaussian image

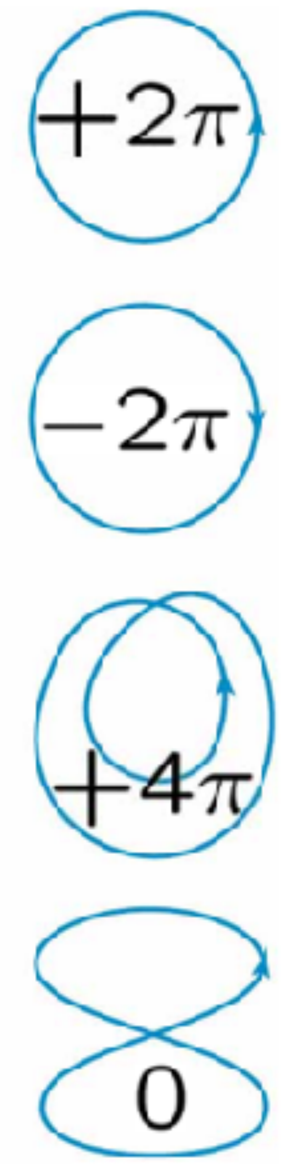


Curvature: Some Intuition

Turning number theorem

For a closed curve, the integral of curvature is an integer multiple of 2π

$$\int_{\Omega} \kappa ds = 2\pi k$$



Take Home Message

In the limit of a refinement sequence, discrete measure of length and curvature **agree** with continuous measures http://cs621.hao-li.com

Thanks!