8.1 Surface Parameterization
Modeling
The creation of a 3D assets surface, including that surface’s color, texture, opacity, and reflectivity (or specularity).
Rango: Creating creature scale textures in ZBrush...
Viewpaint

(Wrinkle Pass)
Color Maps
Wet Maps
bump Maps
Motivation

Normal Mapping

290000 facets

3500 facets

normal map
Motivation

Texture Mapping
Normal Mapping
Detail Transfer

Morphing
Mesh Completion
Editing

Databases
Remeshing
Surface Fitting
Motivation
Motivation
Mesh Parameterization

Find a 1-to-1 mapping between given surface mesh and 2D parameter domain
Spherical Coordinates

\[ \begin{bmatrix} \theta \\ \phi \end{bmatrix} \rightarrow \begin{bmatrix} \sin \theta \sin \phi \\ \cos \theta \sin \phi \\ \cos \phi \end{bmatrix} \]
Desirable Properties

Low distortion

Bijective mapping
Cartography

orthographic

preserves angles
= conformal

stereographic

Mercator

preserves area
= equiareal

Lambert

Floater, Hormann: *Surface Parameterization: A Tutorial and Survey*, Advances in Multiresolution for Geometric Modeling, 2005
More Maps

- Mollweide-Projektion
- Mercator-Projektion
- Zylinderprojektion nach Miller
- Hammer-Aitoff-Projektion
- Peters-Projektion
- Längentreue Azimuthalprojektion
- Stereographische Projektion
- Behrmann-Projektion
- Senkrecht Umgebungs perspektive
- Robinson-Projektion
- Hotine Oblique Mercator-Projektion
- Sinusoidale Projektion
- Gnomonische Projektion
- Flächenfertige Kegelprojektion
- Transverse Mercator-Projektion
- Cassini-Soldner-Projektion
Demo: Parameterization
Recall: Differential Geometry

Parametric surface representation

\[ x : \Omega \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3 \]

\[ (u, v) \rightarrow \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \]

Regular if

- Coordinate functions \( x, y, z \) are smooth
- Tangents are linearly independent

\[ x_u \times x_v \neq 0 \]
A regular parameterization \( x : \Omega \rightarrow S \) is

- **Conformal** (angle preserving), if the angle of every pair of intersecting curves on \( S \) is the same as that of the corresponding pre-images in \( \Omega \).

- **Equiareal** (area preserving) if every part of \( \Omega \) is mapped onto a part of \( S \) with the same area.

- **Isometric** (length preserving), if the length of any arc on \( S \) is the same as that of its pre-image in \( \Omega \).
Distortion Analysis

\[ \mathbf{u} = (u, v) \]

\[ \mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \]

\[ \mathbf{x} = (x, y, z) \]

Jacobian transforms infinitesimal vectors

\[ \mathbf{d} \mathbf{x} = \mathbf{J} \mathbf{d} \mathbf{u} \]

\[ \mathbf{J} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \]

\[ \| \mathbf{d} \mathbf{x} \|^2 = (\mathbf{d} \mathbf{u})^T \mathbf{J}^T \mathbf{J} \mathbf{d} \mathbf{u} = (\mathbf{d} \mathbf{u})^T \mathbf{I} \mathbf{d} \mathbf{u} \]
First Fundamental Form

Characterizes the surface locally

\[ I = \begin{pmatrix} x_u^T x_u & x_u^T x_v \\ x_v^T x_u & x_v^T x_v \end{pmatrix} \]

Allows to measure on the surface

- Angles \( \cos \theta = \frac{(d\mathbf{u}_1^T I d\mathbf{u}_2)}{||d\mathbf{u}_1|| \cdot ||d\mathbf{u}_2||} \)
- Length \( ds^2 = d\mathbf{u}^T I d\mathbf{u} \)
- Area \( dA = \det(I) \, du \, dv \)
A regular parameterization $x(u, v)$ is isometric, iff its first fundamental form is the identity:

$$I(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A surface has an isometric parameterization iff it has zero Gaussian curvature.
Cylinder
A regular parameterization $\mathbf{x}(u, v)$ is conformal, iff its first fundamental form is a scalar multiple of the identity:

$$\mathbf{I}(u, v) = s(u, v) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
A regular parameterization \( \mathbf{x}(u, v) \) is equiareal, iff the determinant of its first fundamental form is 1:

\[
\det(I(u, v)) = 1
\]
An isometric parameterization is conformal and equiareal, and vice versa:

\[ \text{isometric} \iff \text{conformal} + \text{equireal} \]

Isometric is ideal, but rare. In practice, people try to compute:

- Conformal
- Equireal
- Some balance between the two
Harmonic Maps

- A regular parameterization $\mathbf{x}(u, v)$ is harmonic, iff it satisfies

$$\Delta \mathbf{x}(u, v) = 0$$

- Isometric $\Rightarrow$ Conformal $\Rightarrow$ Harmonic

- Easier to compute than conformal, but does not preserve angles
Harmonic Maps

• A harmonic map minimizes the Dirichlet energy

\[
\int_{\Omega} \left\| \nabla x \right\|^2 = \int_{\Omega} \left\| x_u \right\|^2 + \left\| x_v \right\|^2 \, du \, dv
\]

• Variational calculus then tells us that

\[
\Delta x(u, v) = 0
\]

• If \( x : \Omega \rightarrow S \) is harmonic and maps the boundary \( \partial \Omega \) of a convex region \( \Omega \subset \mathbb{R}^2 \) homeomorphically onto the boundary \( \partial S \), then \( x \) is one-to-one.
Parameterization Goal

- Piecewise linear mapping of a discrete 3D triangle mesh onto a planar 2D polygon

- Slightly different situation: Given a 3D mesh, compute the inverse parameterization
Floater’s Parameterization
Floater’s Parameterization

• For Quadrilateral Patch

• Fix the parameters of the boundary vertices on a unit square

• Derive the bijection \( u \) for each of the interior vertices \( v_i \) by solving

\[
\begin{equation}
\begin{aligned}
   u(v_i) &= \sum_{k \in v(i)} \lambda_{i,k} u(v_k) \\
   \text{where } \lambda_{i,k} &\text{ satisfies shape preserving criteria} \\
   \text{and } \sum_{k \in v(i)} \lambda_{i,k} &= 1, \ i = 1,2,\ldots,n
\end{aligned}
\end{equation}
\]
Floater’s Algorithm

- Compute for each $i$ the $\lambda_{i,k}, k \in v(i)$
- Compute a local parameterization for $v(i)$ that preserves the aspect ratio of the angle and length
- Compute $\lambda_{i,k}, k \in v(i)$ that satisfies
  
  \[
  \text{Shape preserving criteria}
  \]
  \[
  \sum_{k \in v(i)} \lambda_{i,k} = 1, \quad i = 1, 2, \ldots, n
  \]
- Solve the sparse equation for $u(v_i), i = 1 \ldots n$

\[
  u(v_i) = \sum_{k \in v(i)} \lambda_{i,k} u(v_k)
  \]
Discrete Harmonic Maps

- Map the boundary $\partial S$ homeomorphically to some (convex) polygon $\partial \Omega$ in the parameter plane.
- Minimize the Dirichlet energy of $u$ by solving the corresponding Euler-Lagrange PDE

$$\Delta_S u = 0$$

- Requires discretization of Laplace-Beltrami
- Compare to surface fairing
Discrete Harmonic Maps

- System of linear equations

\[ \forall v_i \in \mathcal{S} : \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} (u(v_j) - u(v_i)) \]

\[ w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij} \]

- Properties of system matrix:
  - Symmetric + positive definite $\implies$ unique solution
  - Sparse $\implies$ efficient solvers
But...

- Does the same theory hold for discrete harmonic maps as for harmonic maps?
- In other words, is it possible for triangles to flip or become degenerate?
Convex Combination Maps

- If the linear equations are satisfied
  \[ \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} (u(v_j) - u(v_i)) \]
  and if the weights satisfy
  \[ w_{ij} > 0 \land \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} = 1 \]
  then we get a convex combination mapping.
• Each $u(v_i)$ is a convex combination of $u(v_j)$

$$u(v_i) = \sum_{v_j \in N_1(v_i)} w_{ij} u(v_j)$$

• If $u : S \to \Omega$ is a convex combination map that maps the boundary $\partial S$ homeomorphically to the boundary $\partial \Omega$ of a convex region $\Omega \subset \mathbb{R}^2$, then $u$ is one-to-one.
Convex Combination Maps

- Uniform barycentric weights

\[ w_{ij} = \frac{1}{\text{valence}(v_i)} \]

- Cotangent weights ( > 0 if \( \alpha_{ij} + \beta_{ij} < \pi \) )

\[ w_{ij} = \cot(\alpha_{ij}) + \cot(\beta_{ij}) \]

- Mean value weights

\[ w_{ij} = \frac{\tan(\delta_{ij}/2) + \tan(\gamma_{ij}/2)}{\|p_j - p_i\|} \]

(no negative weights, even for obtuse angles)
• Comparison

original mesh  uniform weights  cotan weights (shape preserving)  mean value
Fixing the Boundary

- Choose a simple convex shape
  - Triangle, square, circle
- Distribute points on boundary
  - Use chord length parameterization

Fixed boundary can create high distortion
Open Boundary Mappings

- Include boundary vertices in the optimization
- Produces mappings with lower distortion
Open Boundary Mappings
Need disk-like topology

- Introduce cuts on the mesh
Naive Cut, Numerical Problems
• Split model into number of patches (atlas)
  • because higher genus models cannot be mapped onto plane and/or
  • because distortion, the number of patches will be too high eventually

Texture Atlas Generation

- Split model into number of patches (atlas)
  - because higher genus models cannot be mapped onto plane and/or
  - because distortion, the number of patches will be too high eventually

Non-Planar Domains

seamless, continuous parameterization of genus-0 surfaces
Constrained Parameterizations

• Book, Chapter 5

• Hormann et al.: Mesh Parameterization, Theory and Practice, Siggraph 2007 Course Notes

• Floater and Hormann: Surface Parameterization: a tutorial and survey, advances in multiresolution for geometric modeling, Springer 2005

• Hormann, Polthier, and Sheffer, Mesh Parameterization: Theory and Practice, SIGGRAPH Asia 2008 Course Notes
Next Time

Decimation
http://cs599.hao-li.com

Thanks!