

Spring 2015

CSCI 599: **Digital Geometry Processing**

7.1 **Surface Smoothing**



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<http://cs599.hao-li.com>

Mesh Optimization

Smoothing

- Low geometric noise

Fairing

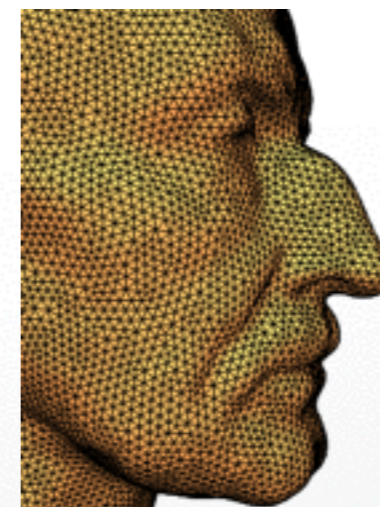
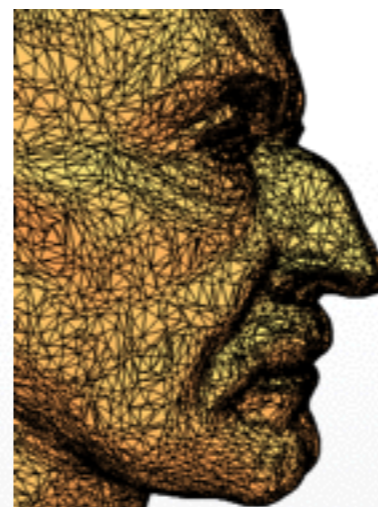
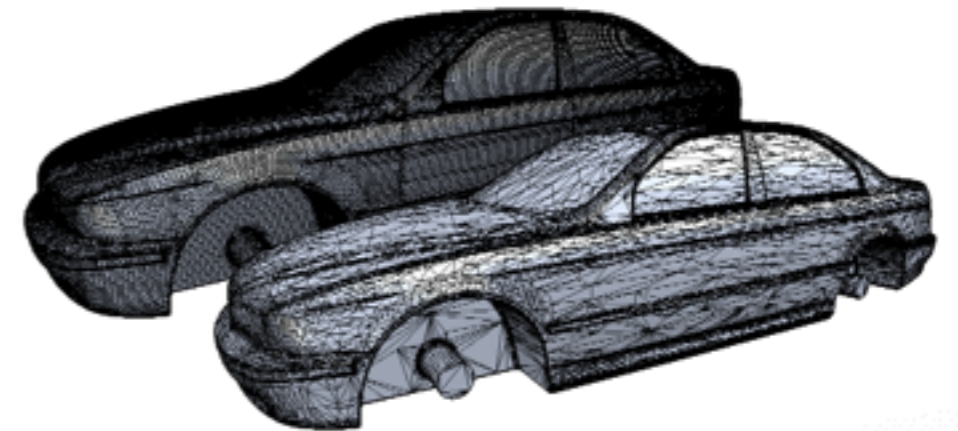
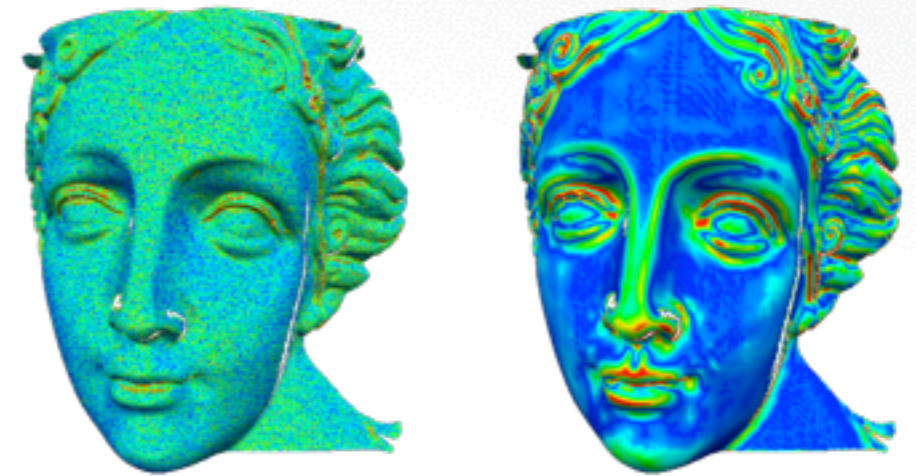
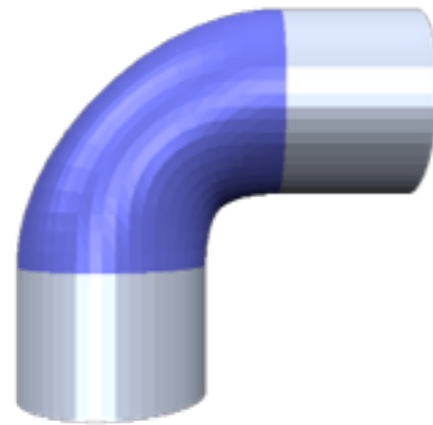
- Simplest shape

Decimation

- Low complexity

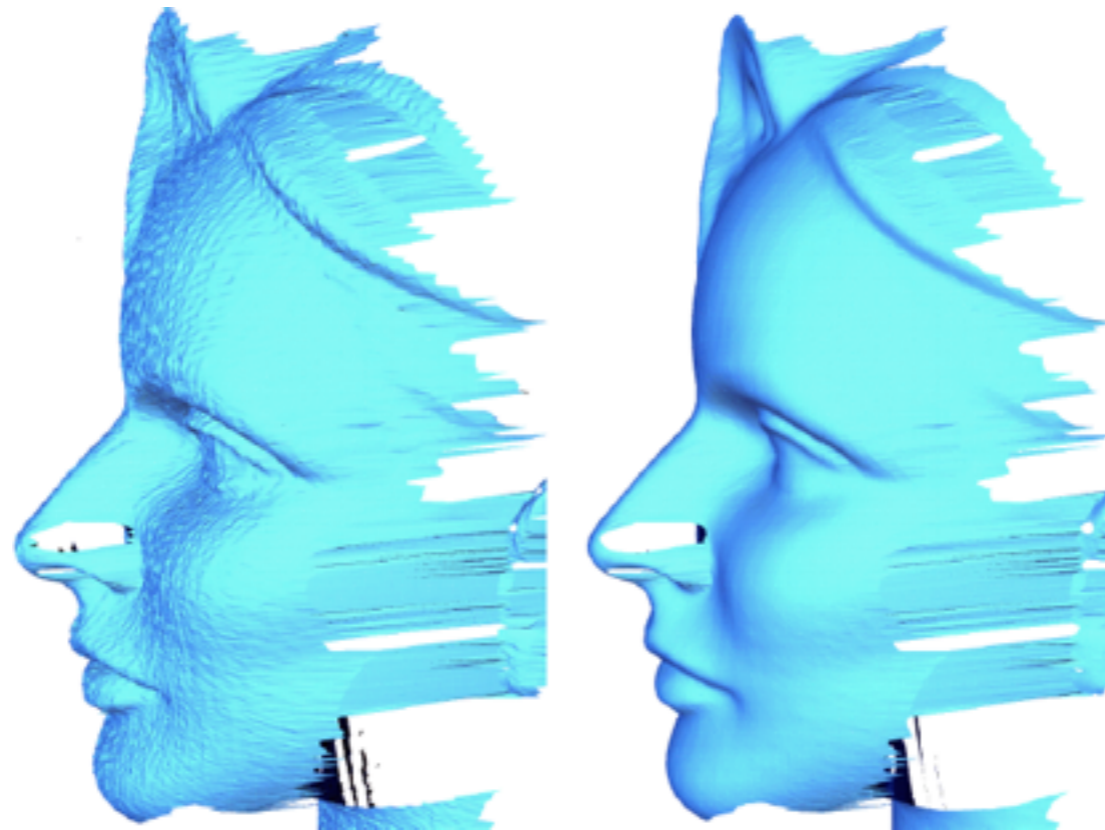
Remeshing

- Triangle Shape



Mesh Smoothing

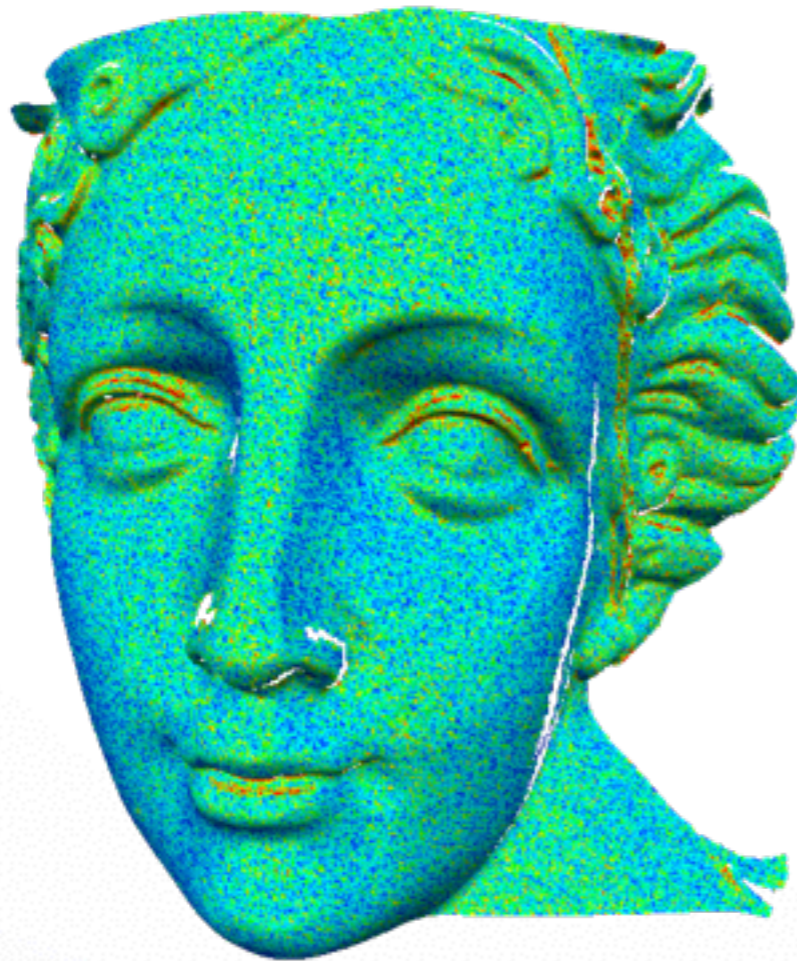
Filter out high frequency noise



Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99

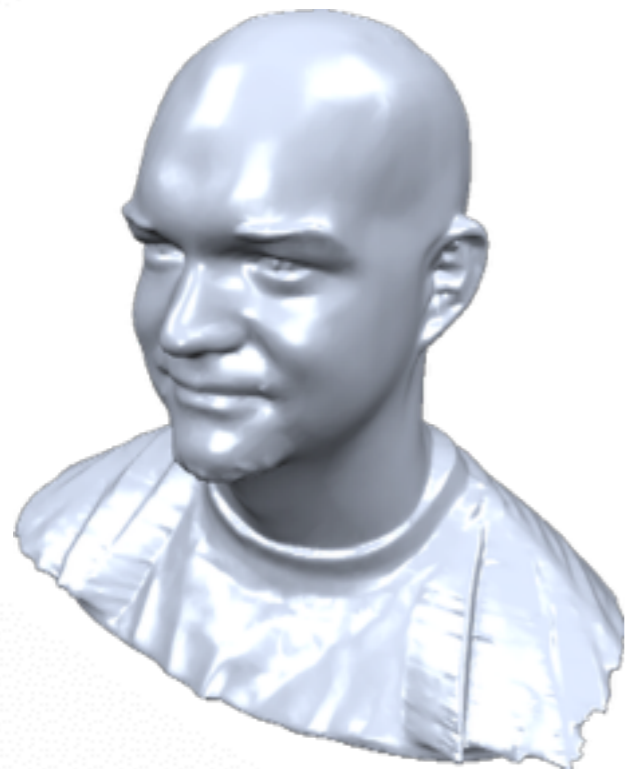
Mesh Smoothing

Filter out high frequency noise

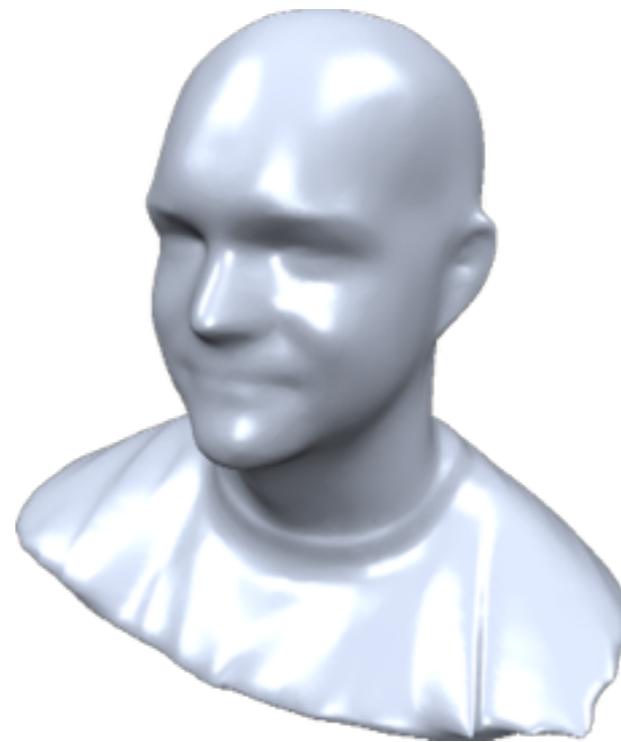


Mesh Smoothing

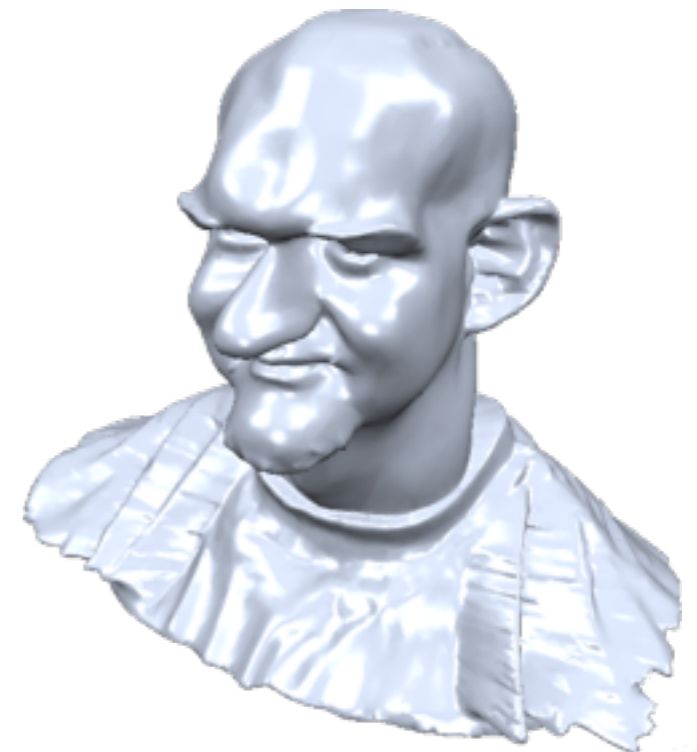
Advanced Filtering



input data



low pass

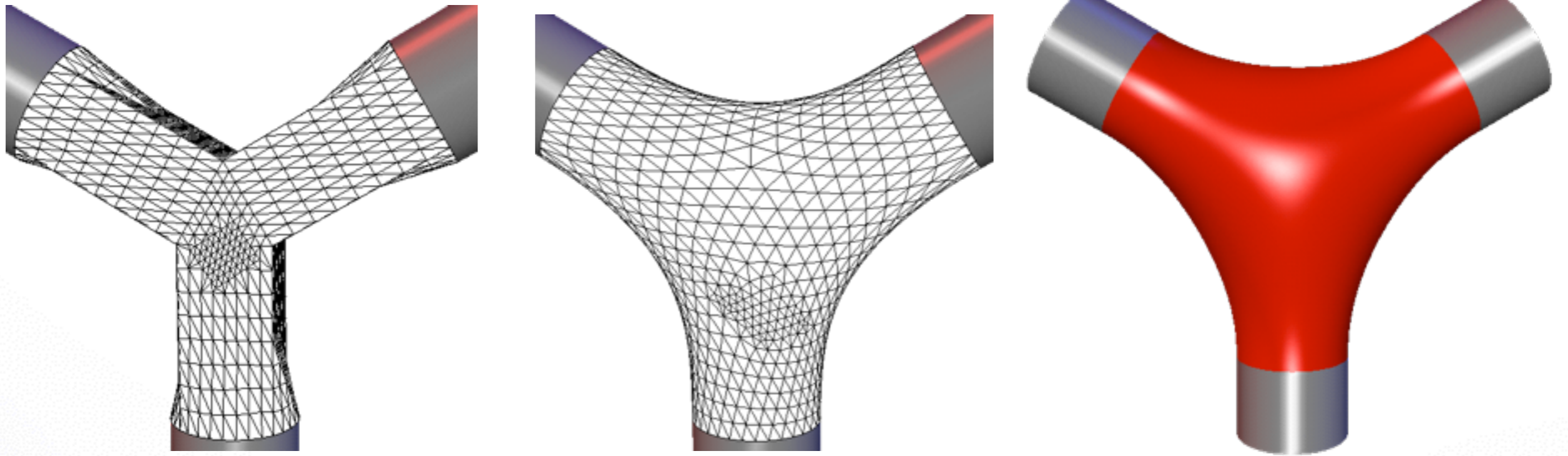


exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*, Eurographics 05

Mesh Smoothing

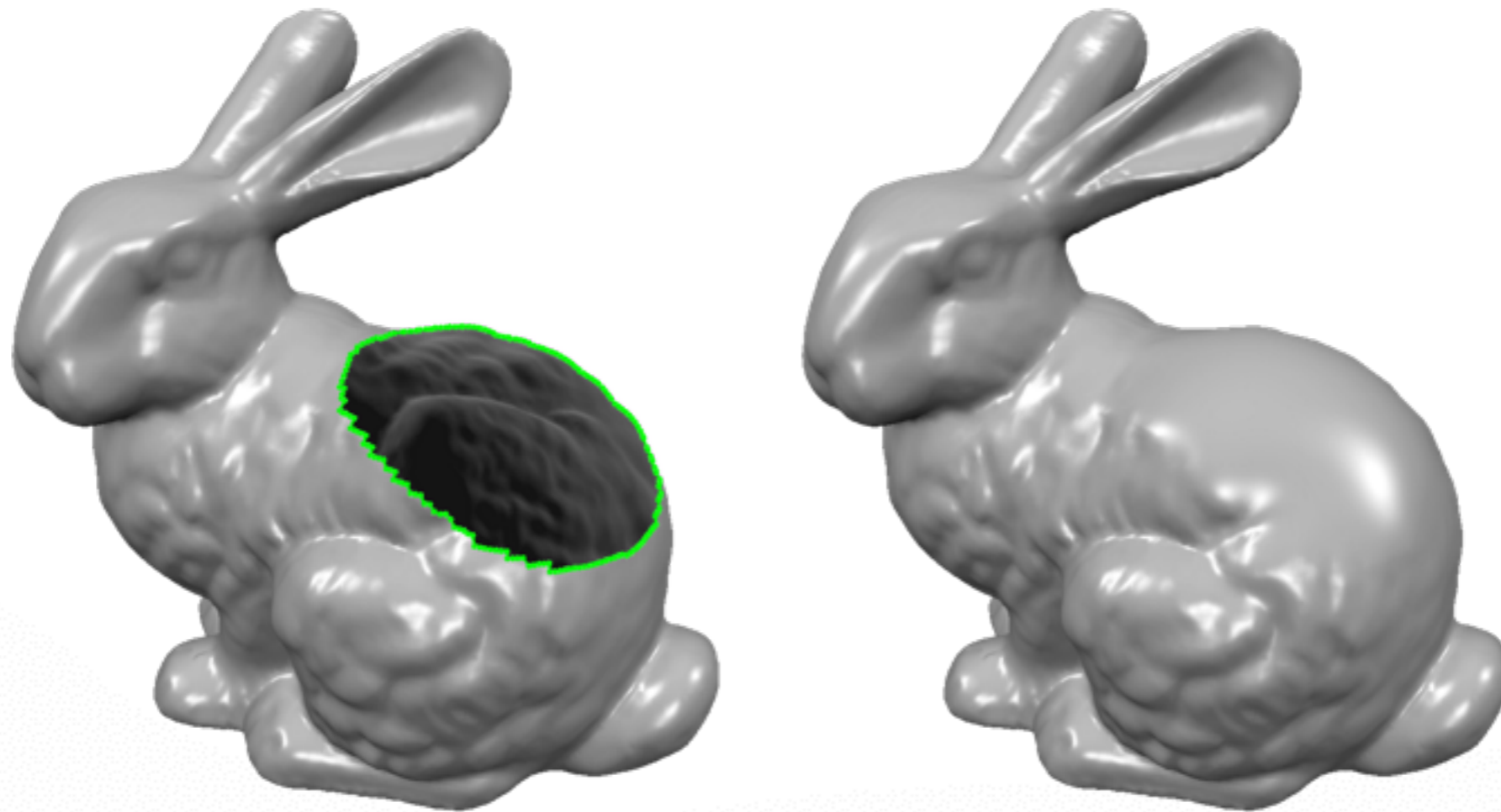
Fair Surface Design



Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*, CAGD 18(4), 2001

Mesh Smoothing

Hole filling with energy-minimizing patches



Outline

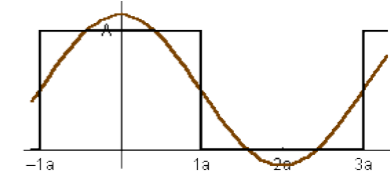
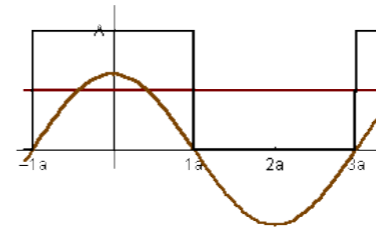
- **Spectral Analysis**
- **Diffusion Flow**
- **Energy Minimization**

Fourier Transform

Represent a function as a weighted sum of sines and cosines



Joseph Fourier 1768 - 1830



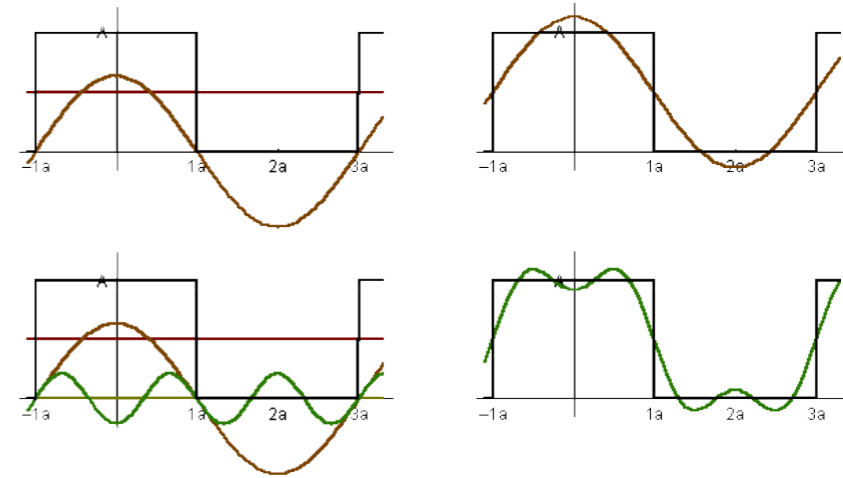
$$f(x) = a_0 + a_1 \cos(x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines



Joseph Fourier 1768 - 1830



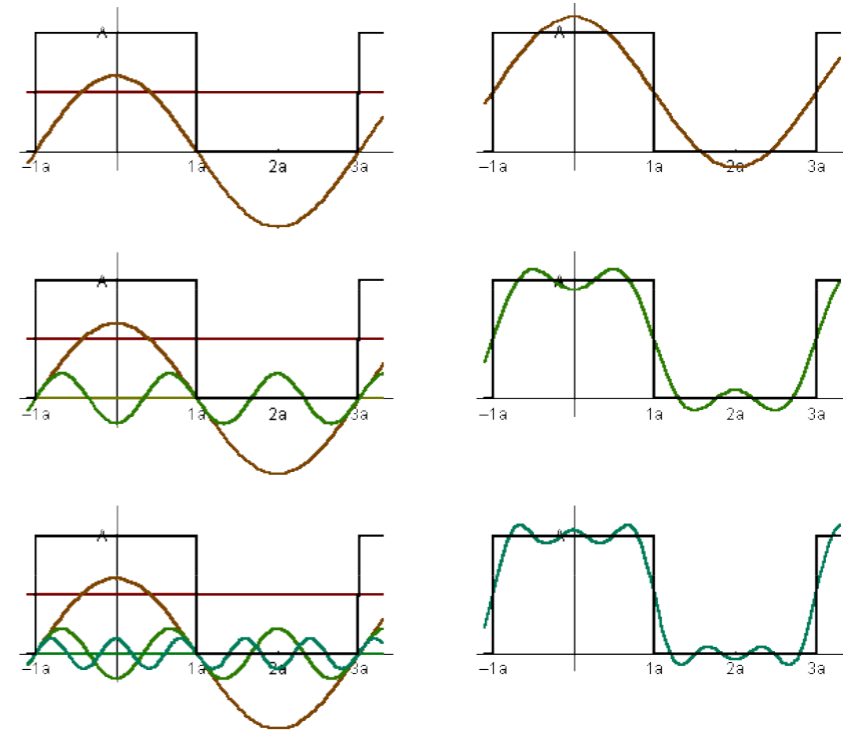
$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines



Joseph Fourier 1768 - 1830



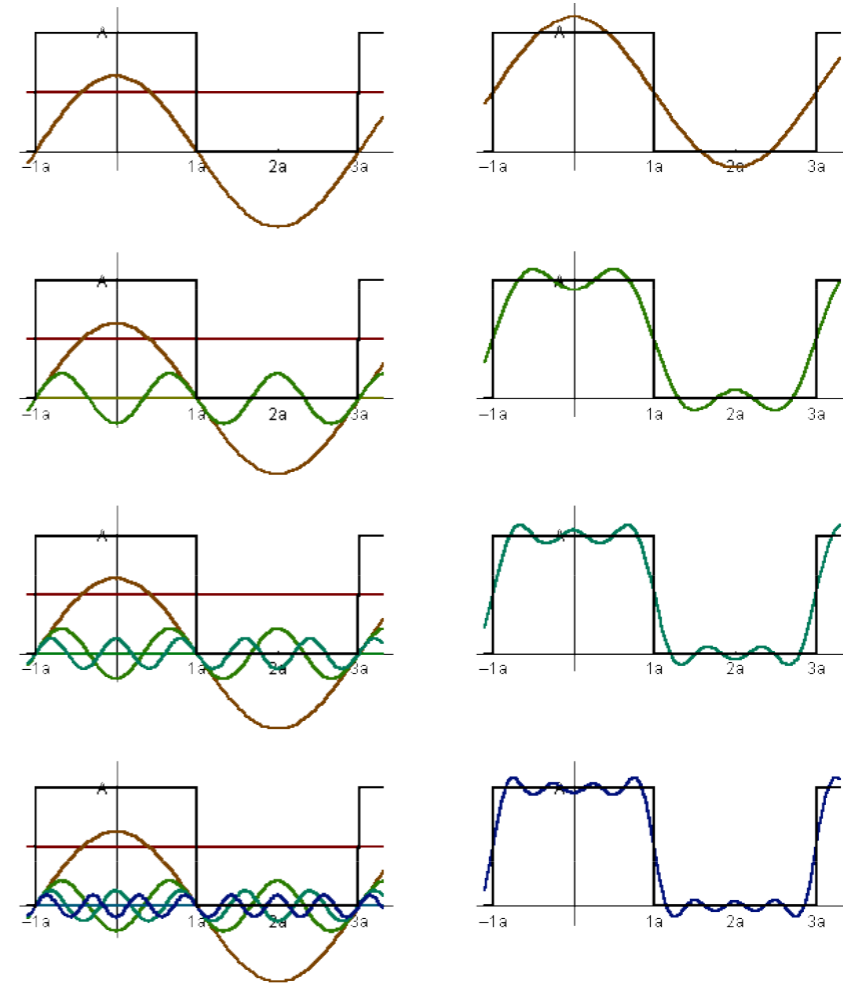
$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x)$$

Fourier Transform

Represent a function as a weighted sum of sines and cosines



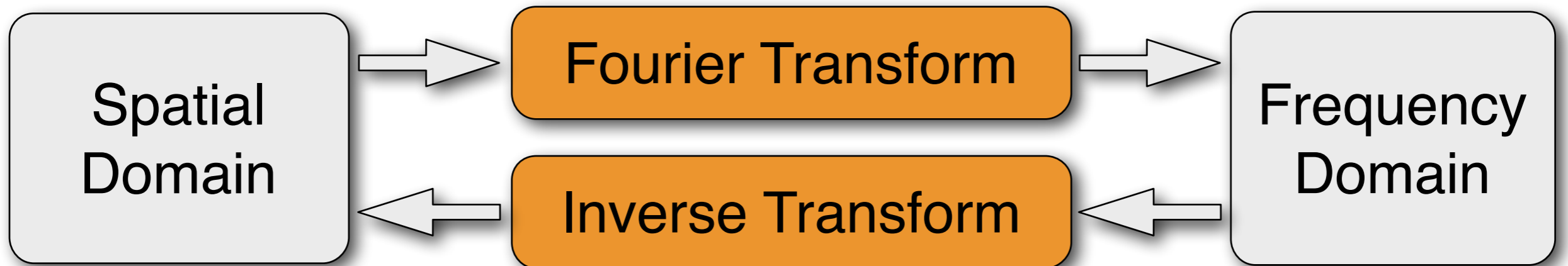
Joseph Fourier 1768 - 1830



$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \dots$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx$$



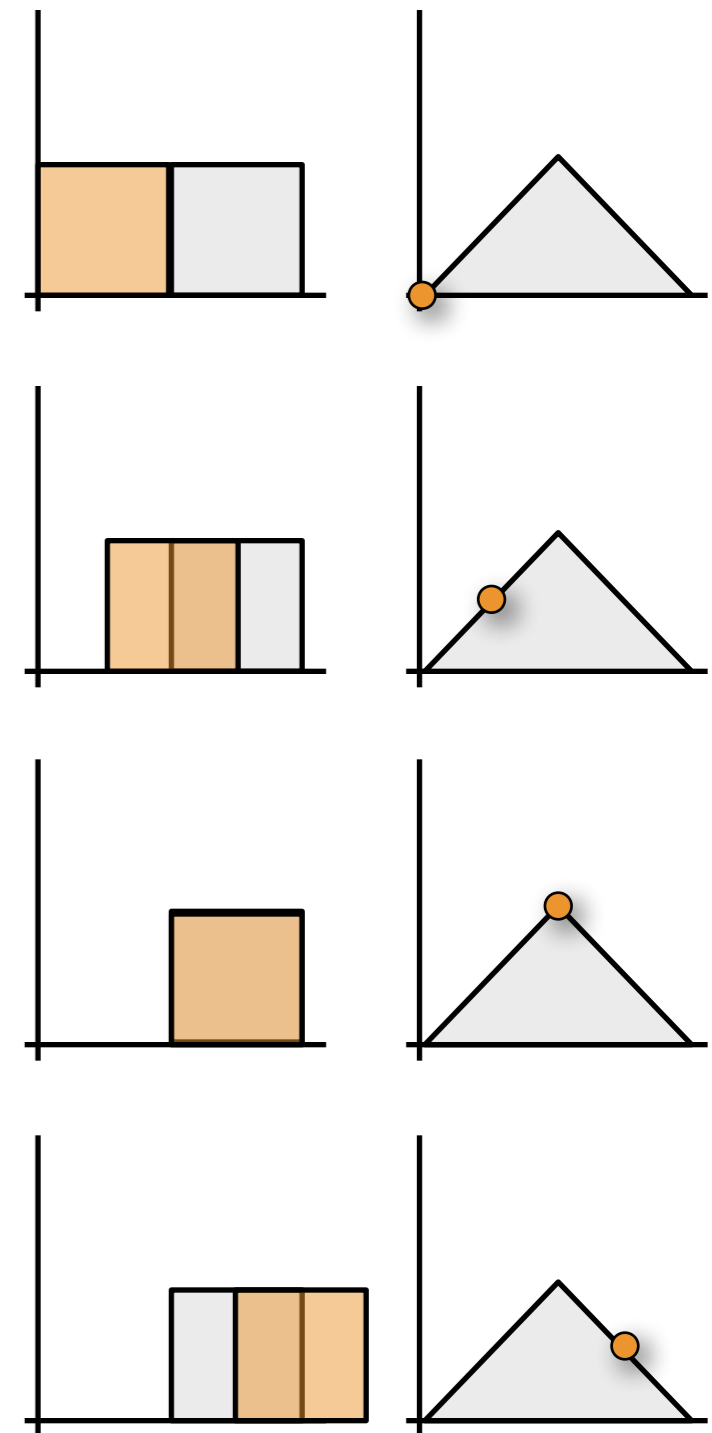
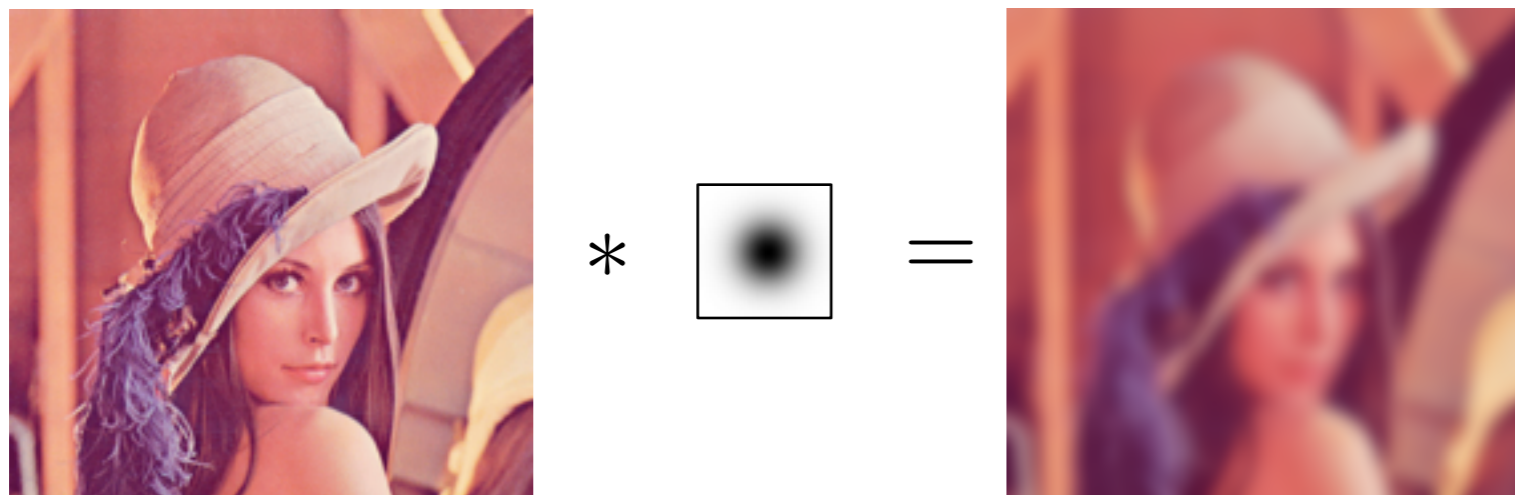
$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Convolution

Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x - y) dy$$

Example: Gaussian blurring



Convolution

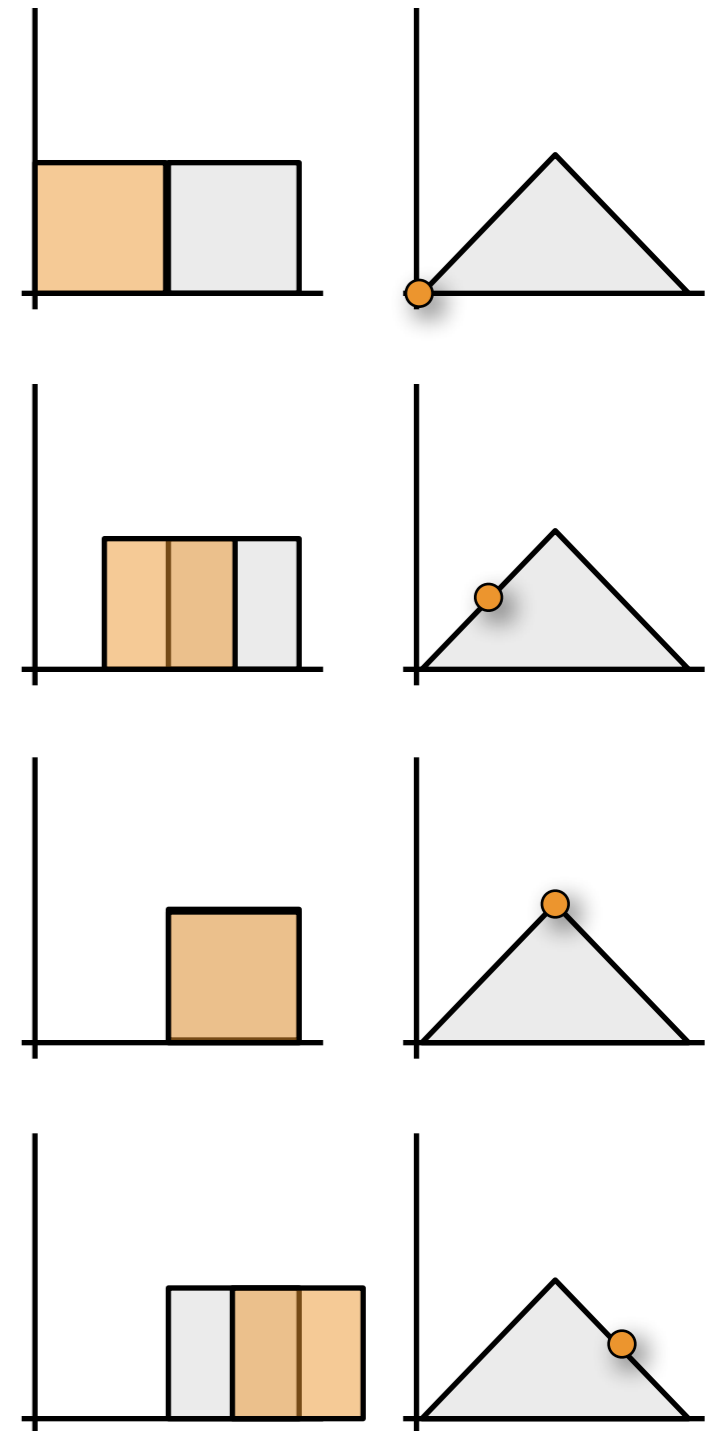
Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x - y) dy$$

Convolution in spatial domain \Leftrightarrow

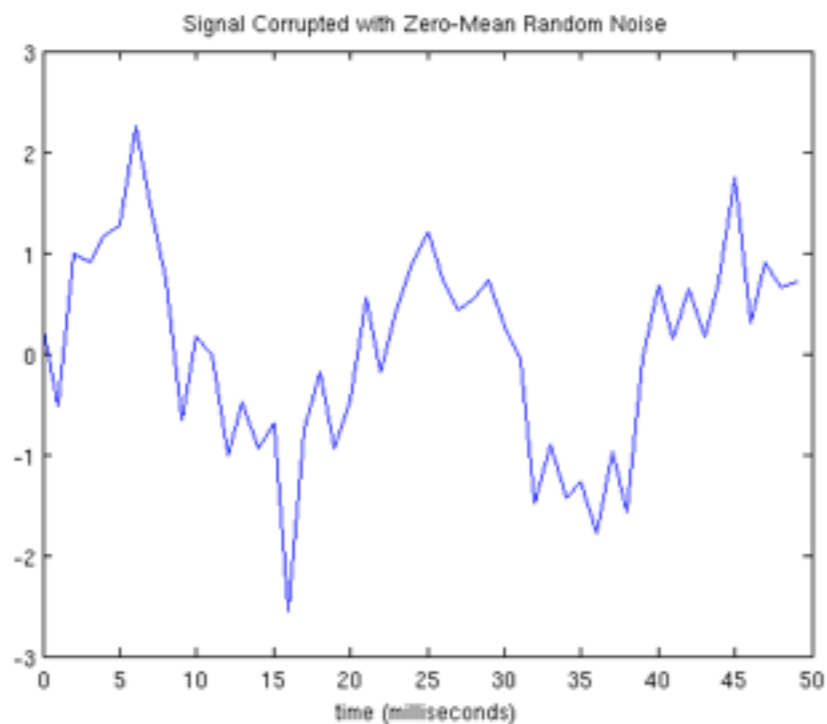
Multiplication in frequency domain

$$H(\omega) = F(\omega) \cdot G(\omega)$$

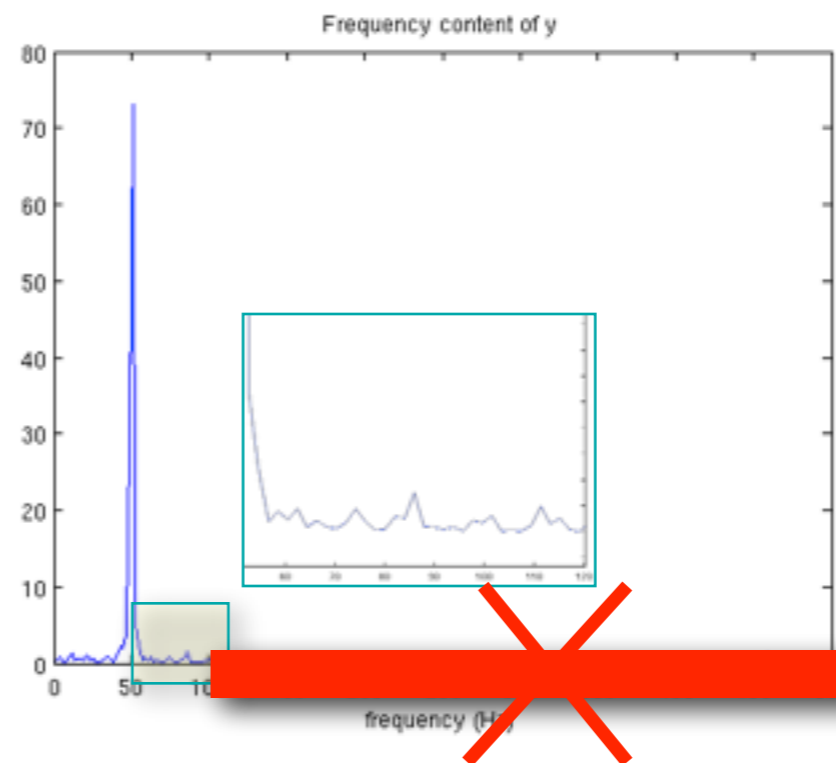


Fourier Analysis

Low-pass filter discards high frequencies



spatial domain



frequency domain

LOW PASS

Fourier Transform

Spatial domain $f(x)$ \rightarrow **Frequency domain** $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx$$

Multiply by low-pass filter $G(\omega)$

$$F(\omega) \leftarrow F(\omega) \cdot G(\omega)$$

Frequency domain $F(\omega)$ \rightarrow **Spatial domain** $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega x} d\omega$$

Fourier Transform

Consider L^2 -function space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx$$

Complex “waves” build an orthonormal basis

$$e_{\omega}(x) := e^{-2\pi i \omega x} = \cos(2\pi \omega x) - i \sin(2\pi \omega x)$$

Fourier transform is a change of basis

$$f(x) = \sum_{\omega=-\infty}^{\infty} \langle f, e_{\omega} \rangle e_{\omega} \, d\omega \quad \longrightarrow \quad f(x) = \int_{-\infty}^{\infty} \langle f, e_{\omega} \rangle e_{\omega} \, d\omega$$

Fourier Analysis on Meshes?

- Only applicable to parametric patches
- Generalize frequency to the discrete setting
- Complex waves are Eigenfunctions of Laplace

$$\Delta(e^{2\pi i\omega x}) = \frac{d^2}{dx^2} e^{2\pi i\omega x} = -(2\pi\omega)^2 e^{2\pi i\omega x}$$

Use Eigenfunctions of discrete Laplace-Beltrami

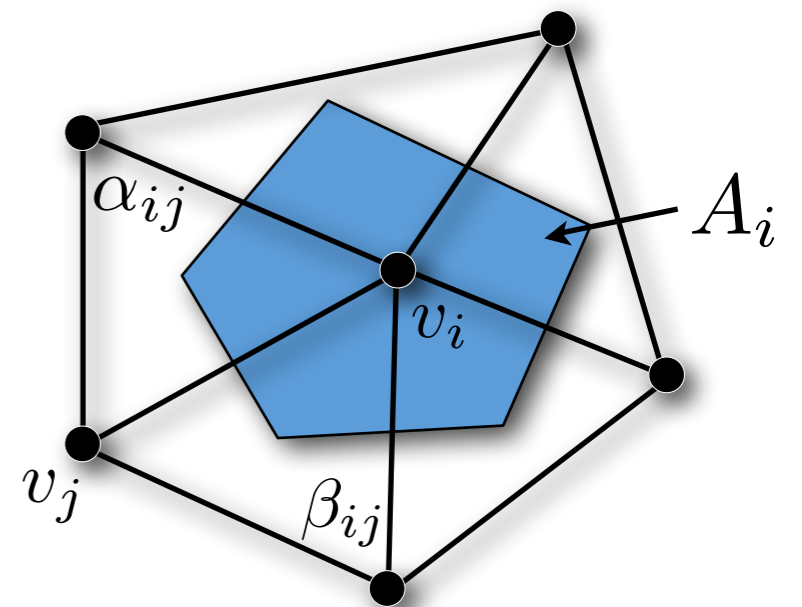
Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \dots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

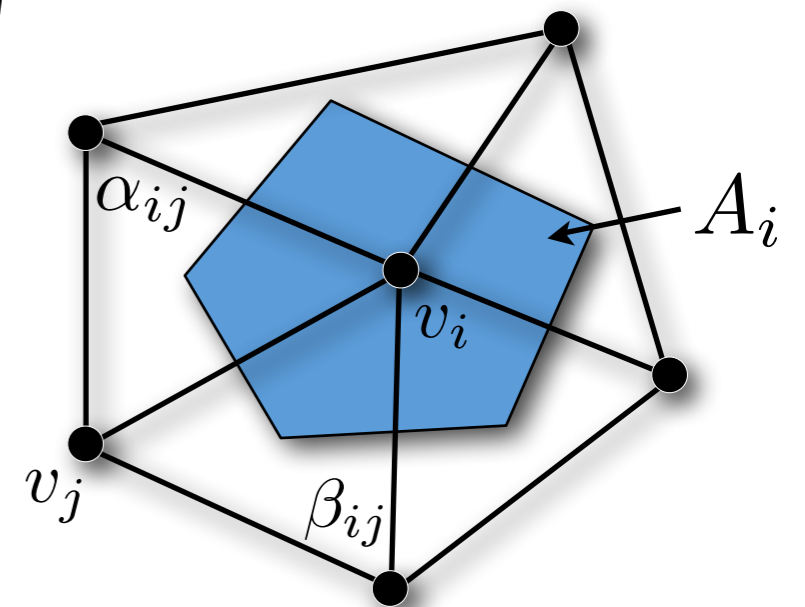
$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij}) (f(v_j) - f(v_i))$$



Discrete Laplace-Beltrami

- Discrete Laplace Operator (per mesh)
- Sparse matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} \vdots \\ \Delta_S f(v_i) \\ \vdots \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \vdots \\ f(v_i) \\ \vdots \end{pmatrix}$$

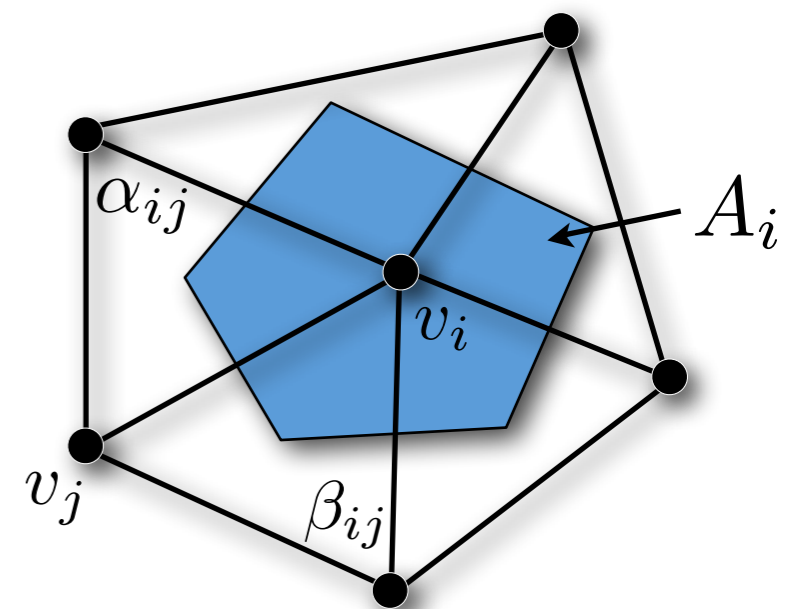


Discrete Laplace-Beltrami

- Discrete Laplace Operator (per mesh)
- Sparse matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$

$$\mathbf{M}_{ij} = \begin{cases} \cot\alpha_{ij} + \cot\beta_{ij}, & i \neq j, j \in \mathcal{N}_1(v_i) \\ -\sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{D} = \text{diag}\left(\dots, \frac{1}{2A_i}, \dots\right)$$



Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \dots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

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- Discrete Laplace-Beltrami matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$
 - Eigenvectors are **natural vibrations**
 - Eigenvalues are **natural frequencies**

Discrete Laplace-Beltrami



- Discrete Laplace-Beltrami matrix $\mathbf{L} = \mathbf{DM} \in \mathbb{R}^{n \times n}$
 - Eigenvectors are **natural vibrations**
 - Eigenvalues are **natural frequencies**

Spectral Analysis

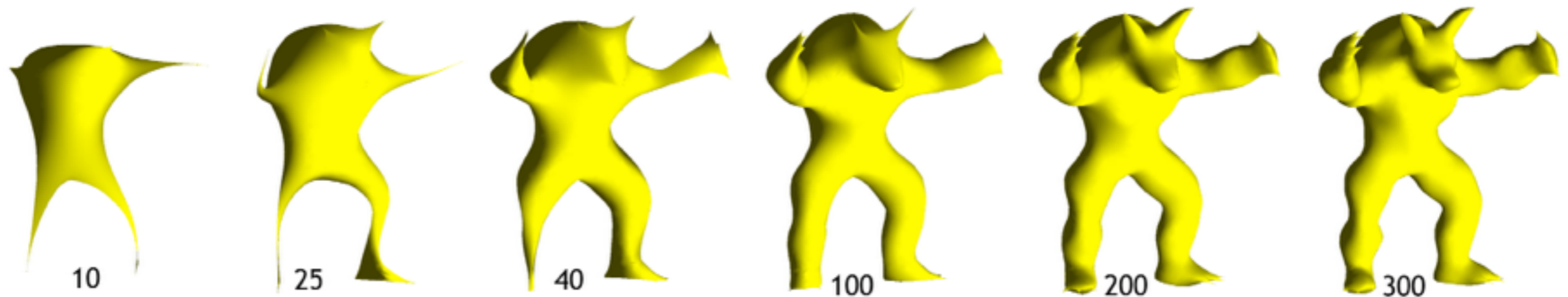
- Setup Laplace-Beltrami matrix \mathbf{L}
- Compute k smallest eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$
- Reconstruct mesh from those (component-wise)

$$\begin{aligned} \mathbf{x} &:= [x_1, \dots, x_n] & \mathbf{y} &:= [y_1, \dots, y_n] & \mathbf{z} &:= [z_1, \dots, z_n] \\ \mathbf{x} &\leftarrow \sum_{i=1}^k (\mathbf{x}^T \mathbf{e}_i) \mathbf{e}_i & \mathbf{y} &\leftarrow \sum_{i=1}^k (\mathbf{y}^T \mathbf{e}_i) \mathbf{e}_i & \mathbf{z} &\leftarrow \sum_{i=1}^k (\mathbf{z}^T \mathbf{e}_i) \mathbf{e}_i \end{aligned}$$

Spectral Analysis

- Setup Laplace-Beltrami matrix \mathbf{L}
- Compute k smallest eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$
- Reconstruct mesh from those (component-wise)

Too complex for
large meshes!



Bruno Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006

Outline

- **Spectral Analysis**
- **Diffusion Flow**
- **Energy Minimization**

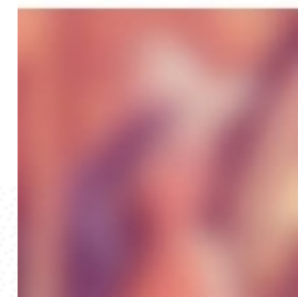
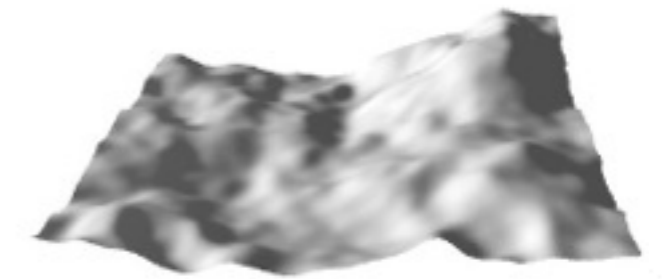
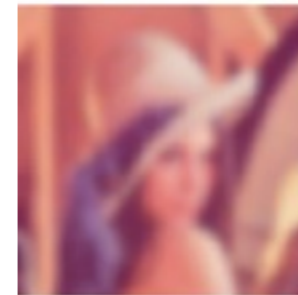
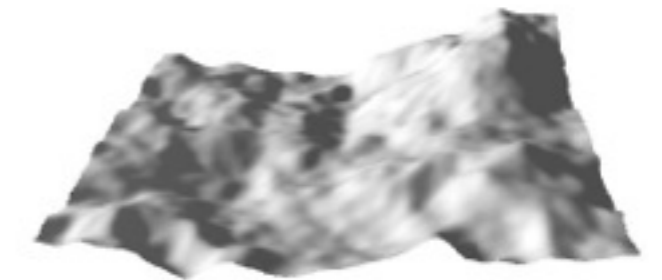
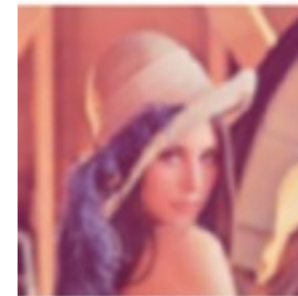
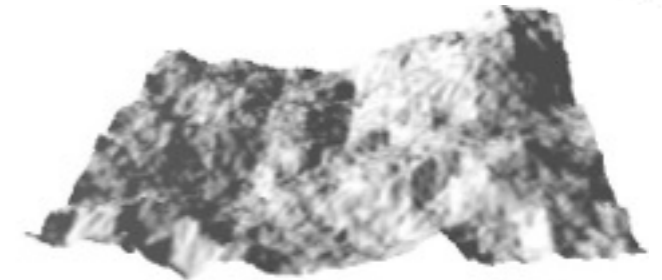
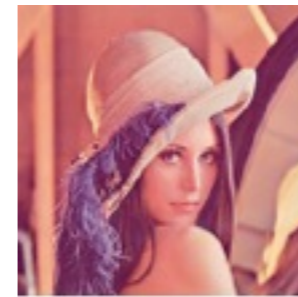
Diffusion Flow on Height Fields

Diffusion equation (this one is heat equation)

diffusion constant

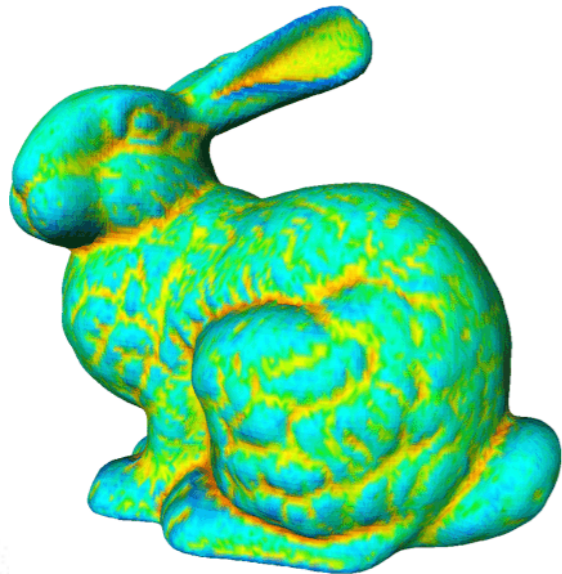
$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

Laplace operator

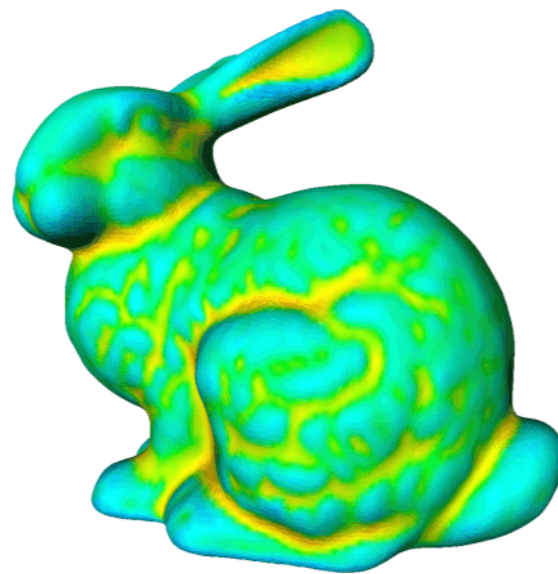


Diffusion Flow on Meshes

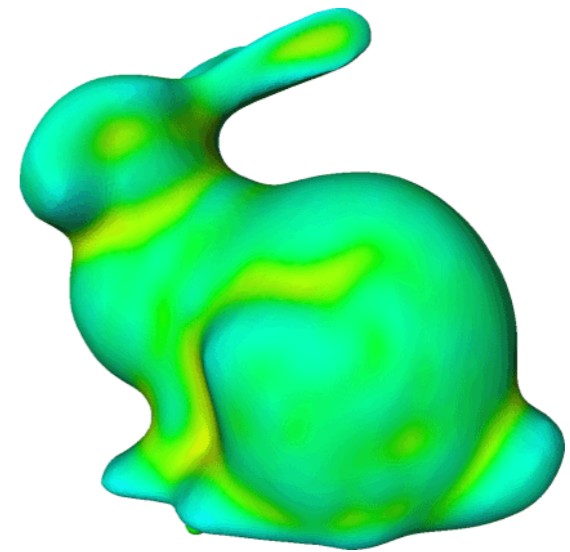
Iterate $\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$



0 Iterations



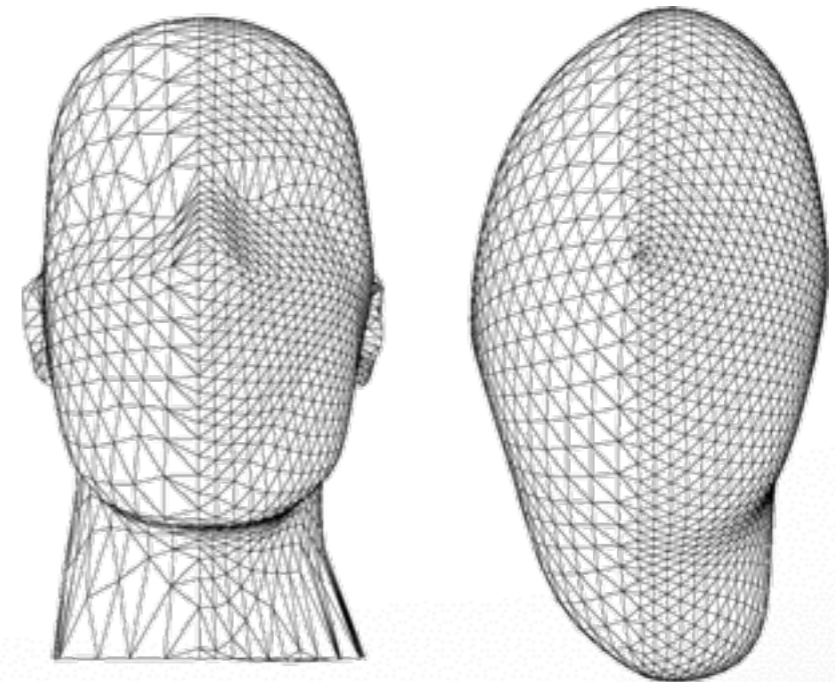
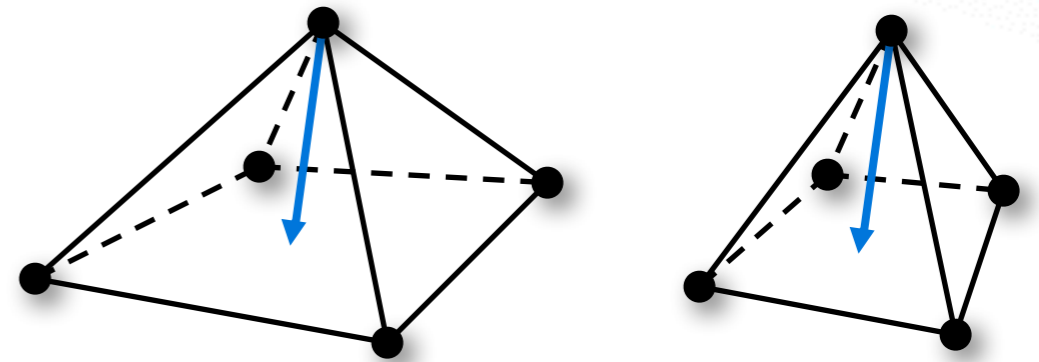
5 Iterations



20 Iterations

Uniform Laplace Discretization

- Smooths geometry and triangulation
- Can be non-zero even for planar triangulation
- Vertex drift can lead to distortions
- Might be desired for mesh regularization



Desbrun et al., Siggraph 1999

Mean **Curvature Flow**

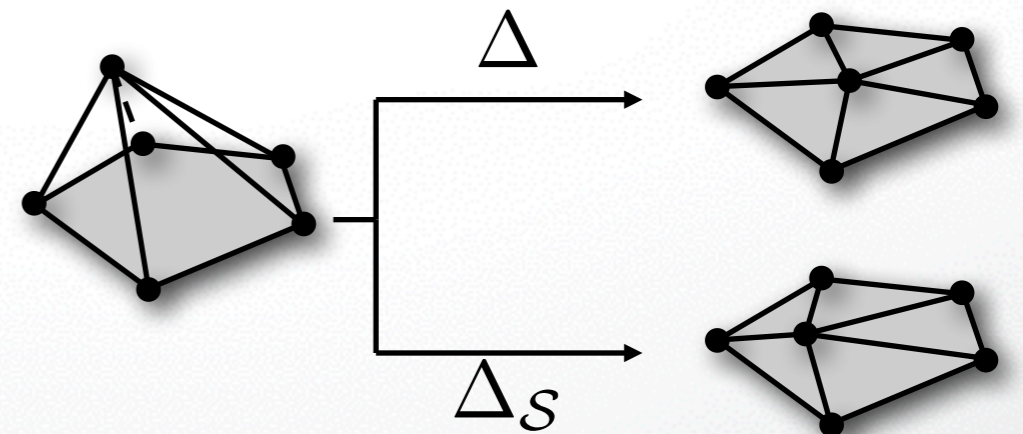
- Use diffusion flow with Laplace-Beltrami

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta_S \mathbf{p}$$

- Laplace-Beltrami is parallel to surface normal

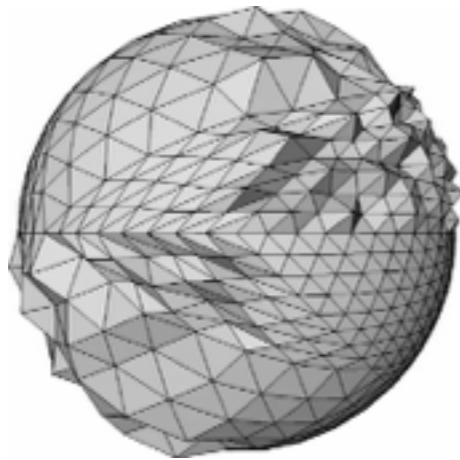
$$\frac{\partial \mathbf{p}}{\partial t} = -2\lambda H \mathbf{n}$$

Avoids vertex drift on surface

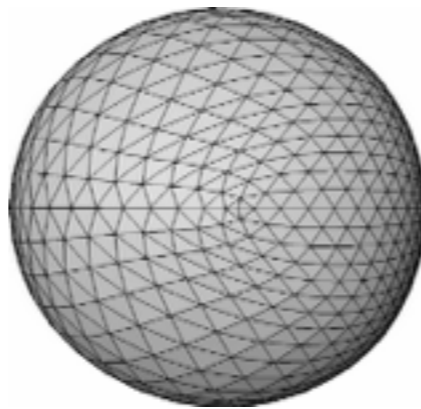


Mean Curvature Flow

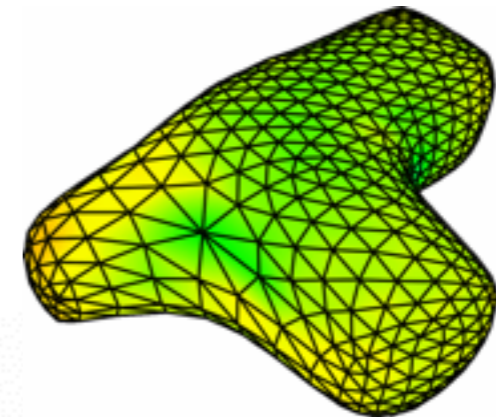
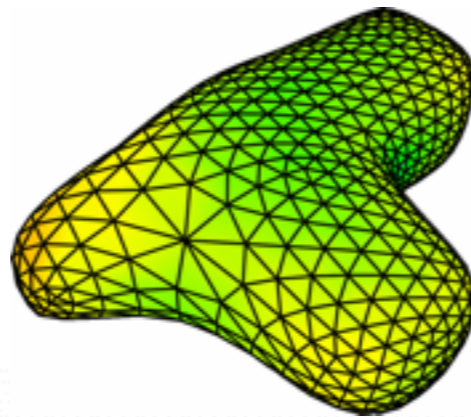
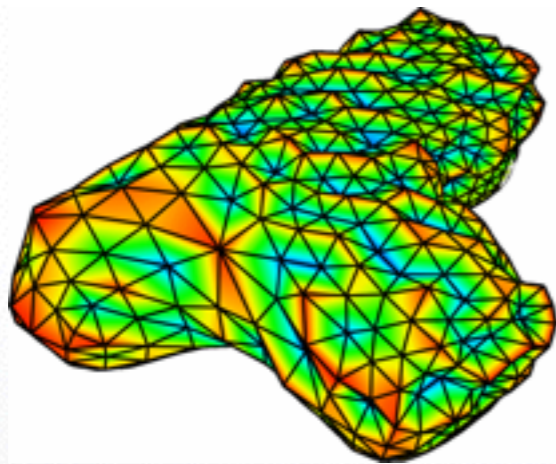
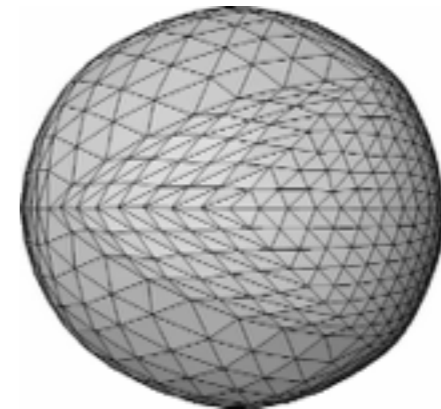
input data



uniform Laplace



Laplace-Beltrami



Numerical **Integration**

- Write update $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$ in matrix notation

$$\mathbf{P}^{(t)} = \left(\mathbf{p}_1^{(t)}, \dots, \mathbf{p}_n^{(t)} \right)^T \in \mathbb{R}^{n \times 3}$$

- Corresponds to explicit integration

$$\mathbf{P}^{(t+1)} = (\mathbf{I} + \lambda \mathbf{L}) \mathbf{P}^{(t)}$$

Requires small λ
for stability!

- Implicit integration is unconditionally stable

$$(\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

(backward Euler method)

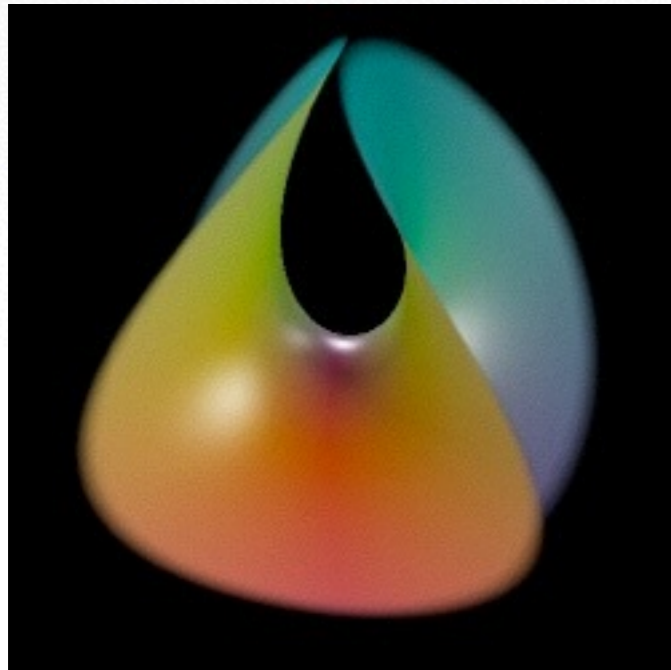
Outline

- **Spectral Analysis**
- **Diffusion Flow**
- **Energy Minimization**

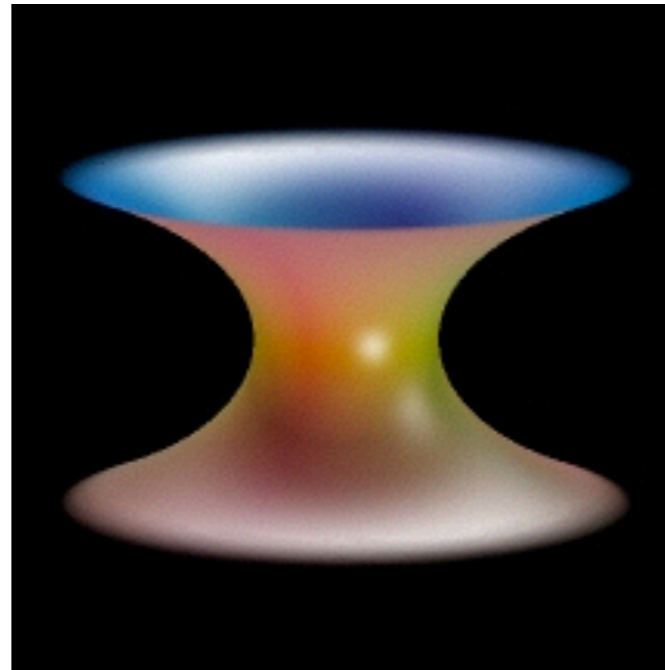
Fairness

- Idea: Penalize “unaesthetic behavior”
- Measure fairness
 - Principle of the simplest shape
 - Physical interpretation
- Minimize some fairness functional
 - Surface area, curvature
 - Membrane energy, thin plate energy

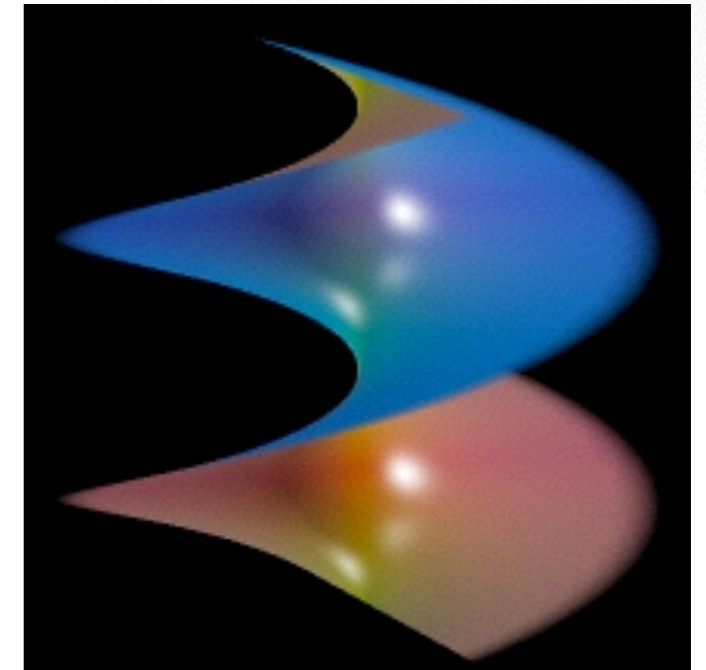
Minimal Surfaces



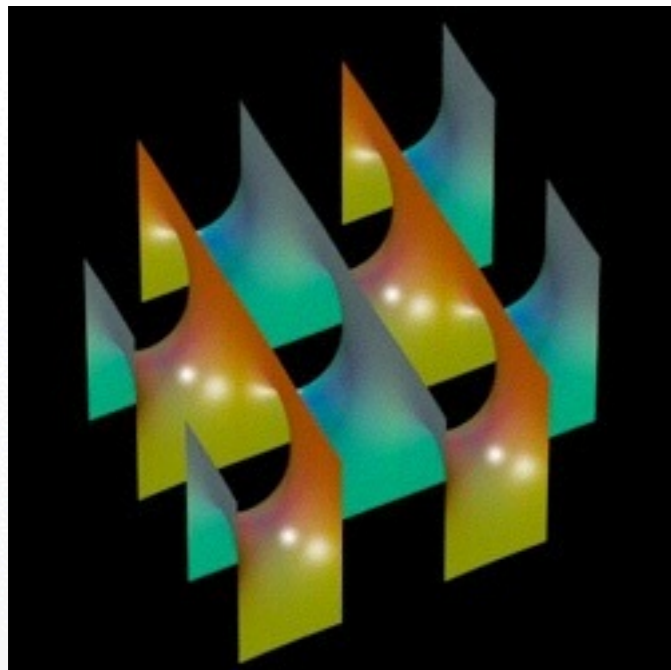
Enneper's Surface



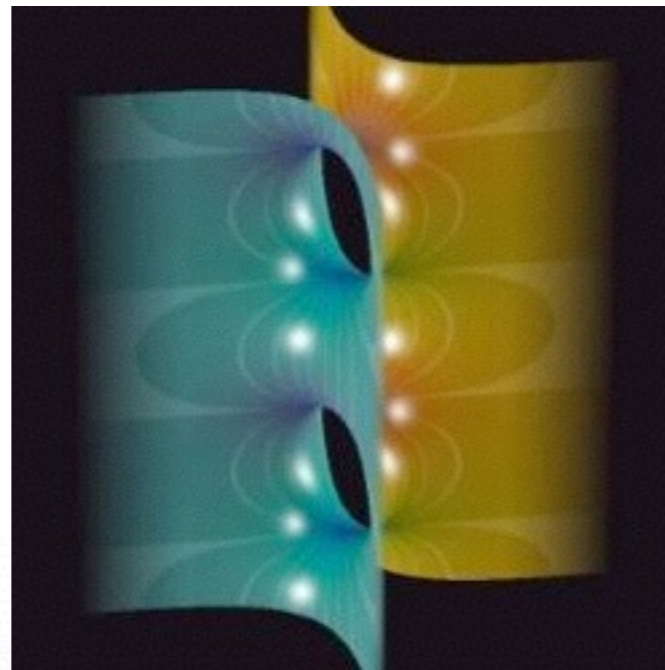
Catenoid



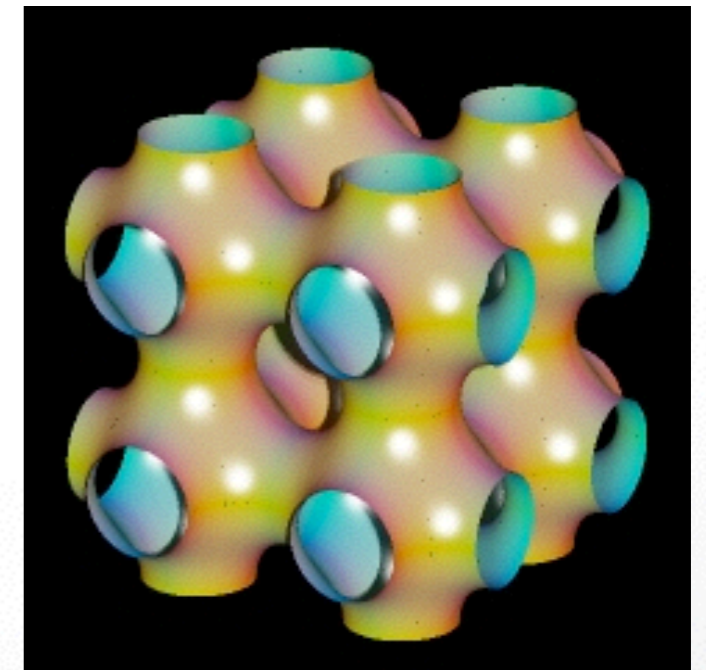
Helicoid



Scherk's First Surface

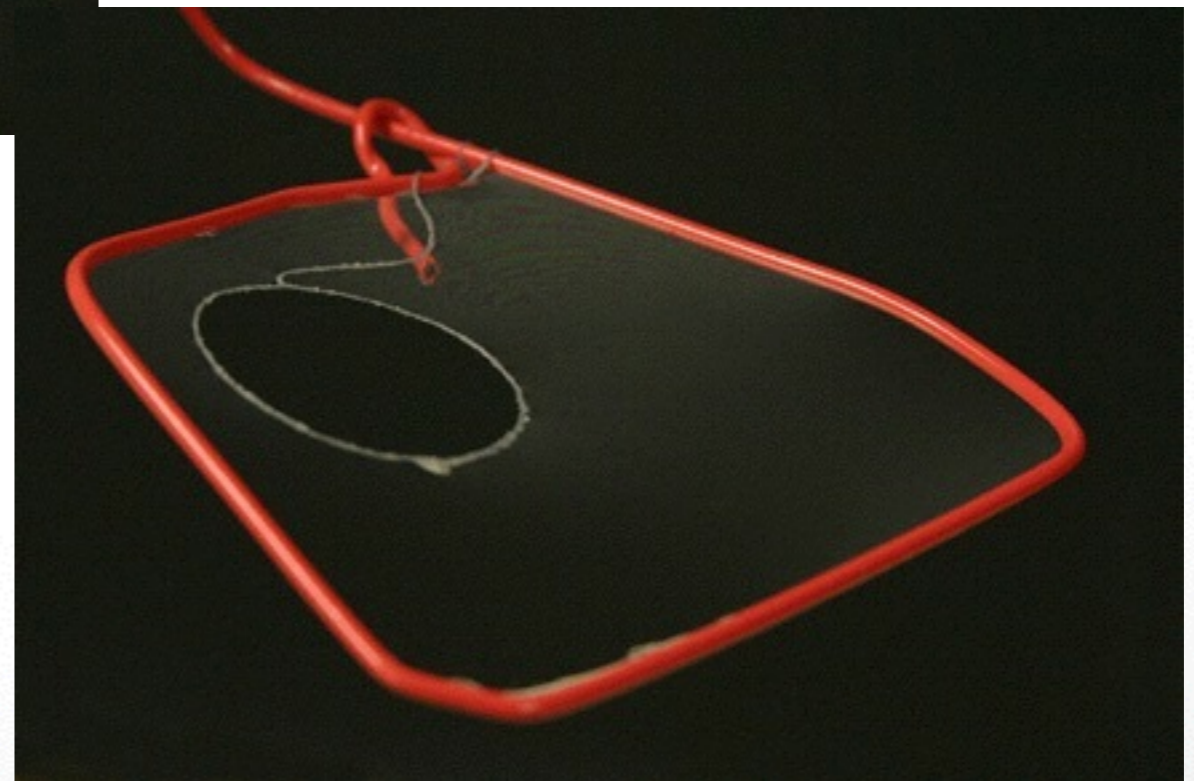
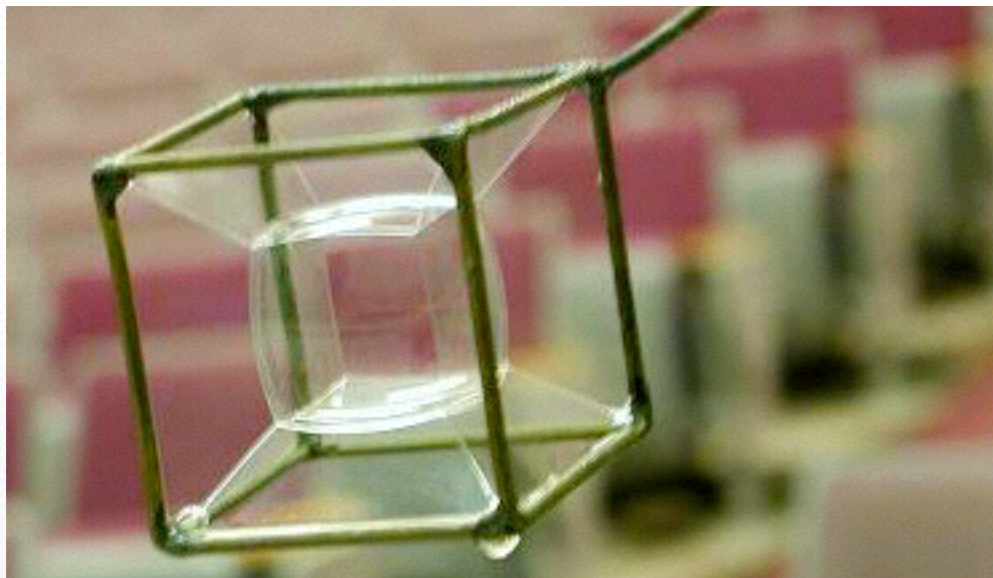
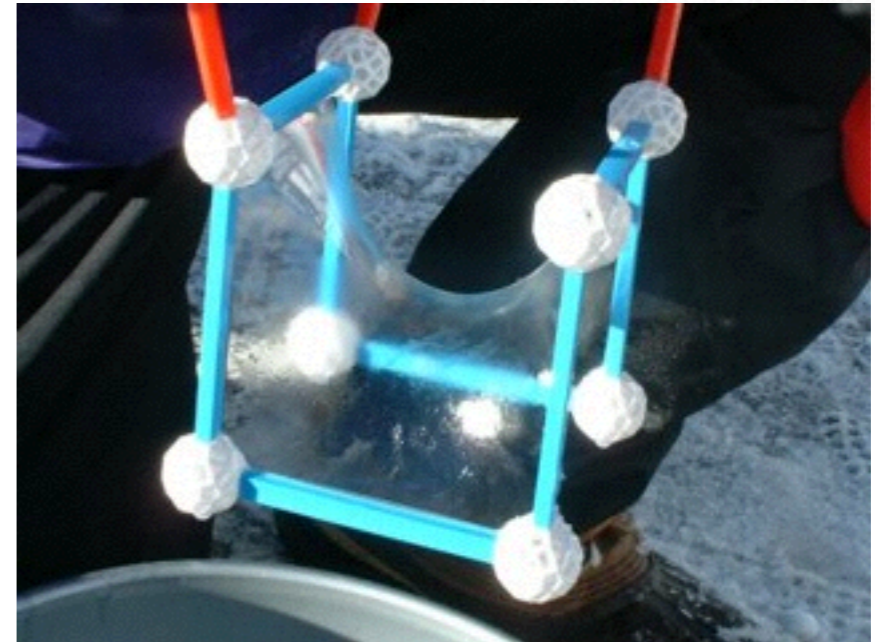
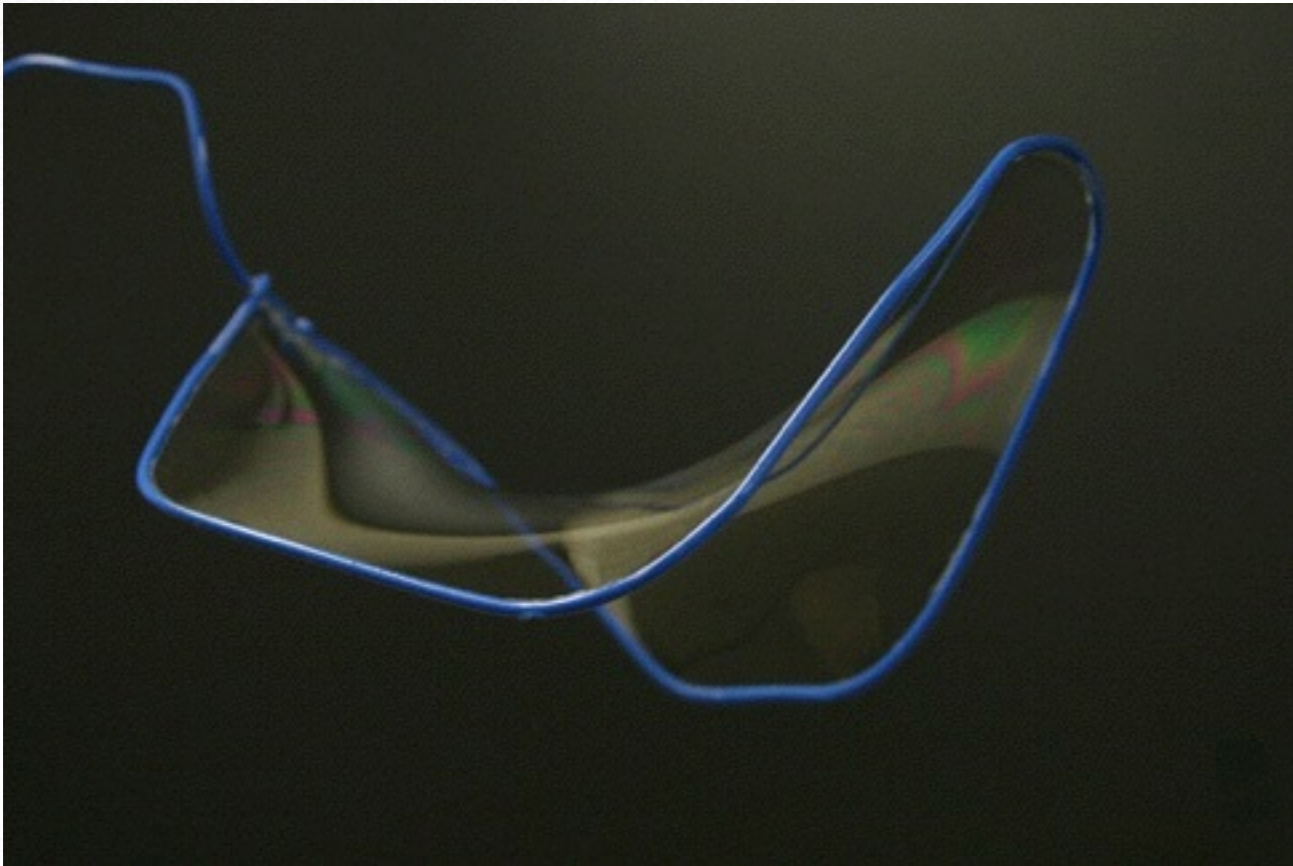


Scherk's Second Surface



Schwarz P Surface

Soap Films



Surface Area

$$\begin{aligned} dA &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, du \, dv \\ \text{infinitesimal} & \\ \text{Area} &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} \, du \, dv \\ &= \sqrt{EG - F^2} \, du \, dv \end{aligned}$$

cross product → determinant with unit vectors → area

Non-Linear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} dA \rightarrow \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

- Thin-plate surface (curvature)

$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 dA \rightarrow \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \quad \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

- Too complex... simplify energies

Membrane Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface Area)

$$\int_{\Omega} \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, dudv \rightarrow \min$$

Variational Calculus in 1D

- 1D membrane energy

$$L(f) = \int_a^b f'^2(x) \, dx \rightarrow \min$$

- Add test function u with $u(a) = u(b) = 0$

$$L(f + \lambda u) = \int_a^b (f' + \lambda u')^2 \, dx = \int_a^b f'^2 + 2\lambda f' u' + \lambda^2 u'^2 \, dx$$

- If f minimizes L , the following has to vanish

$$\left. \frac{\partial L(f + \lambda u)}{\partial \lambda} \right|_{\lambda=0} = \int_a^b 2f' u' \, dx \stackrel{!}{=} 0$$

Variational Calculus in 1D

- Has to vanish for any u with $u(a) = u(b) = 0$

$$\int_a^b f' u' = \underbrace{[f' u]_a^b}_{=0} - \int_a^b f'' u \stackrel{!}{=} 0 \quad \forall u$$

$$\int_0^1 f' g = [fg]_0^1 - \int_0^1 fg'$$

- Only possible if

$$f'' = \Delta f = 0$$

Euler-Lagrange equation

Bivariate Variational Calculus

- Find minimum of functional

$$\operatorname{argmin}_f \int_{\Omega} L(f_{uu}, f_{vv}, f_u, f_v, f, u, v)$$

- Euler-lagrange PDE defines the minimizer

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial u} \frac{\partial L}{\partial f_u} - \frac{\partial}{\partial v} \frac{\partial L}{\partial f_v} + \frac{\partial^2}{\partial u^2} \frac{\partial L}{\partial f_{uu}} + \frac{\partial^2}{\partial u \partial v} \frac{\partial L}{\partial f_{uv}} + \frac{\partial^2}{\partial v^2} \frac{\partial L}{\partial f_{vv}} = 0$$

Again, subject to suitable boundary constraints

Bivariate Variational Calculus

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_{\Omega} \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, dudv \rightarrow \min$$

- Variational calculus

$$\Delta \mathbf{p} = 0$$

Thin-Plate Surface

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

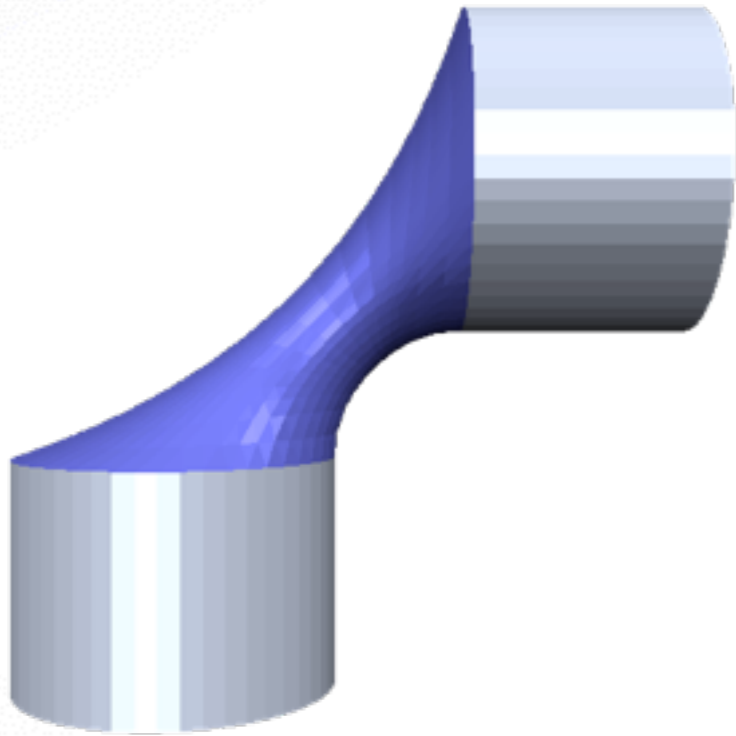
- Thin-plate energy (curvature)

$$\int_{\Omega} \|\mathbf{p}_{uu}\|^2 + 2 \|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \, dudv \rightarrow \min$$

- Variational calculus

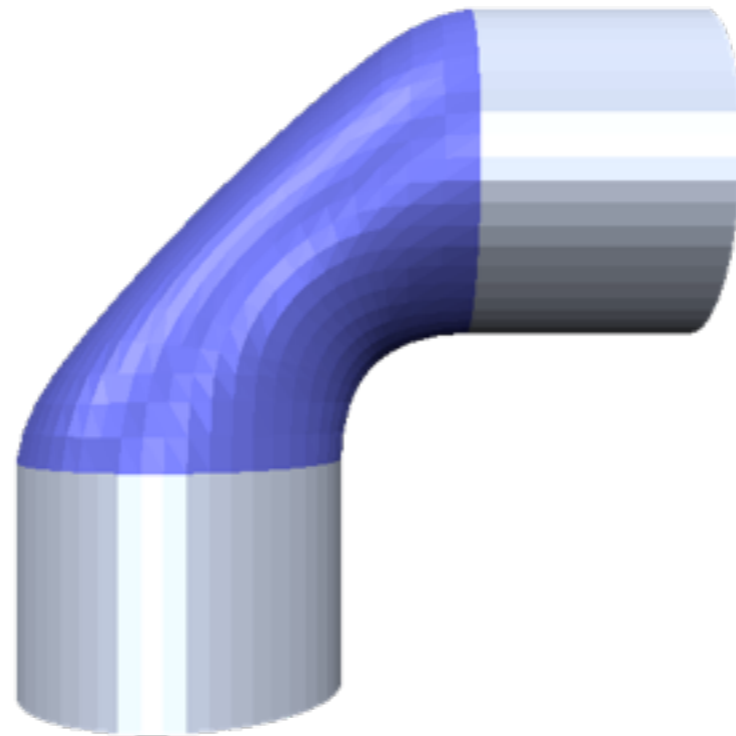
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



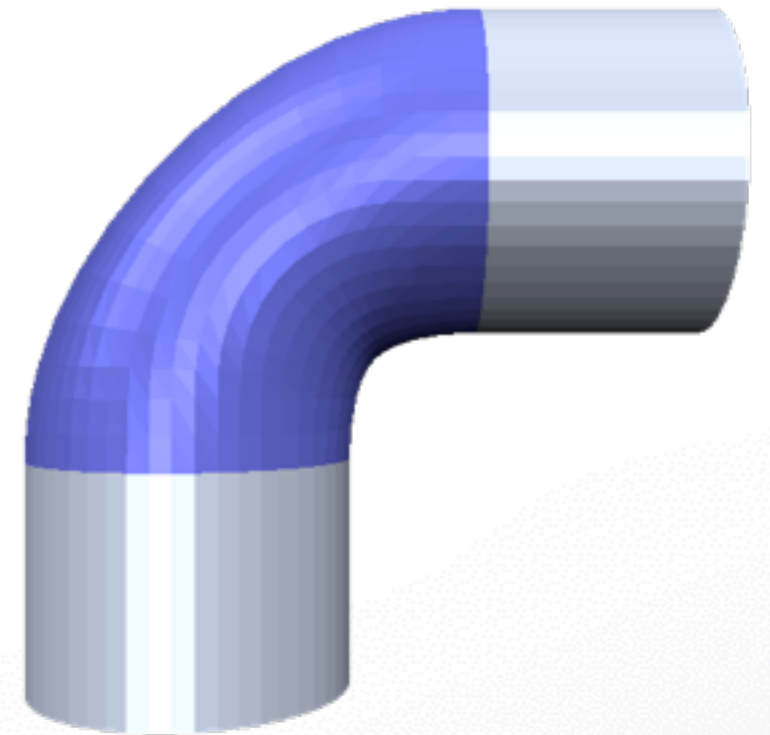
Membrane

$$\Delta_S \mathbf{p} = 0$$



Thin Plate

$$\Delta_S^2 \mathbf{p} = 0$$



$$\Delta_S^3 \mathbf{p} = 0$$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_S^k \mathbf{p} = 0$$

- They are stationary surfaces of Laplacian flow

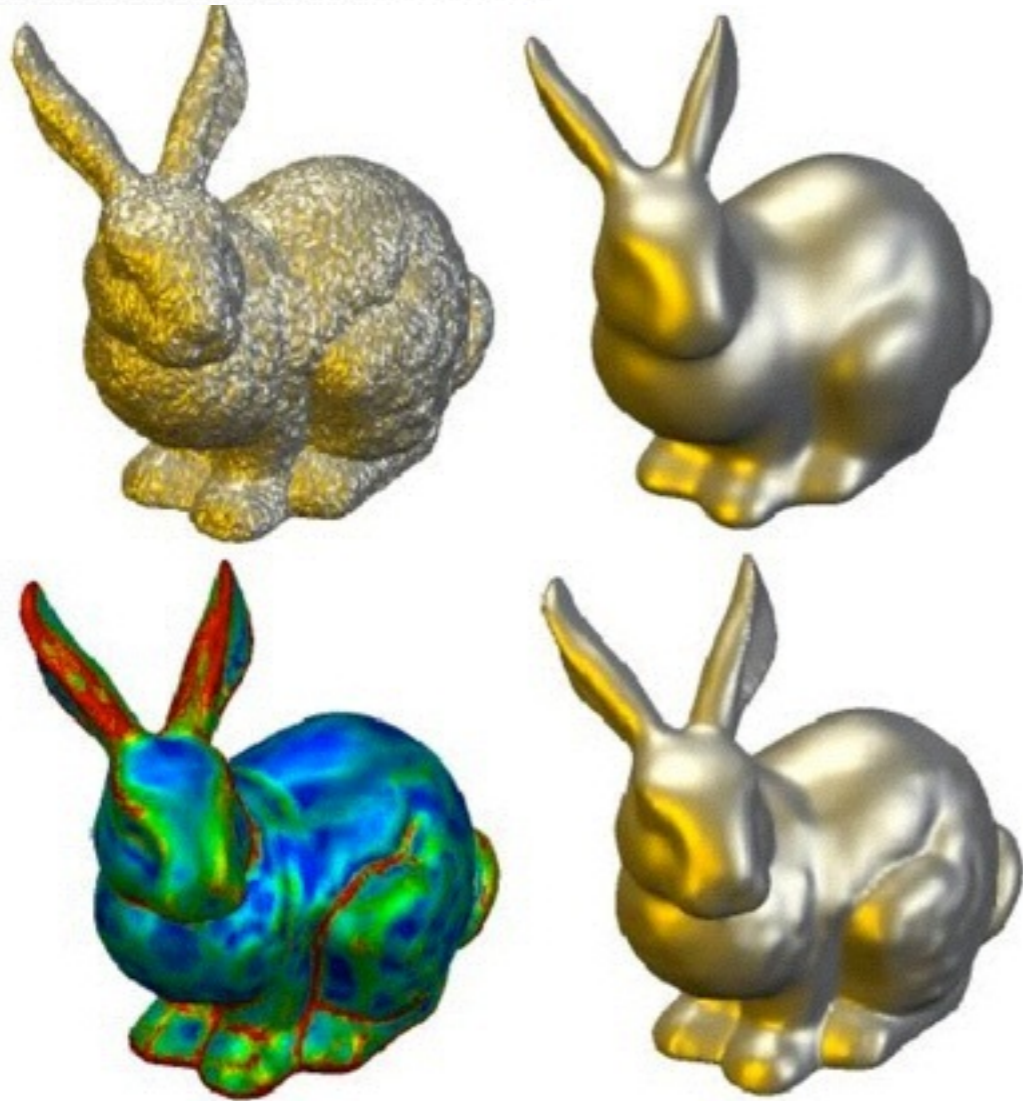
$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_S^k \mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

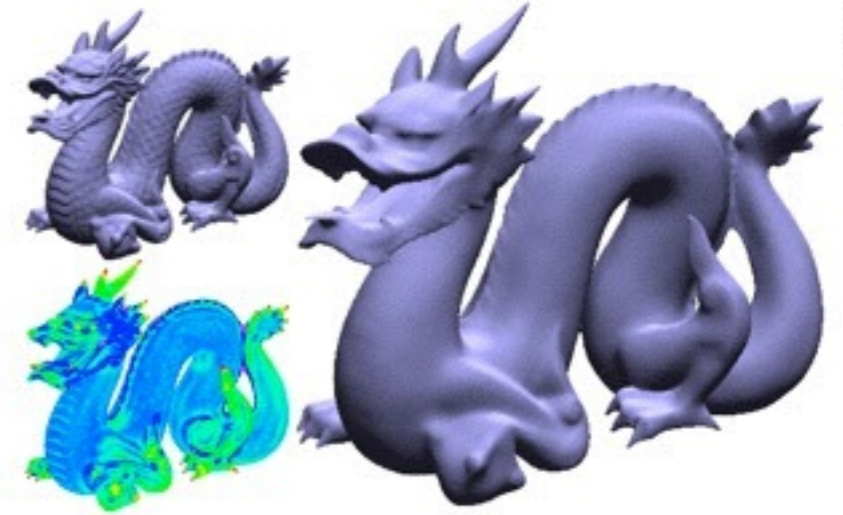
Literature

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- Taubin: A signal processing approach to fair surface design, SIGGRAPH 1996
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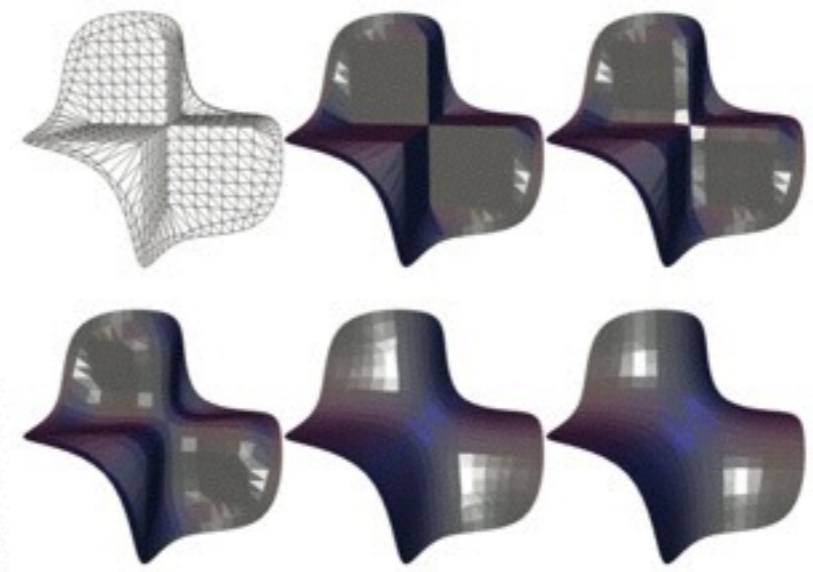
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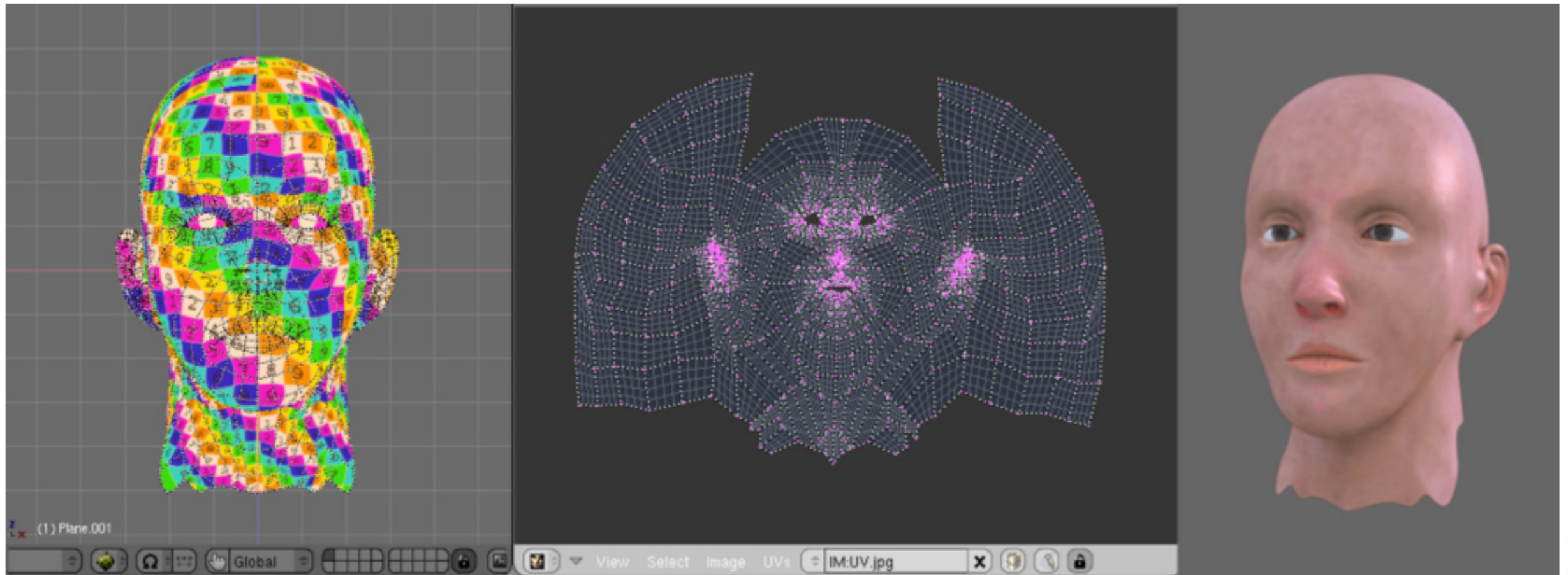


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Next Time



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Parameterization

<http://cs599.hao-li.com>

Thanks!

