Spring 2015

CSCI 599: Digital Geometry Processing

4.1 Discrete Differential Geometry





Outline

- Discrete Differential Operators
- Discrete Curvatures
- Mesh Quality Measures

Differential Operators on Polygons

Differential Properties

- Surface is sufficiently differentiable
- Curvatures \rightarrow 2nd derivatives

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Polygonal Meshes

- Piecewise linear approximations of smooth surface
- Focus on Discrete Laplace Beltrami Operator
- Discrete differential properties defined over $\mathcal{N}(\mathbf{x})$

Local Averaging

Local Neighborhood $\,\mathcal{N}(\mathbf{x})$ of a point \mathbf{x}

- often coincides with mesh vertex v_i
- n-ring neighborhood $\mathcal{N}_n(v_i)$ or local geodesic ball

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Neighborhood size

- Large: smoothing is introduced, stable to noise
- Small: fine scale variation, sensitive to noise

Local Averaging: 1-Ring







Barycentric cell (barycenters/edgemidpoints)

Voronoi cell (circumcenters) tight error bound Mixed Voronoi cell (circumcenters/midpoint) better approximation

Barycentric Cells

Connect edge midpoints and triangle barycenters

- Simple to compute
- Area is 1/3 o triangle areas
- Slightly wrong for obtuse triangles



Mixed Cells

Connect edge midpoints and

- Circumcenters for non-obtuse triangles
- Midpoint of opposite edge for obtuse triangles
- Better approximation, more complex to compute...



Normal Vectors

Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume regular parameterization

$$\mathbf{x}_u imes \mathbf{x}_v
eq \mathbf{0}$$
 normal exists

≬n

 \mathbf{x}_v

 \mathbf{X}_{u}

$$\mathbf{n}(T) = \frac{(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)}{\|(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)\|}$$

$$T = (\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_j)$$

$$\mathbf{n}(T) = \frac{(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)}{\|(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)\|}$$

$$T = (\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_j)$$

$$\mathbf{n}(v) = \frac{\sum_{T \in \mathcal{N}_1(v)} \alpha_T \mathbf{n}(T)}{\left\| \sum_{T \in \mathcal{N}_1(v)} \alpha_T \mathbf{n}(T) \right\|}$$

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$$\alpha_T = 1 \qquad \qquad \alpha_T = |T| \qquad \qquad \alpha_T = \theta_T$$

tessellated cylinder

$$\alpha_T = 1$$
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Simple Curvature Discretization

Simple Curvature Discretization

Laplace Operator on Meshes

Extend finite differences to meshes?

• What weights per vertex/edge?

Uniform Laplace

Uniform discretization

• What weights per vertex/edge?

Properties

- depends only on connectivity
- simple and efficient

Uniform Laplace

Uniform discretization

$$\Delta_{\mathrm{uni}} \mathbf{x}_i := \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (\mathbf{x}_j - \mathbf{x}_i) \approx -2H\mathbf{n}$$

Properties

- depends only on connectivity
- simple and efficient
- bad approximation for irregular triangulations
 - can give non-zero H for planar meshes
 - tangential drift for mesh smoothing

Discrete Gradient of a Function

- Defined on piecewise linear triangle
- Important for parameterization and deformation

triangle $(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)$

piecewise linear function $f(\mathbf{u}) = f_i B_i(\mathbf{u}) + f_j B_j(\mathbf{u}) + f_k B_k(\mathbf{u})$ $\mathbf{u} = (u, v)$ $f_i = f(\mathbf{x}_i)$

linear basis functions for barycentric interpolation on a triangle

piecewise linear function $f(\mathbf{u}) = f_i B_i(\mathbf{u}) + f_j B_j(\mathbf{u}) + f_k B_k(\mathbf{u})$ $\mathbf{u} = (u, v)$

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gradient of linear function $\nabla f(\mathbf{u}) = f_i \nabla B_i(\mathbf{u}) + f_j \nabla B_j(\mathbf{u}) + f_k \nabla B_k(\mathbf{u})$

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partition of unity $B_i(\mathbf{u}) + B_j(\mathbf{u}) + B_k(\mathbf{u}) = 1$

gradients of basis $\nabla B_i(\mathbf{u}) + \nabla B_j(\mathbf{u}) + \nabla B_k(\mathbf{u}) = 0$

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gradient of linear function $\nabla f(\mathbf{u}) = (f_j - f_i)\nabla B_j(\mathbf{u}) + (f_k - f_i)\nabla B_k(\mathbf{u})$

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with appropriate normalization: $\nabla B_i(\mathbf{u}) = \frac{(\mathbf{x}_k - \mathbf{x}_j)^{\perp}}{2A_T}$

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discrete gradient of a piecewiese linear function within T

Laplace-Beltrami

divergence theorem

$$\int_{A_i} \operatorname{div} \mathbf{F}(\mathbf{u}) \, \mathrm{d}A = \int_{\partial A_i} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \, \mathrm{d}s$$

divergence theorem

$$\int_{A_i} \operatorname{div} \mathbf{F}(\mathbf{u}) \, \mathrm{d}A = \int_{\partial A_i} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \, \mathrm{d}s$$

vector-valued function ${f F}$ local averaging domain $A_i = A(v_i)$ boundary ∂A_i

divergence theorem

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$$I) ds$$

$$\int_{A_i} \Delta f(\mathbf{u}) \, \mathrm{d}A = \int_{A_i} \mathrm{div} \nabla f(\mathbf{u}) \, \mathrm{d}A = \int_{\partial A_i} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \, \mathrm{d}s$$

average Laplace-Beltrami

$$\int_{A_i} \Delta f(\mathbf{u}) \, \mathrm{d}A \ = \ \int_{A_i} \mathrm{div} \nabla f(\mathbf{u}) \, \mathrm{d}A \ = \ \int_{\partial A_i} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \, \mathrm{d}s$$

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gradient is constant and local Voronoi passes through a,b:

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gradient is constant and local Voronoi passes through a,b:

$$\begin{split} & \int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \mathrm{d}s \ = \ \nabla f(\mathbf{u}) \cdot (\mathbf{a} - \mathbf{b})^{\perp} \\ & \text{over triangle} \end{split} = \ & \frac{1}{2} \nabla f(\mathbf{u}) \cdot (\mathbf{x}_j - \mathbf{x}_k)^{\perp} \end{split}$$

discrete gradient

$$\nabla f(\mathbf{u}) = (f_j - f_i) \frac{(\mathbf{x}_i - \mathbf{x}_k)^{\perp}}{2A_T} + (f_k - f_i) \frac{(\mathbf{x}_j - \mathbf{x}_i)^{\perp}}{2A_T}$$

average Laplace-Beltrami within a triangle

$$\int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) ds = (f_j - f_i) \frac{(\mathbf{x}_i - \mathbf{x}_k)^{\perp} \cdot (\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{4A_T} \\ + (f_k - f_i) \frac{(\mathbf{x}_j - \mathbf{x}_i)^{\perp} \cdot (\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{4A_T}$$

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 $\int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \mathrm{d}s = \frac{1}{2} \left(\cot \gamma_k (f_j - f_i) + \cot \gamma_j (f_k - f_i) \right)$

average Laplace-Beltrami over averaging region

$$\int_{A_i} \Delta f(\mathbf{u}) dA = \frac{1}{2} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{i,j} + \cot \beta_{i,j}) (f_j - f_i)$$

average Laplace-Beltrami over averaging region

$$\int_{A_i} \Delta f(\mathbf{u}) dA = \frac{1}{2} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{i,j} + \cot \beta_{i,j}) (f_j - f_i)$$

$$\Delta f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{i,j} + \cot \beta_{i,j}) (f_j - f_i)$$

Cotangent discretization

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} \left(\cot \alpha_{ij} + \cot \beta_{ij} \right) \left(f(v_j) - f(v_i) \right)$$

Cotangent discretization

$$\Delta_{\mathcal{S}} f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} \left(\cot \alpha_i + \cot \beta_i \right) \left(f(v_i) - f(v) \right)$$

Problems

- weights can become negative
- depends on triangulation

Still the most widely used discretization

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- Discrete Differential Operators
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- Mesh Quality Measures

How to discretize curvature on a mesh?

- Zero curvature within triangles
- Infinite curvature at edges / vertices
- Point-wise definition doesn't make sense

Approximate differential properties at point x as average over local neighborhood A(x)

- x is a mesh vertex
- $A(\mathbf{x})$ within one-ring neighborhood

How to discretize curvature on a mesh?

- Zero curvature within triangles
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Approximate differential properties at point x as average over local neighborhood A(x)

$$K(v) \approx \frac{1}{A(v)} \int_{A(v)} K(\mathbf{x}) \, \mathrm{d}A$$

Which curvatures to discretize?

- Discretize Laplace-Beltrami operator
- Laplace-Beltrami gives us mean curvature H
- Discretize Gaussian curvature K
- From H and K we can compute κ_1 and κ_2

 $\begin{array}{ll} \text{mean}\\ \text{curvature}\\ \end{array}$ Laplace-Beltrami $\Delta_{\mathcal{S}}\mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}}\mathbf{x} = -2H\mathbf{n} \end{array}$

Discrete Gaussian Curvature

Gauss-Bonnet

$$\int K = 2\pi\chi \qquad \qquad \chi = 2 - 2g$$

Discrete Gauss Curvature

$$K = (2\pi - \sum_{j} \theta_{j})/A$$

Verify via Euler-Poincaré

$$V - E + F = 2(1 - g)$$

Mean curvature (absolute value)

$$H = \frac{1}{2} \left\| \Delta_{\mathcal{S}} \mathbf{x} \right\|$$

Gaussian curvature

$$K = (2\pi - \sum_{j} \theta_{j})/A$$

Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - K} \qquad \kappa_2 = H - \sqrt{H^2 - K}$$

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Visual inspection of "sensitive" attributes

• Specular shading

- Specular shading
- Reflection lines

- Specular shading
- Reflection lines
 - differentiability one order lower than surface
 - can be efficiently computed using GPU

- Specular shading
- Reflection lines
- Curvature
 - Mean curvature

- Specular shading
- Reflection lines
- Curvature
 - Gauss curvature

Smoothness

• Low geometric noise

Smoothness

• Low geometric noise

Fairness

• Simplest shape

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• Low geometric noise

Fairness

• Simplest shape

Adaptive tesselation

• Low complexity

Smoothness

• Low geometric noise

Fairness

• Simplest shape

Adaptive tesselation

• Low complexity

Triangle shape

Numerical Robustness

Mesh Optimization

Smoothness

Smoothing

Fairness

• Fairing

Adaptive tesselation

• Decimation

Triangle shape

Remeshing

Summary

Invariants as overarching theme

- shape does not depend on Euclidean motions (no stretch)
 - metric & curvatures
- smooth continuous notions to discrete notions
 - generally only as **averages**
- different ways to derive same equations
 - DEC: discrete exterior calculus, FEM, abstract measure theory.

Literature

- Book: Chapter 3
- Taubin: A signal processing approach to fair surface design, SIGGRAPH 1996
- Desbrun et al. : Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow, SIGGRAPH 1999
- Meyer et al.: Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, VisMath 2002
- Wardetzky et al.: Discrete Laplace Operators: No free lunch, SGP 2007

Next Time

3D Scanning

http://cs599.hao-li.com

Thanks!

