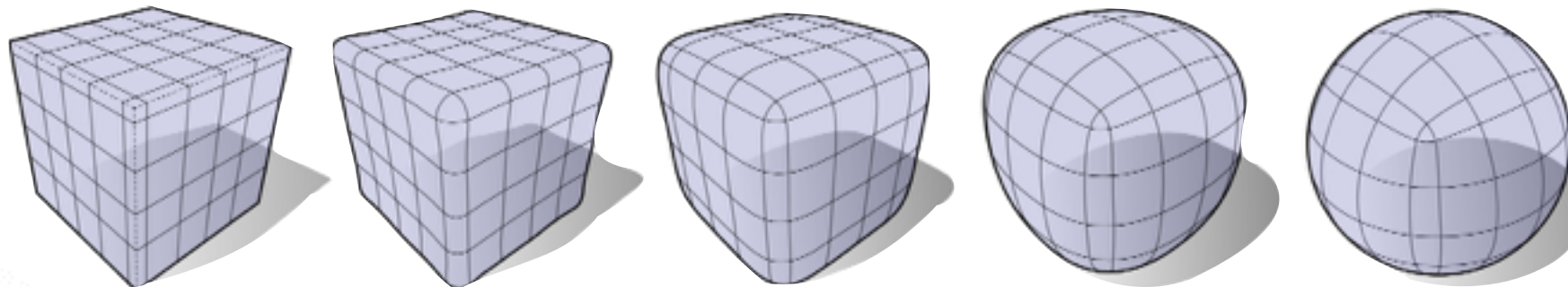


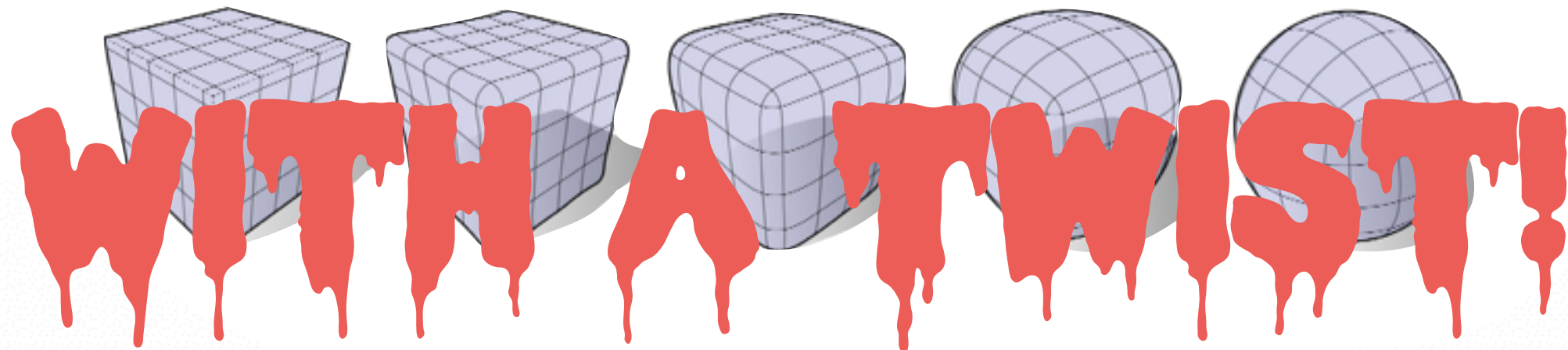
## 3.1 Classic Differential Geometry



Hao Li

<http://cs599.hao-li.com>

## 3.1 Classic Differential Geometry



Hao Li

<http://cs599.hao-li.com>

# Administrative

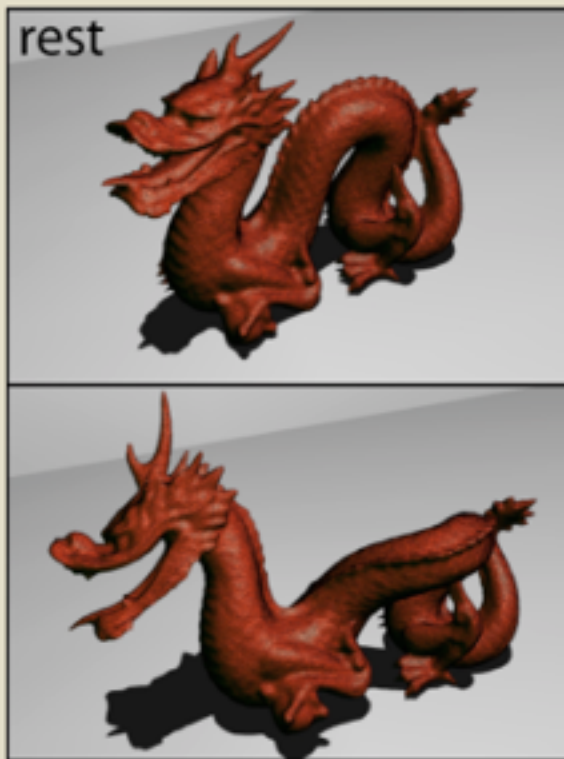
- **Exercise handouts:** 11:59 PM on Monday
- Office hours later from 2pm to 3pm

# Some Updates: [run.usc.edu/vega](http://run.usc.edu/vega)

Another awesome free library with half-edge data-structure

By Prof. Jernej Barbic

## Vega FEM

[MAIN](#)[DOWNLOAD/FAQ](#)[SCREENSHOTS](#)[ABOUT](#)

JURIJ VEGA (1754-1802)



## VEGA FEM LIBRARY

**USC**  
**Viterbi**  
School of Engineering

**NEW:** Vega FEM 2.0 released on Oct 8, 2013. New features described below.

Vega is a computationally efficient and stable C/C++ physics library for three-dimensional deformable object simulation. It is designed to model large deformations, including geometric and material nonlinearities, and can also efficiently simulate linear systems. Vega is open-source and free. It is released under the **3-clause BSD license**, which means that it can be used freely both in academic research and in commercial applications.

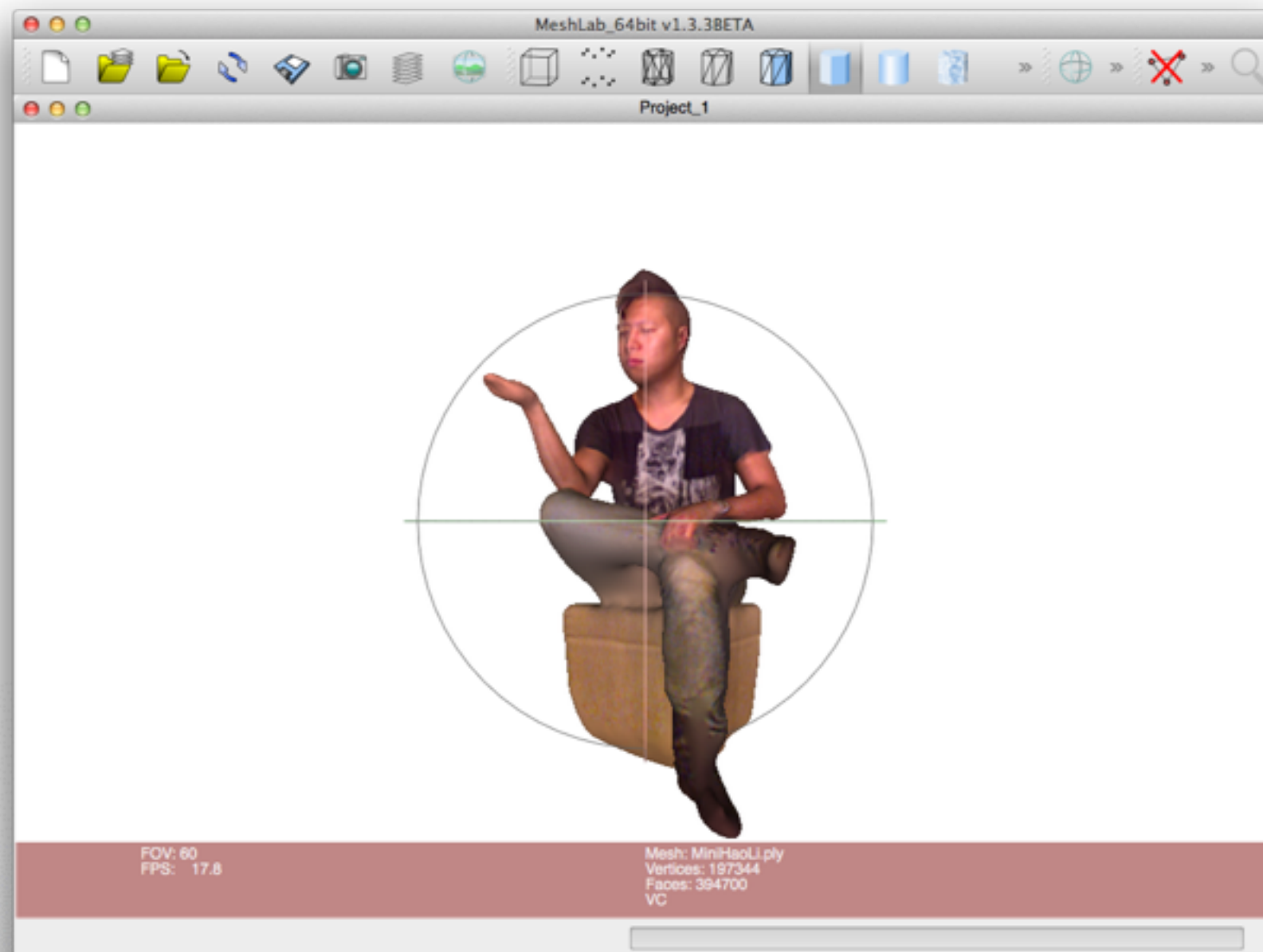
Vega implements several widely used methods for simulation of large deformations of 3D **solid** deformable objects:

- co-rotational linear FEM elasticity [MG04]; it can also compute the exact tangent stiffness matrix [Bar12] (similar to [CPSS10]),
- linear FEM elasticity [Sha90],
- invertible isotropic nonlinear FEM models [ITF04, TSIF05],

# FYI

## MeshLab

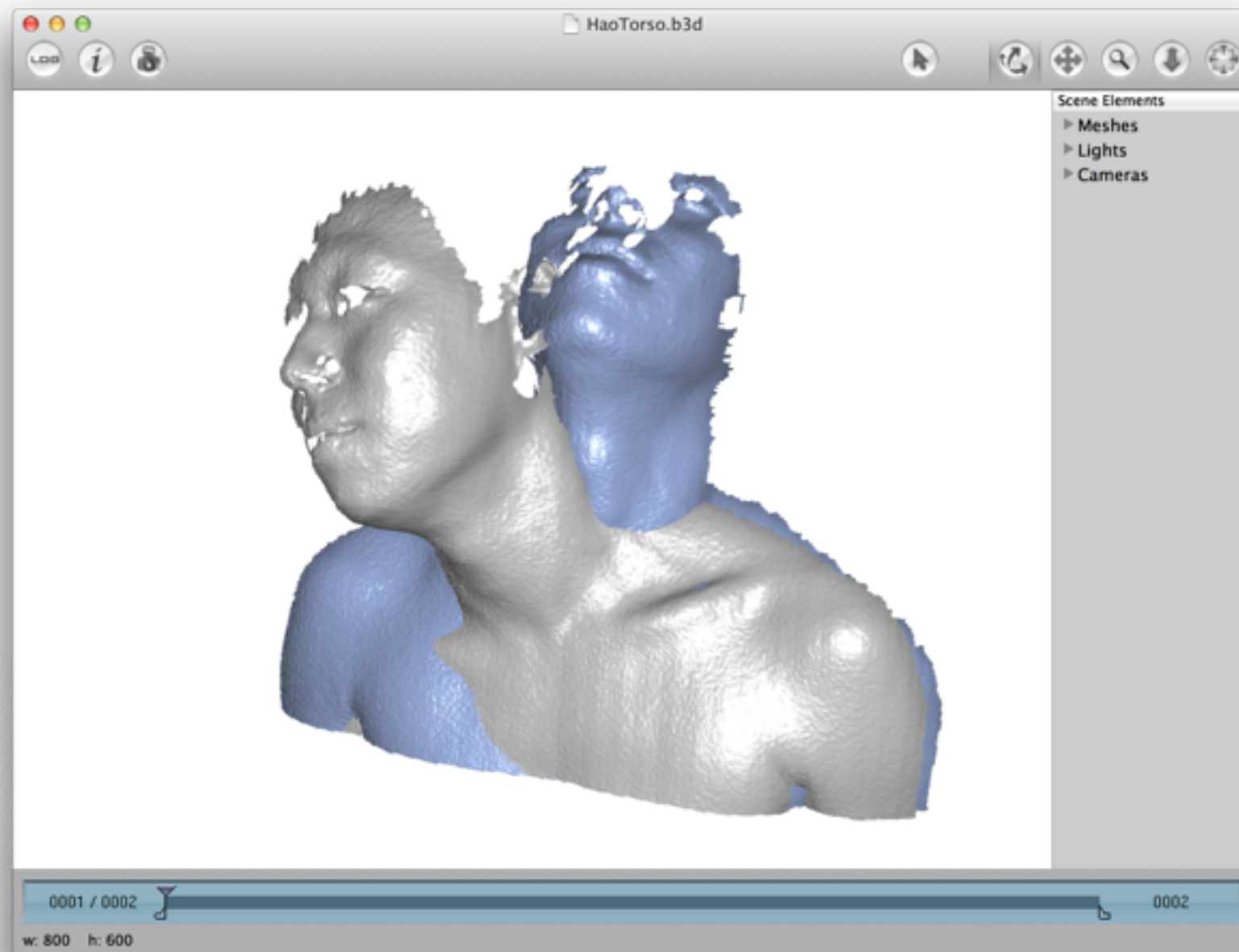
Popular Mesh Processing Software ([meshlab.sourceforge.net](http://meshlab.sourceforge.net))



# FYI

## BeNTO3D

Mesh Processing Framework for Mac ([www.bento3d.com](http://www.bento3d.com))



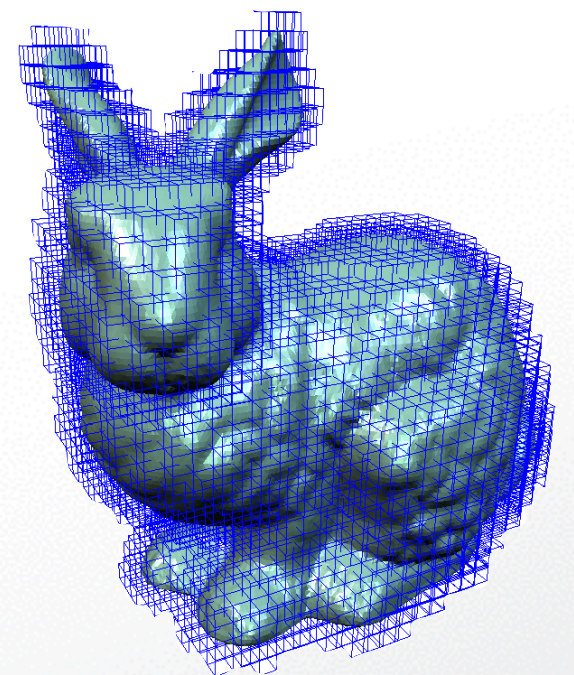
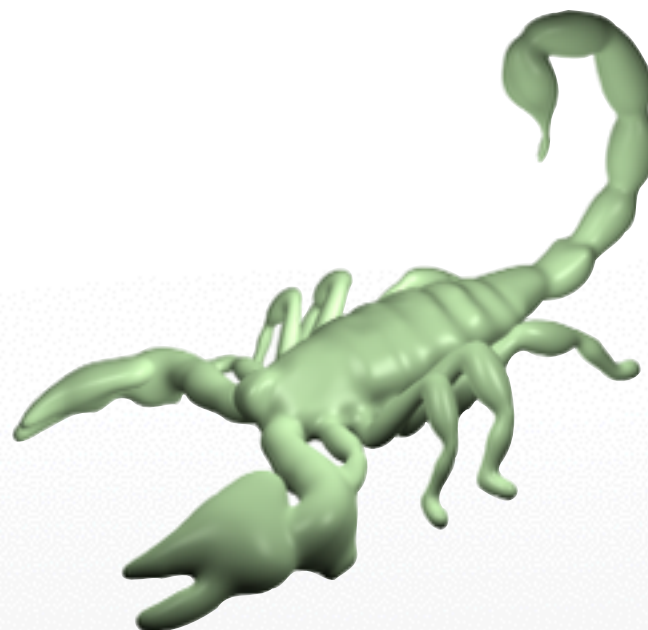
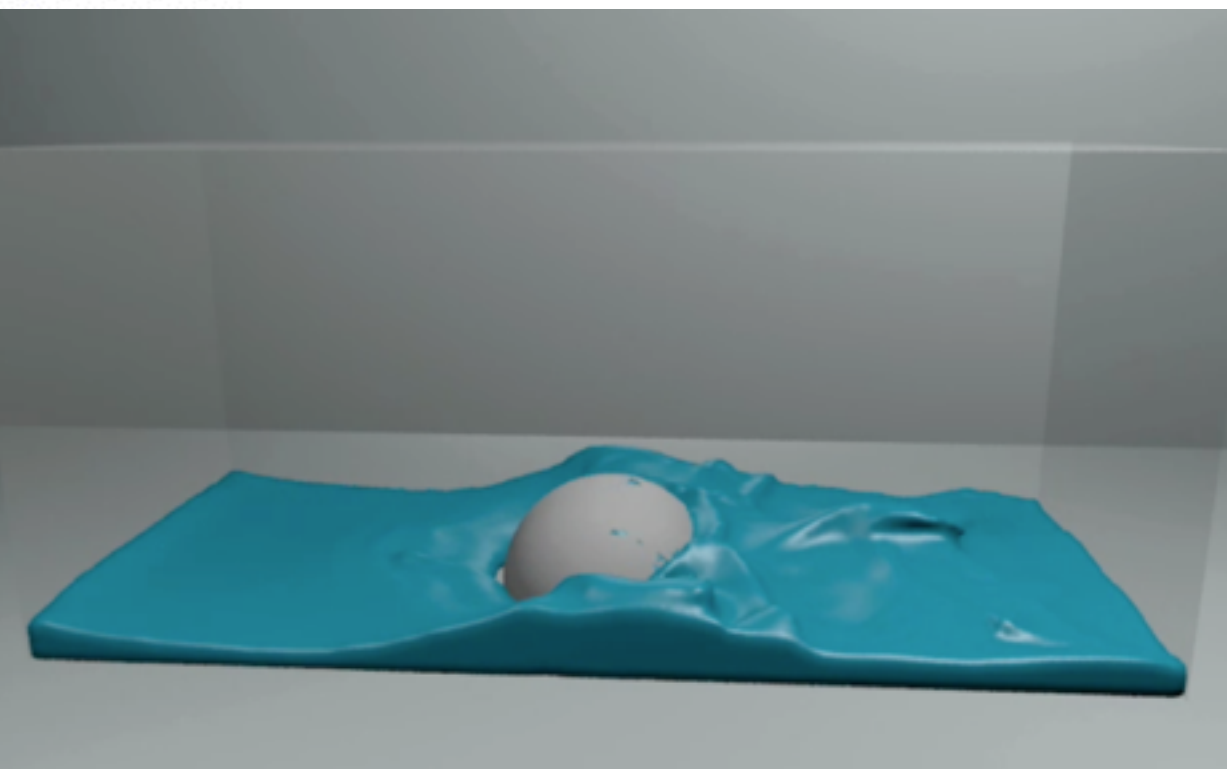
# Last Time

## Discrete Representations

- Explicit (parametric, polygonal meshes)
- Implicit Surfaces (SDF, grid representation)
- Conversions
  - $E \rightarrow I$ : Closest Point, SDF, Fast Marching
  - $I \rightarrow E$ : Marching Cubes Algorithm

Geometry

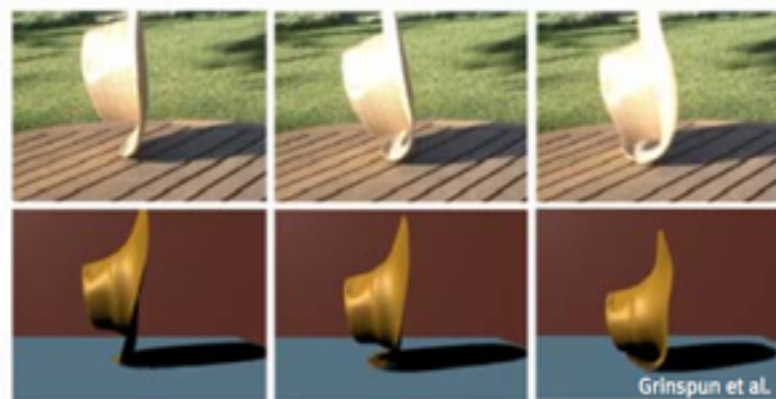
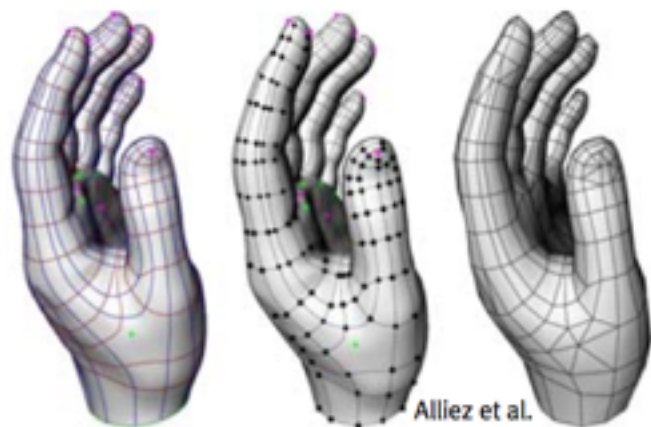
Topology



# Differential Geometry

## Why do we care?

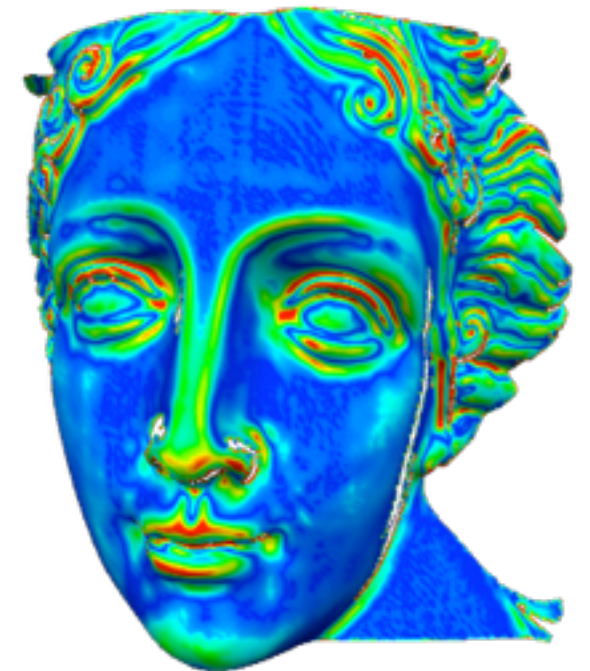
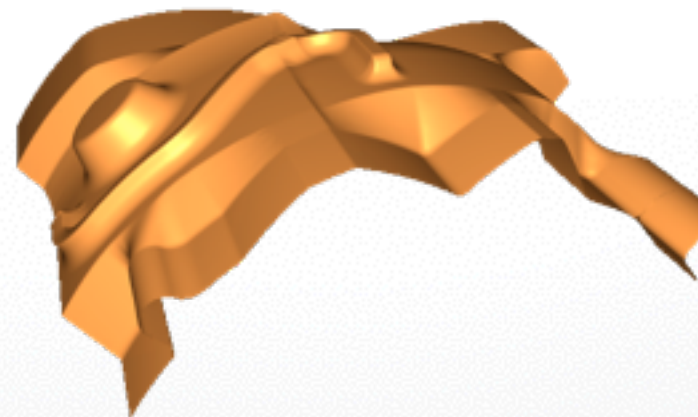
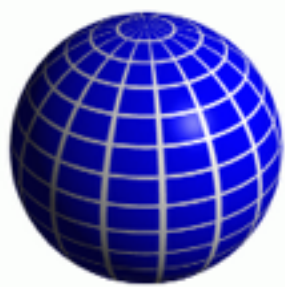
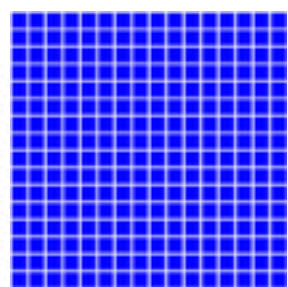
- Geometry of surfaces
- Mother tongue of physical theories
- Computation: processing / simulation



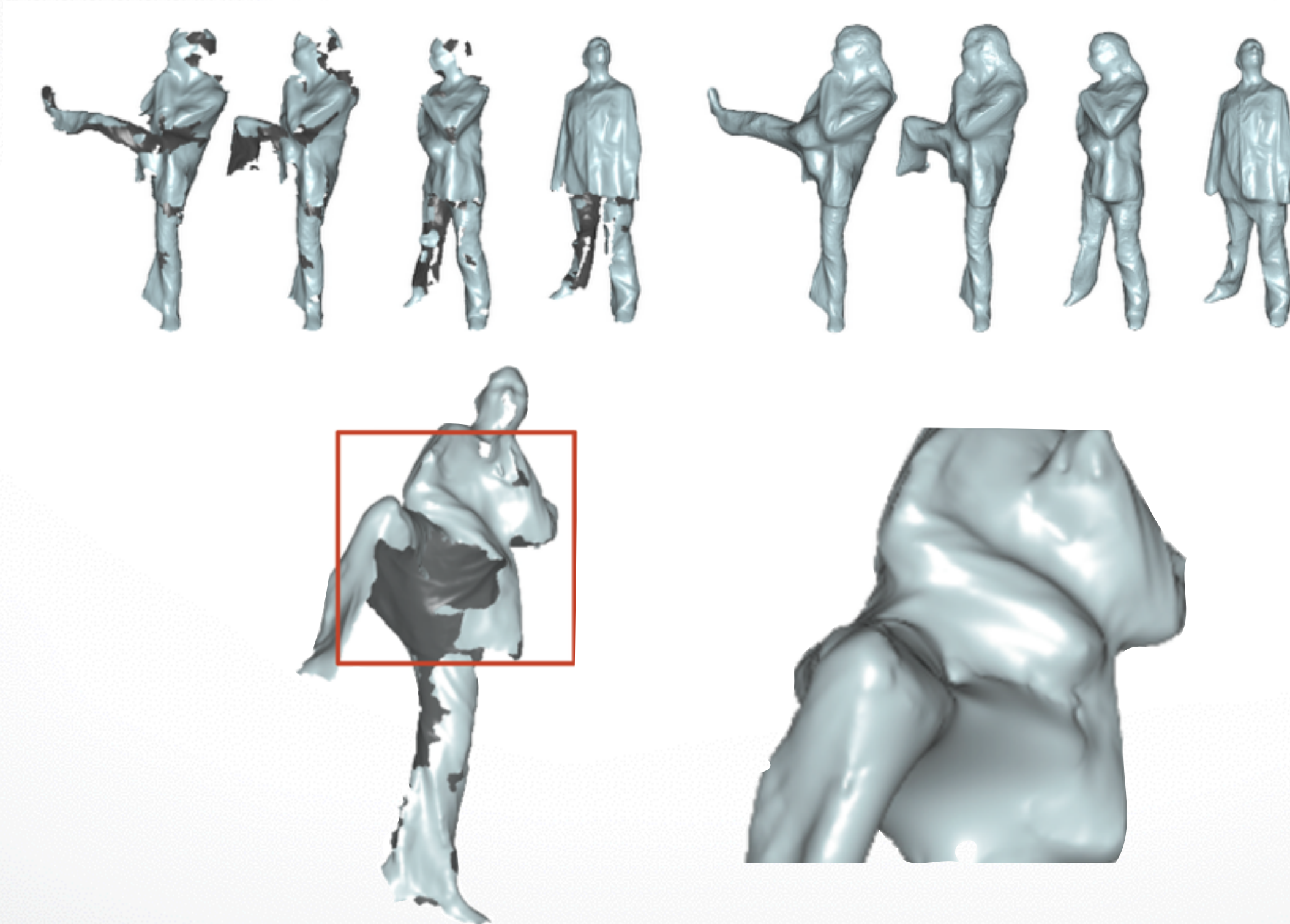
# Motivation

## We need differential geometry to compute

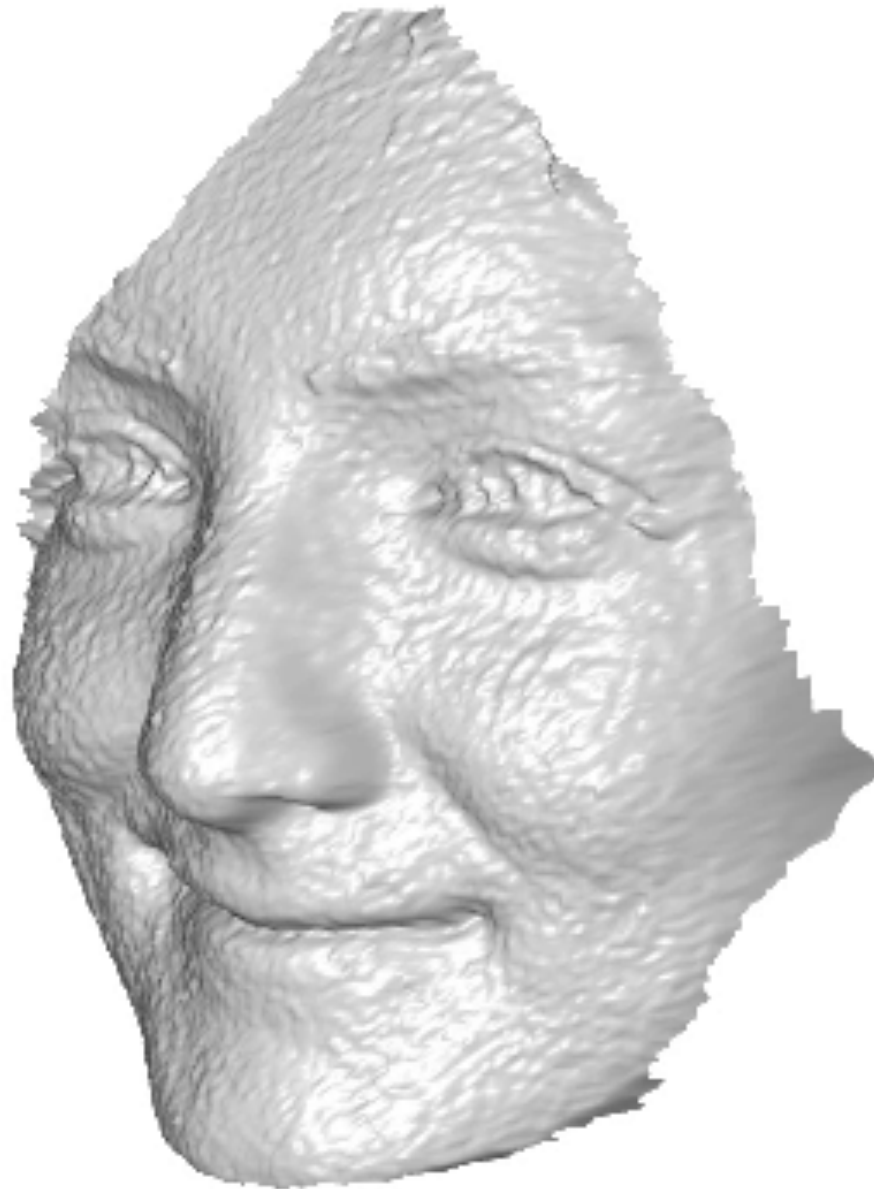
- surface curvature
- parameterization distortion
- deformation energies



# Applications: 3D Reconstruction



# Applications: Head Modeling



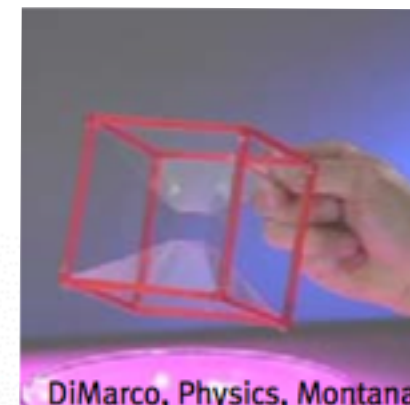
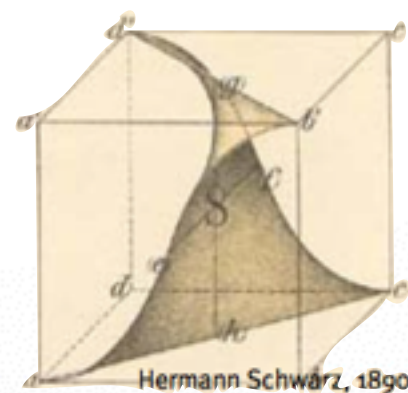
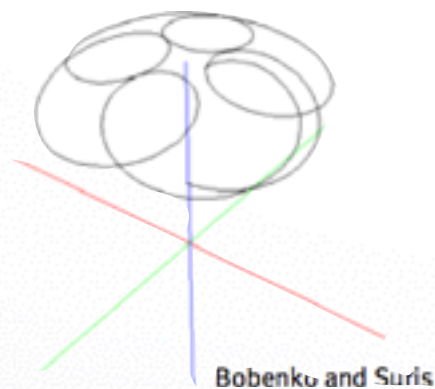
# Applications: Facial Animation



# Motivation

## Geometry is the key

- studied for centuries (Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether...)
- mostly differential geometry
  - differential and integral calculus
- invariants and symmetries



# Getting Started

## How to apply DiffGeo ideas?

- surfaces as a collection of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - **discrete** vs. **discretized**

**Let's look at that first**

# Getting Started

## What characterizes structure(s)?

- What is shape?
  - Euclidean Invariance
- What is physics?
  - Conservation/Balance Laws
- What can we measure?
  - area, curvature, mass, flux, circulation



# Getting Started

## Invariant descriptors

- quantities invariant under a set of transformations

## Intrinsic descriptor

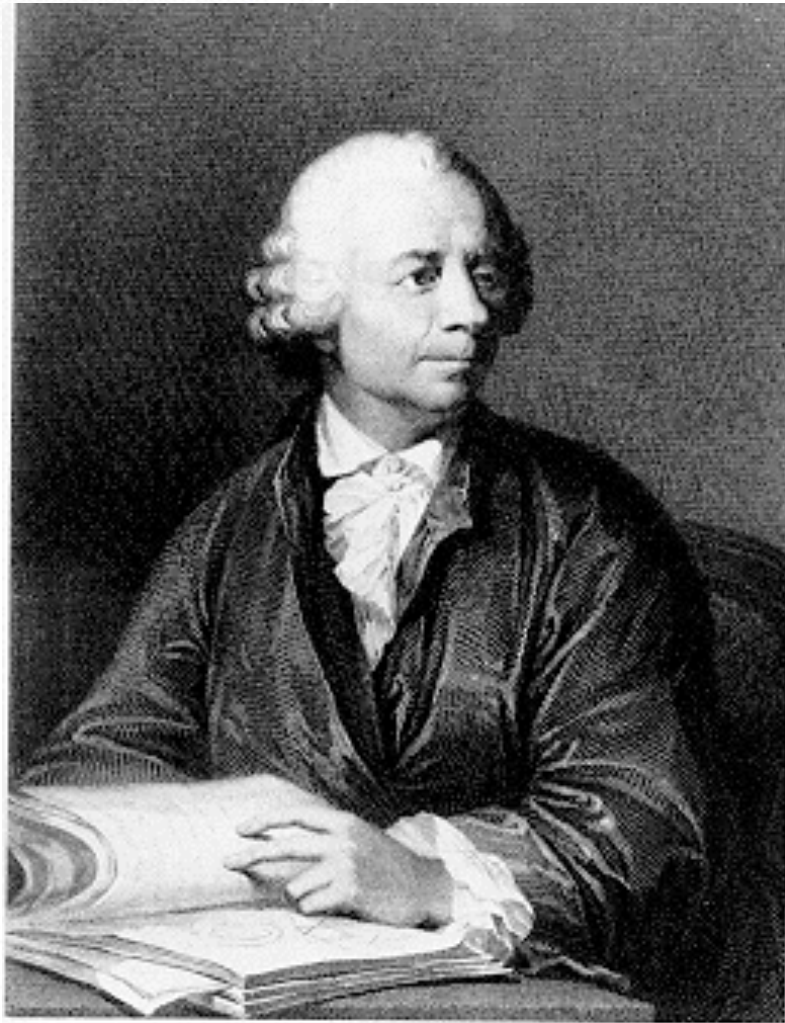
- quantities which do not depend on a coordinate frame

# Outline

- **Parametric Curves**
- Parametric Surfaces

**Formalism & Intuition**

# Differential Geometry



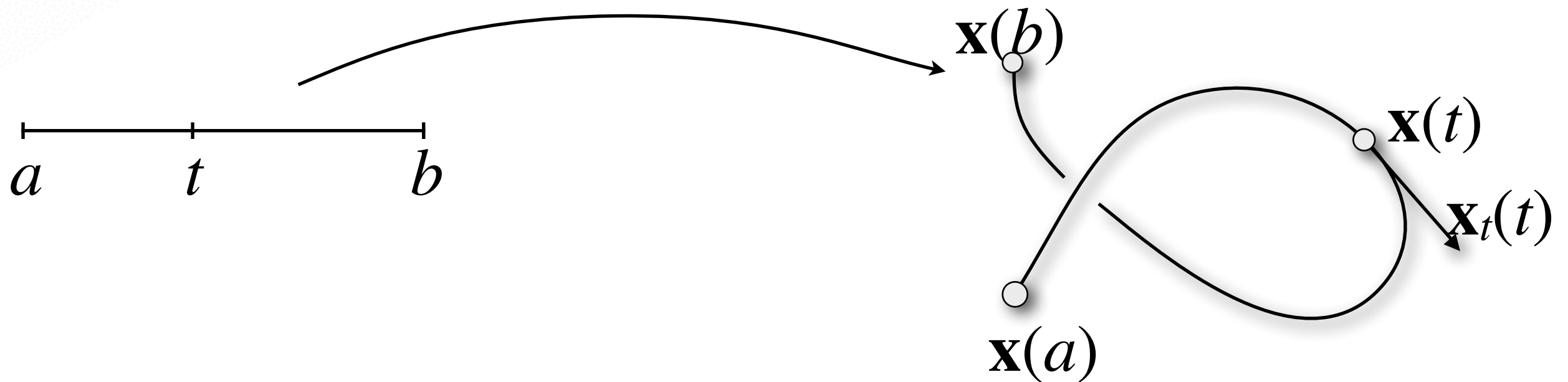
Leonard Euler (1707-1783)



Carl Friedrich Gauss (1777-1855)

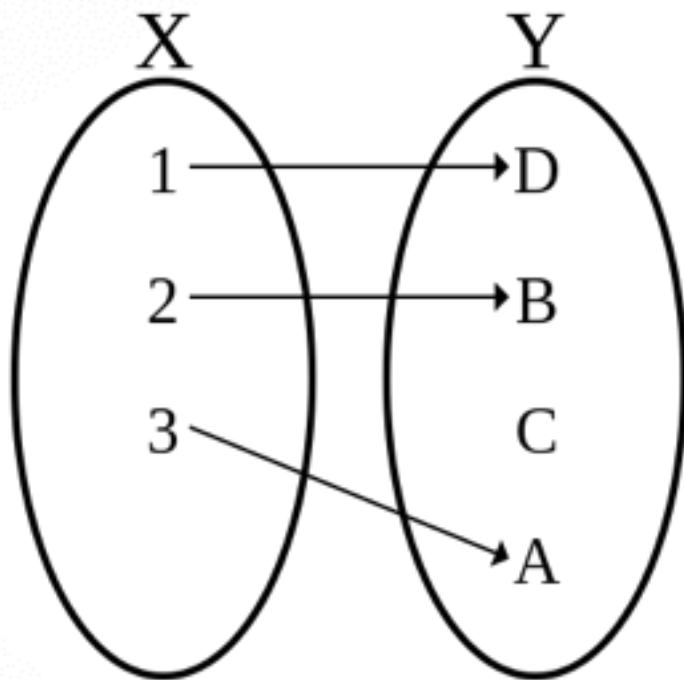
# Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$

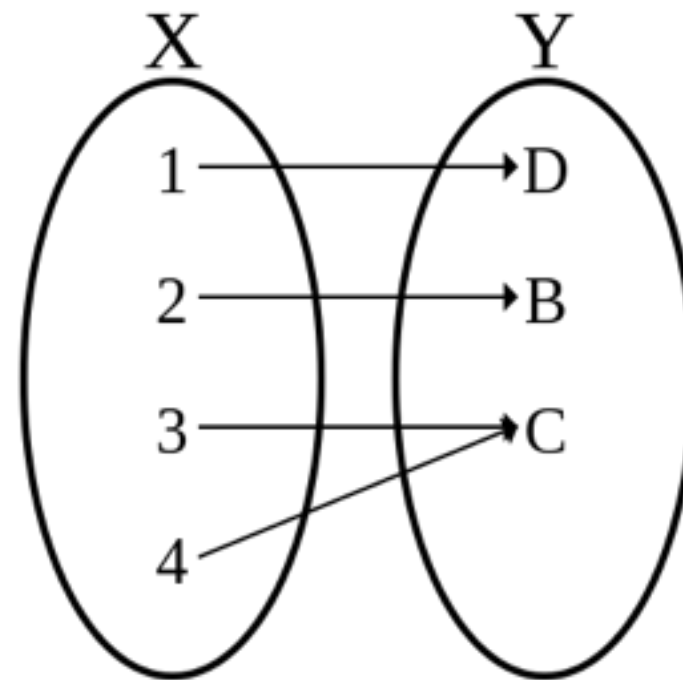


$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \mathbf{x}_t(t) := \frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} dx(t)/dt \\ dy(t)/dt \\ dz(t)/dt \end{pmatrix}$$

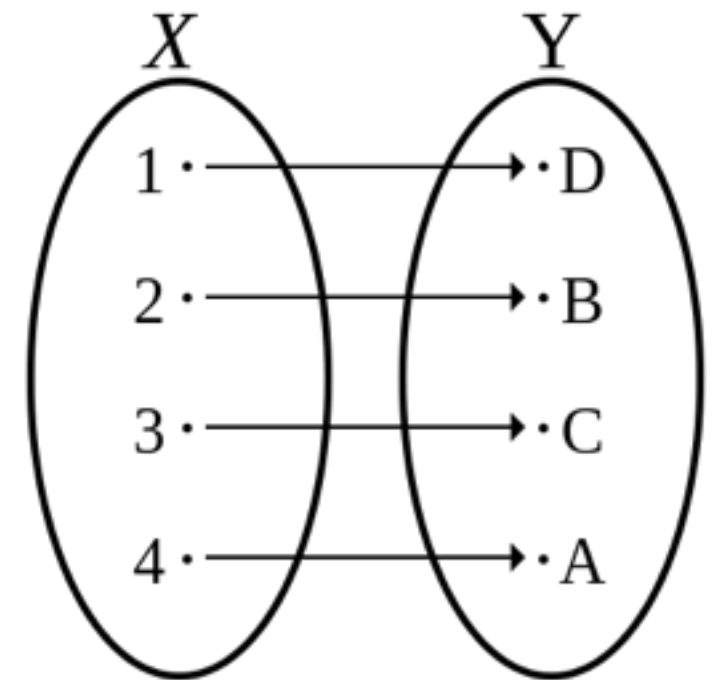
# Recall: **Mappings**



Injective



Surjective



Bijjective

**NO SELF-INTERSECTIONS**

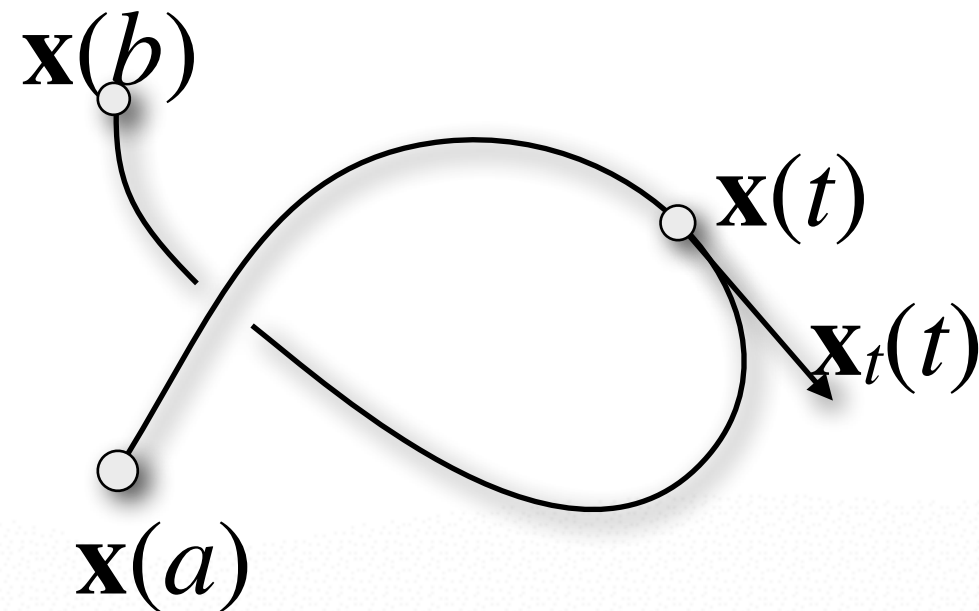
**SELF-INTERSECTIONS**

**AMBIGUOUS PARAMETERIZATION**

# Parametric Curves

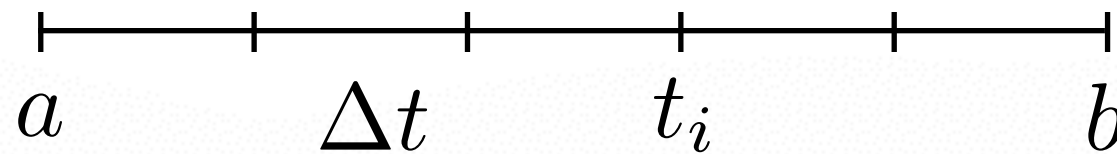
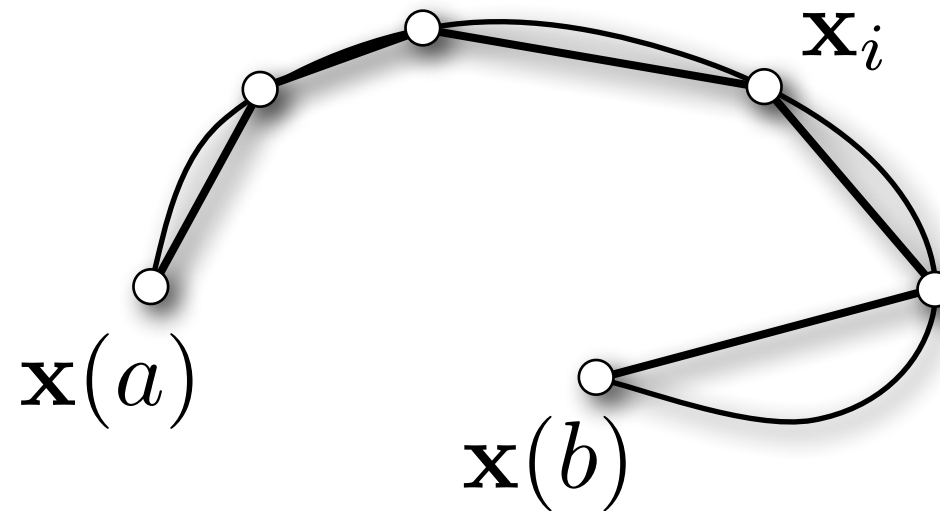
**A parametric curve  $\mathbf{x}(t)$  is**

- simple:  $\mathbf{x}(t)$  is injective (no self-intersections)
- differentiable:  $\mathbf{x}_t(t)$  is defined for all  $t \in [a, b]$
- regular:  $\mathbf{x}_t(t) \neq 0$  for all  $t \in [a, b]$



# Length of a Curve

**Let**  $t_i = a + i\Delta t$  **and**  $\mathbf{x}_i = \mathbf{x}(t_i)$



# Length of a Curve

## Polyline chord length

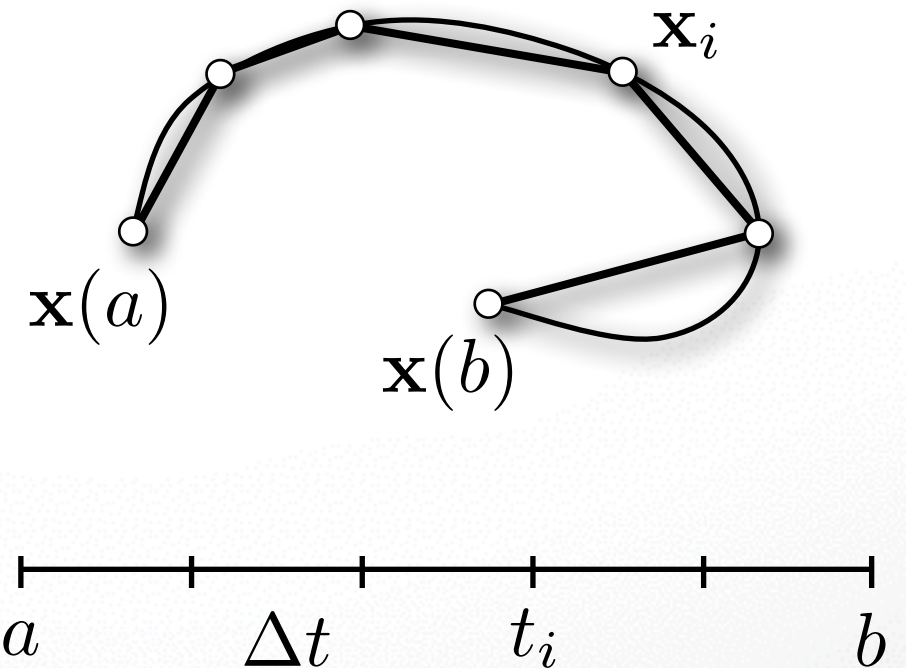
$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

norm change

## Curve arc length ( $\Delta t \rightarrow 0$ )

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

length =  
integration of infinitesimal change  
× norm of speed



# Re-Parameterization

## Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

## Re-parameterization w.r.t. $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3, \quad t \mapsto \mathbf{x}(u(t))$$

## Derivative (chain rule)

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du} \frac{du}{dt} = \mathbf{x}_u(u(t)) u_t(t)$$

# Re-Parameterization

## Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] \quad , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

$$\Rightarrow \mathbf{g}(\phi(t)) = \mathbf{f}(t)$$

# Arc Length Parameterization

Mapping of parameter domain:

$$t \mapsto s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

Parameter  $s$  for  $\mathbf{x}(s)$  equals length from  $\mathbf{x}(a)$  to  $\mathbf{x}(s)$

$$\mathbf{x}(s) = \mathbf{x}(s(t)) \quad ds = \|\mathbf{x}_t\| dt$$

same infinitesimal change

Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

defines orthonormal frame

# The Frenet Frame

## Taylor expansion

$$\mathbf{x}(t + h) = \mathbf{x}(t) + \mathbf{x}_t(t) h + \frac{1}{2} \mathbf{x}_{tt}(t) h^2 + \frac{1}{6} \mathbf{x}_{ttt}(t) h^3 + \dots$$

for convergence analysis and approximations

## Define local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ (Frenet frame)

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

tangent

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

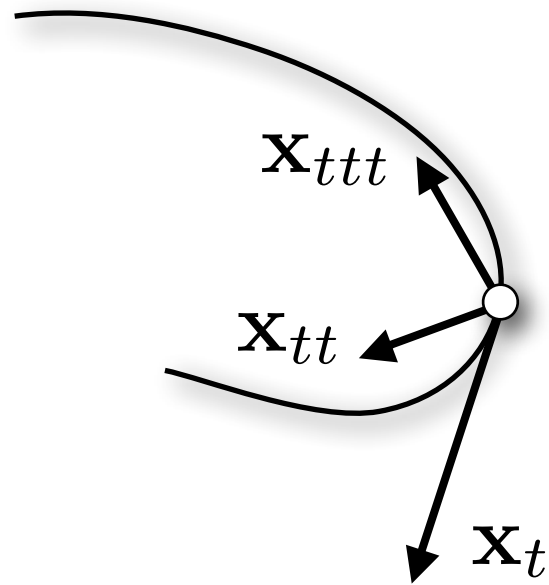
main normal

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

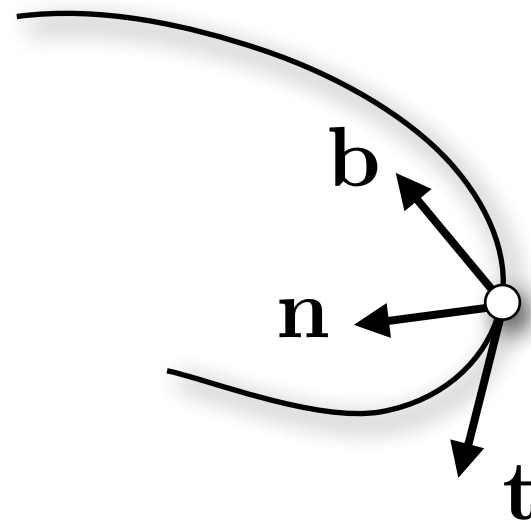
binormal

# The Frenet Frame

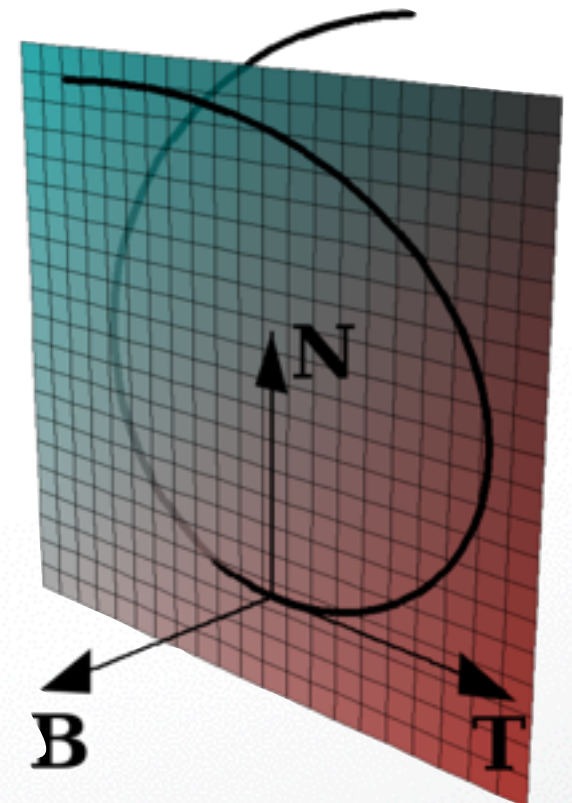
## Orthonormalization of local frame



local affine frame



Frenet frame



# The Frenet Frame

**Frenet-Serret: Derivatives w.r.t. arc length  $s$**

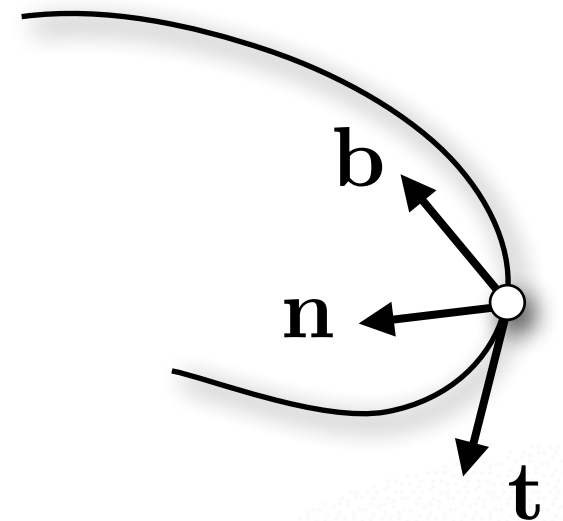
$$\begin{aligned} \mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n} \end{aligned}$$

**Curvature (deviation from straight line)**

$$\kappa = \|\mathbf{x}_{ss}\|$$

**Torsion (deviation from planarity)**

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$



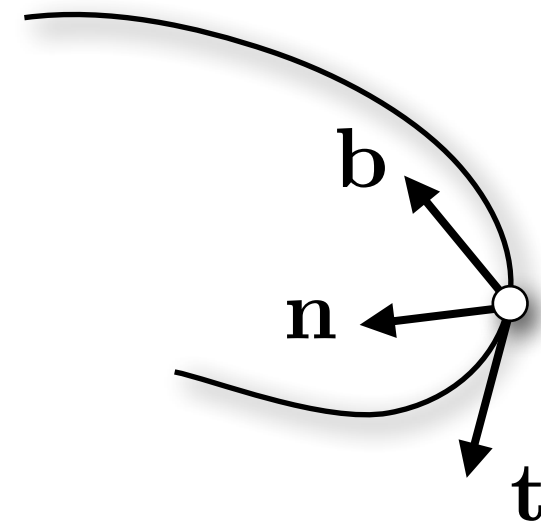
# Curvature and Torsion

## Planes defined by $\mathbf{x}$ and two vectors:

- osculating plane: vectors  $\mathbf{t}$  and  $\mathbf{n}$
- normal plane: vectors  $\mathbf{n}$  and  $\mathbf{b}$
- rectifying plane: vectors  $\mathbf{t}$  and  $\mathbf{b}$

## Osculating circle

- second order contact with curve
- center  $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
- radius  $1/\kappa$

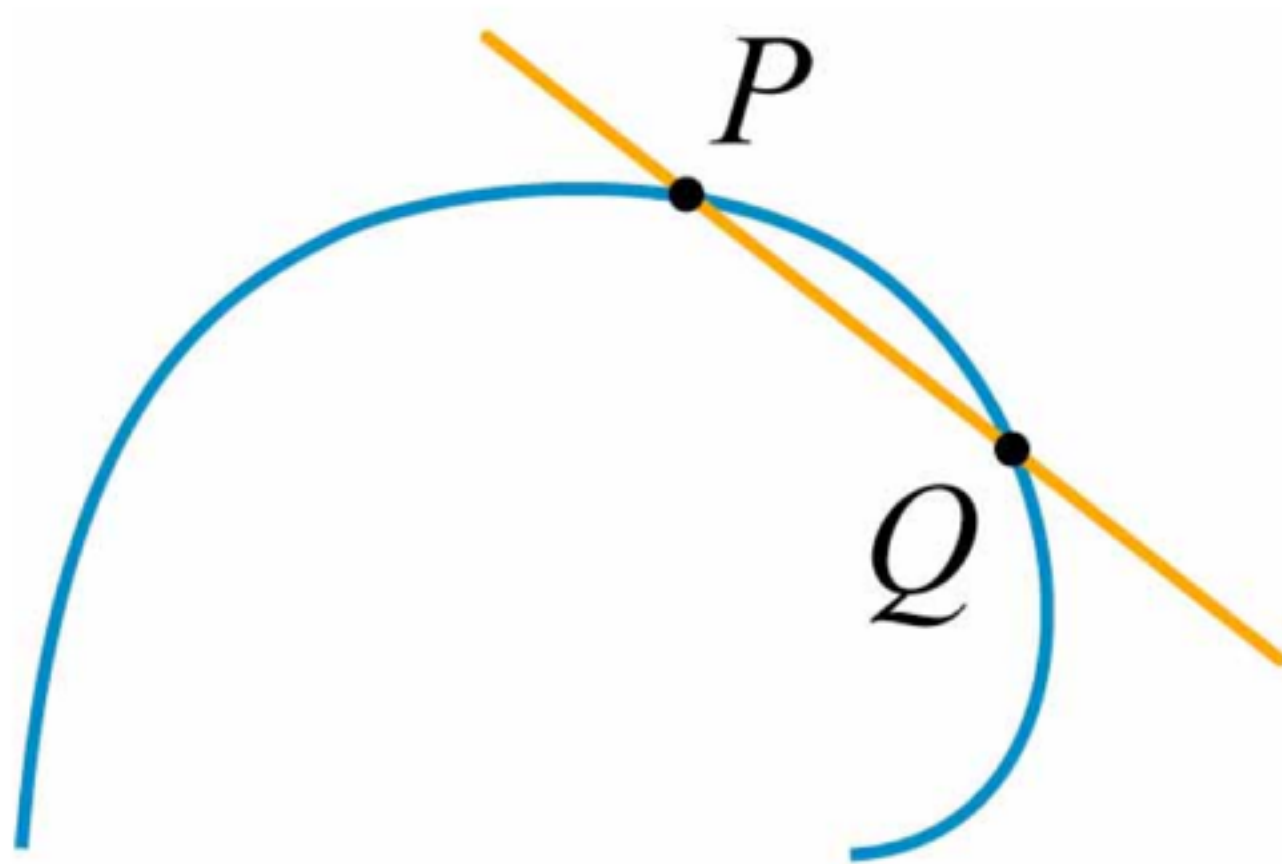


# Curvature and Torsion

- **Curvature**: Deviation from straight line
- **Torsion**: Deviation from planarity
- Independent of parameterization
  - **intrinsic** properties of the curve
- Euclidean invariants
  - **invariant** under rigid motion
- Define curve **uniquely** up to a rigid motion

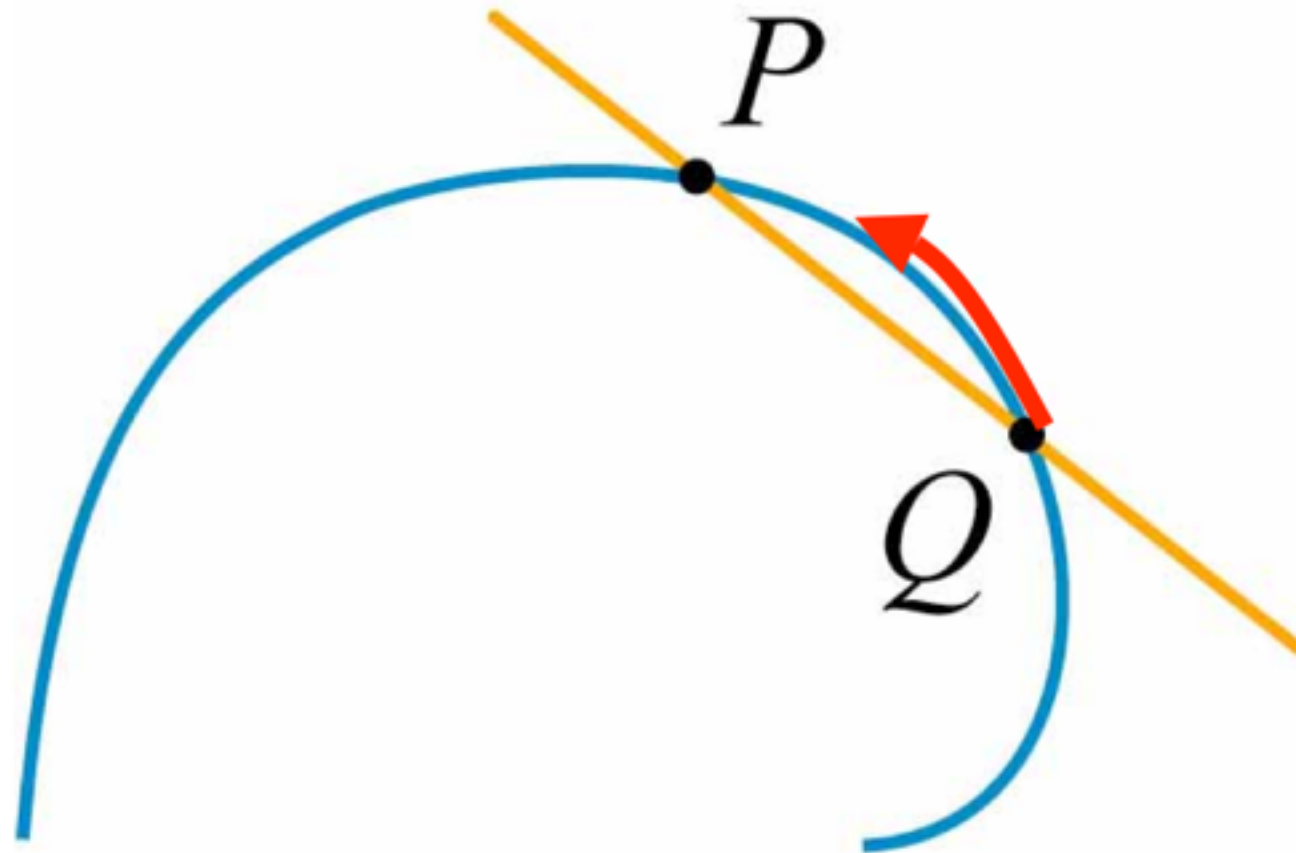
# Curvature: Some Intuition

A line through two points on the curve (Secant)



# Curvature: Some Intuition

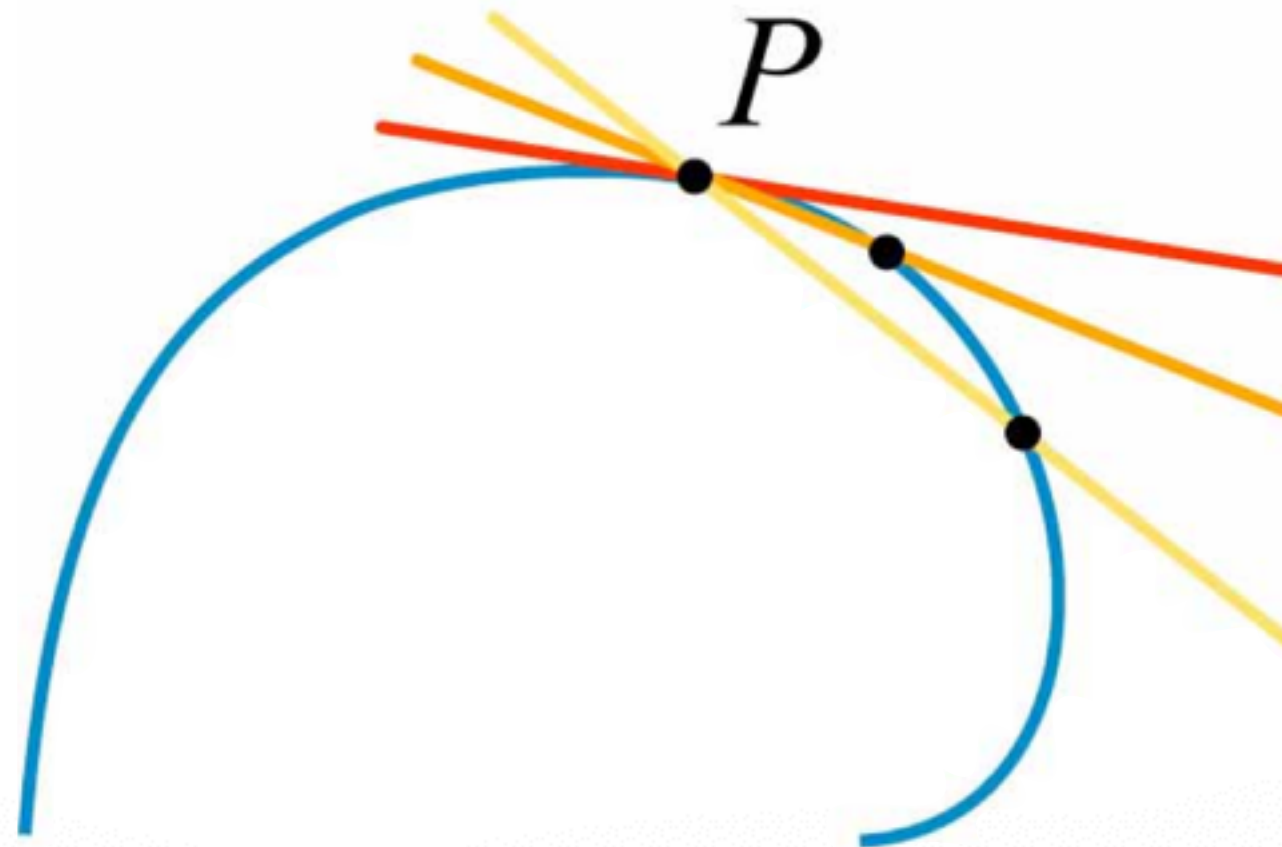
A line through two points on the curve (Secant)



# Curvature: Some Intuition

## Tangent, the first approximation

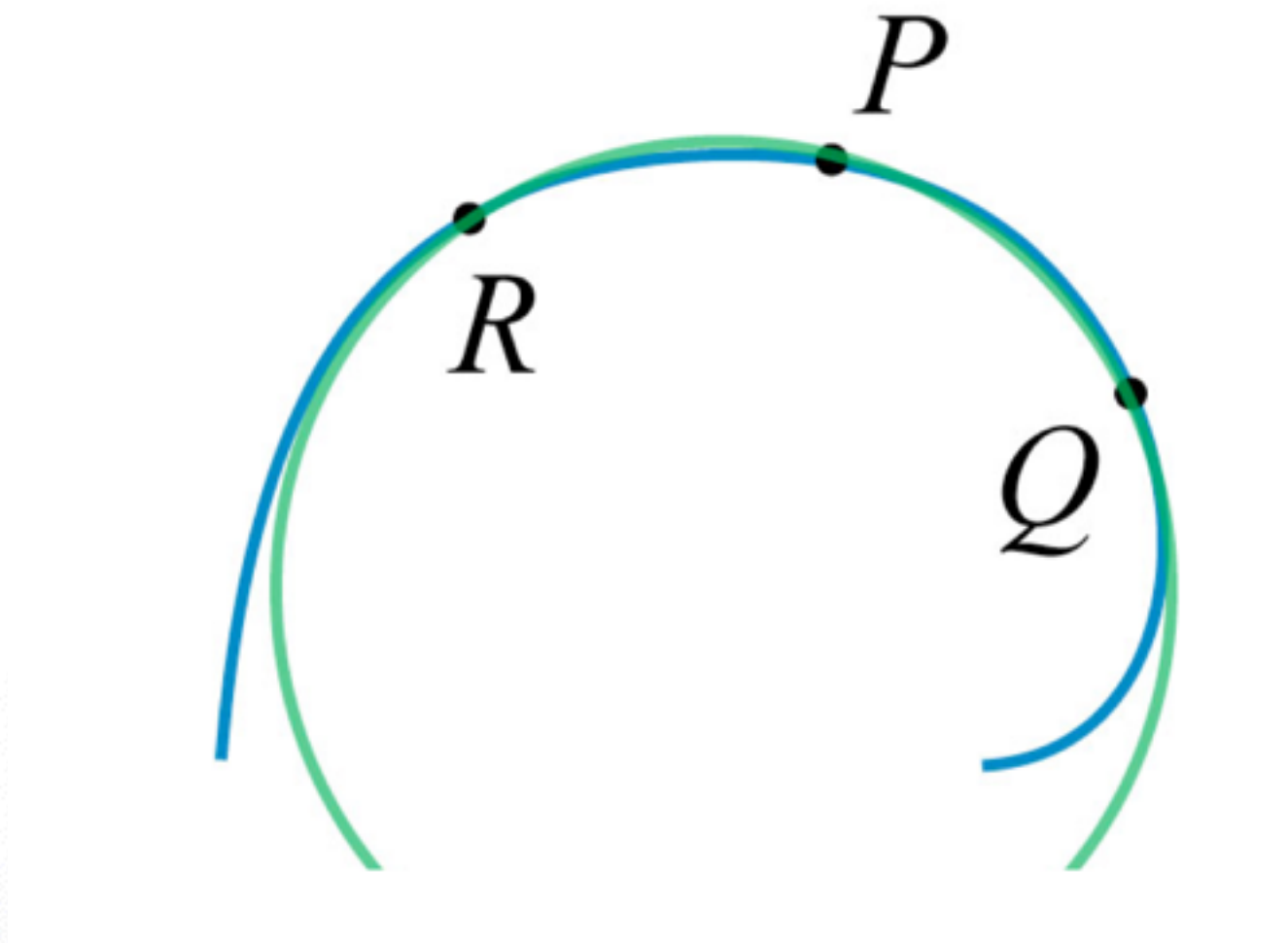
limiting secant as the two points come together



# Curvature: Some Intuition

## Circle of curvature

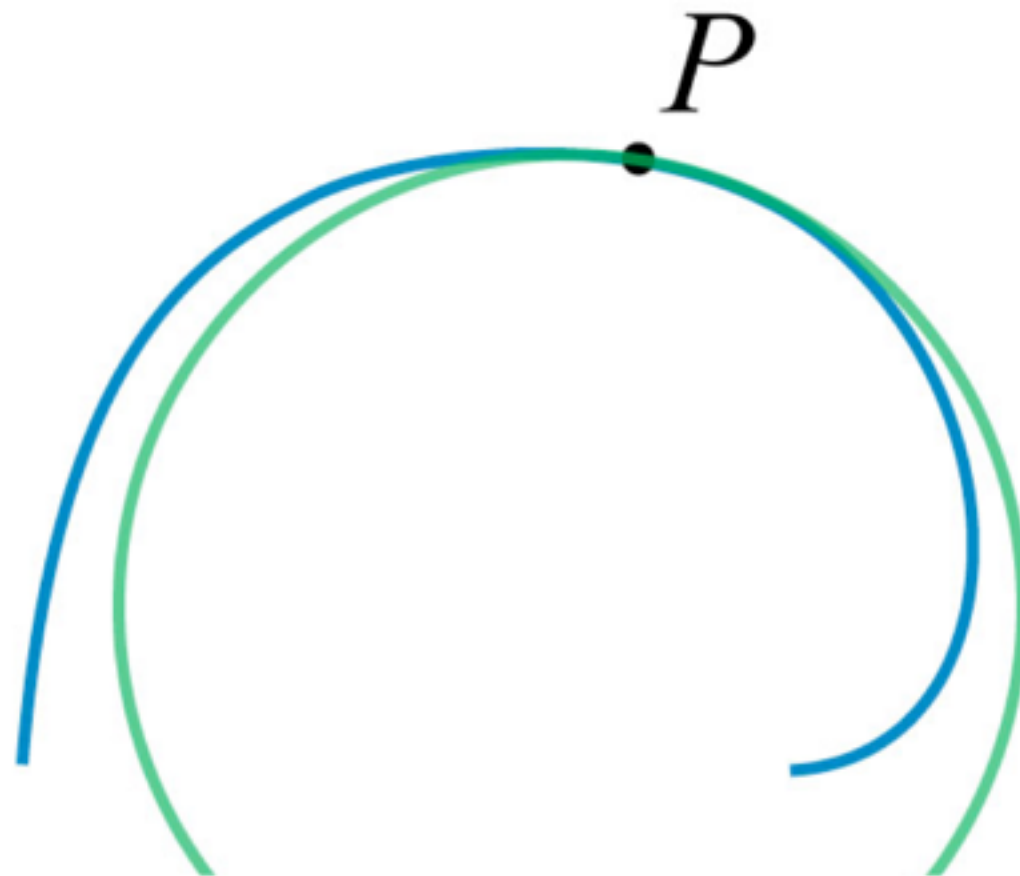
Consider the circle passing through 3 points of the curve



# Curvature: Some Intuition

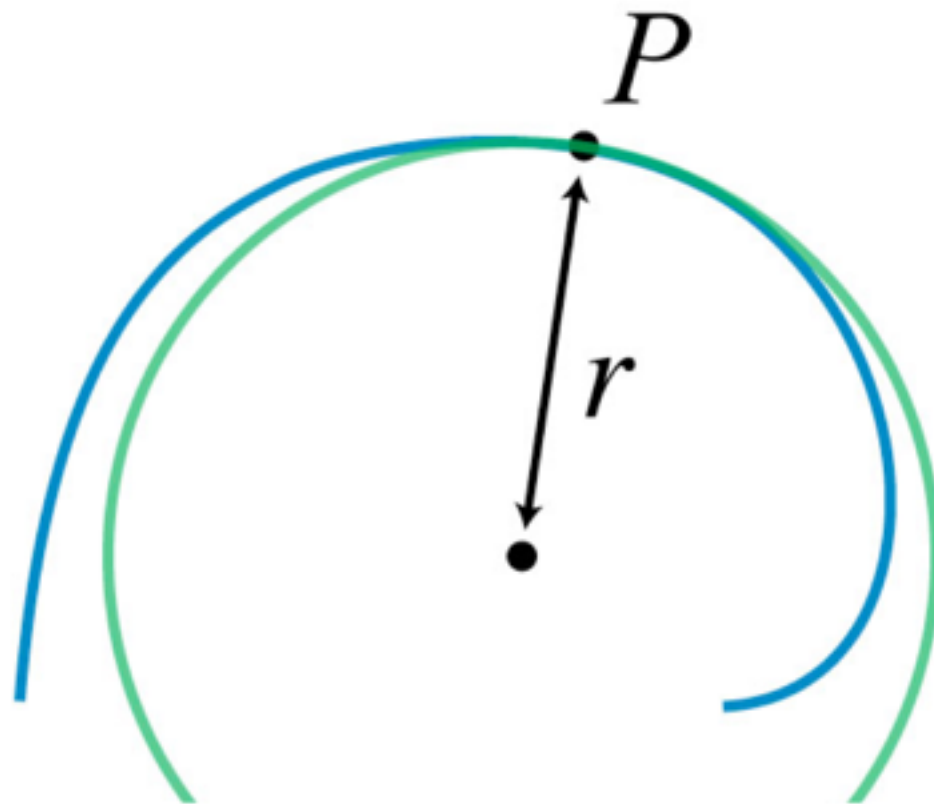
## Circle of curvature

The limiting circle as three points come together



# Curvature: Some Intuition

Radius of curvature  $r$

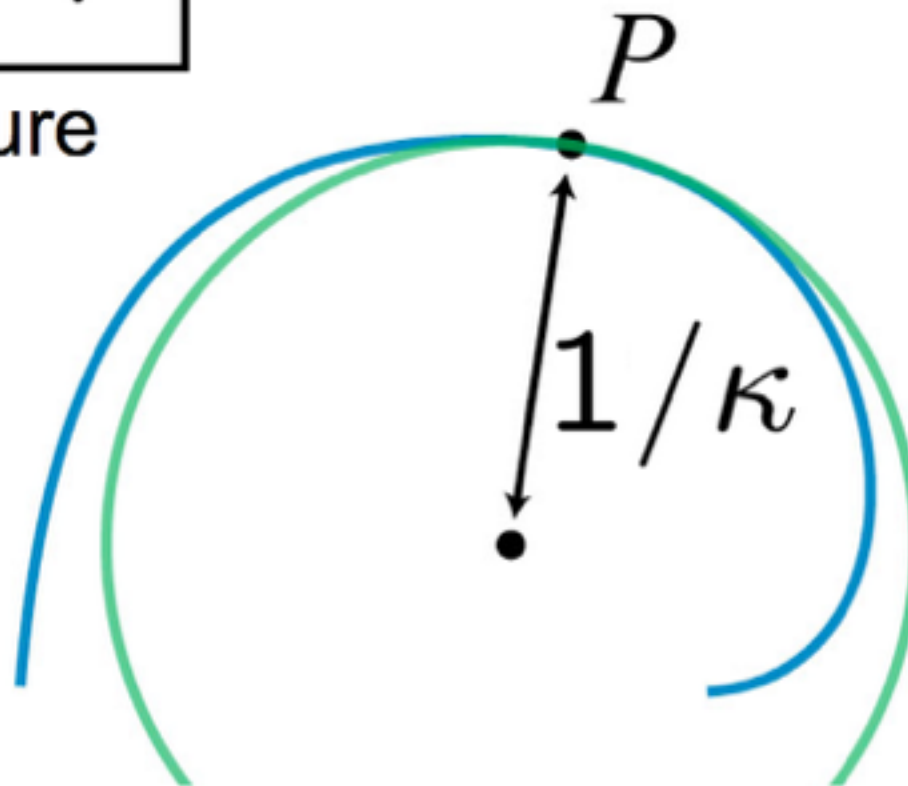


# Curvature: Some Intuition

Radius of curvature  $r$

$$\kappa = \frac{1}{r}$$

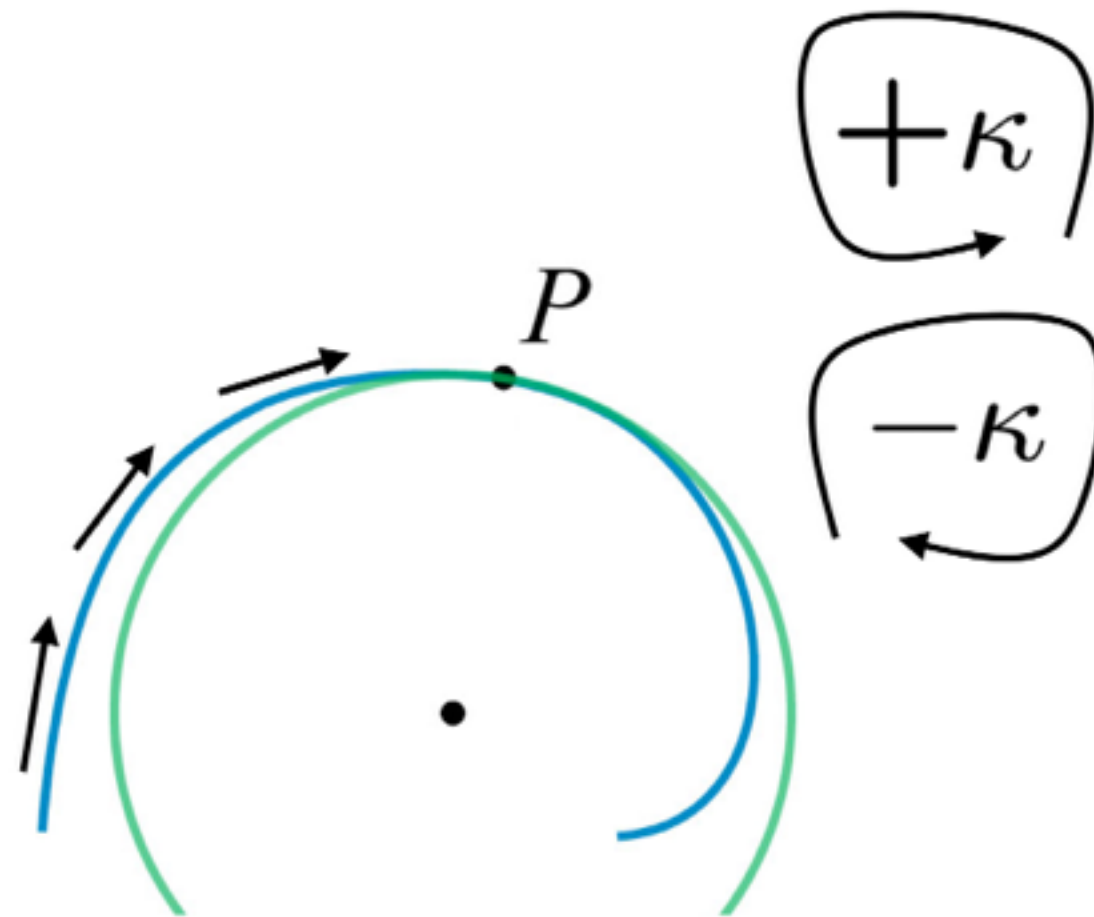
Curvature



# Curvature: Some Intuition

## Signed curvature

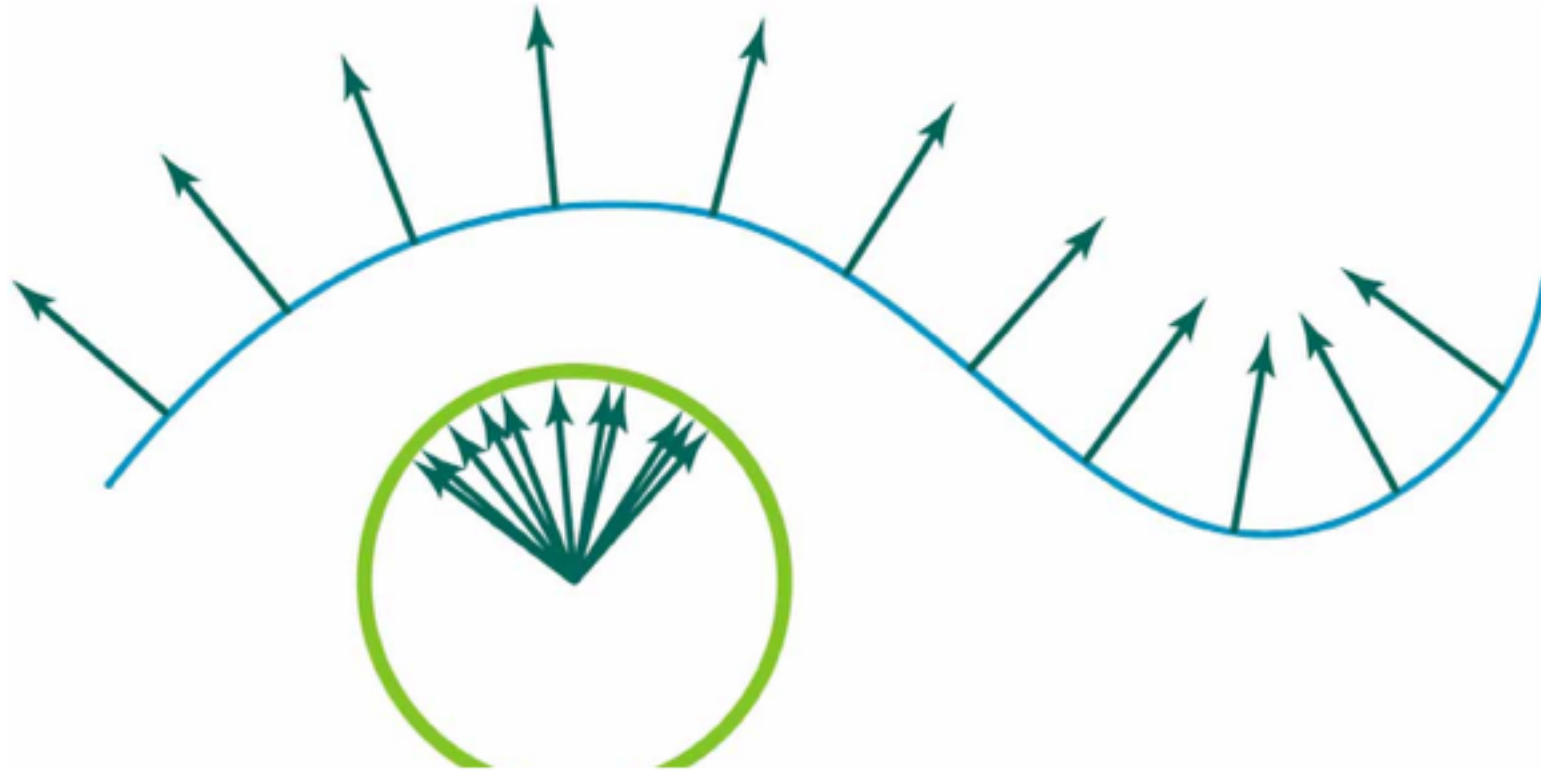
Sense of traversal along curve



# Curvature: Some Intuition

## Gauß map $\hat{n}(\mathbf{x})$

Point on curve maps to point on unit circle



# Curvature: Some Intuition

## Shape operator (Weingarten map)

Change in normal as we slide along curve

negative directional derivative  $D$  of Gauß map

$$S(\mathbf{v}) = -D_{\mathbf{v}}\hat{\mathbf{n}}$$



describes directional curvature

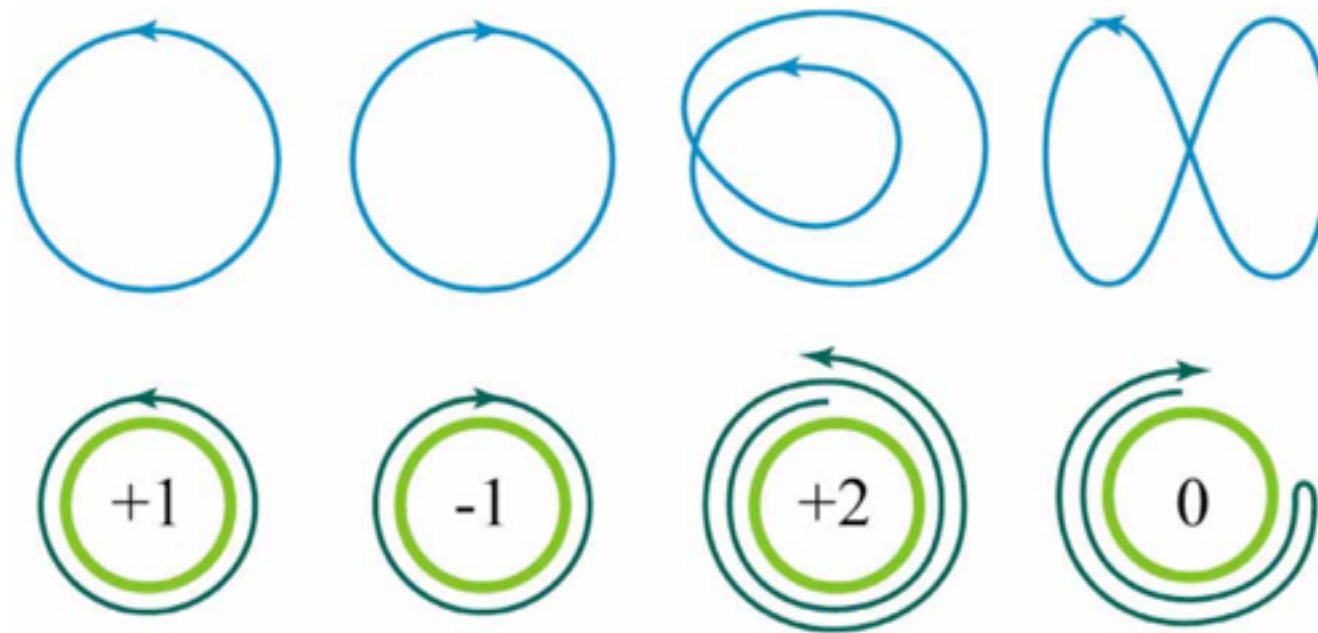
using normals as degrees of freedom

→ accuracy/convergence/implementation (discretization)

# Curvature: Some Intuition

## Turning number, $k$

Number of orbits in Gaussian image

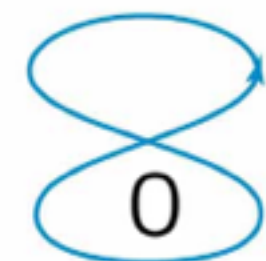
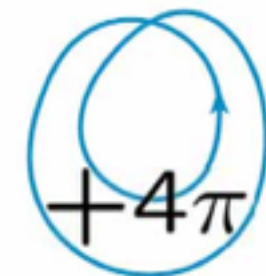
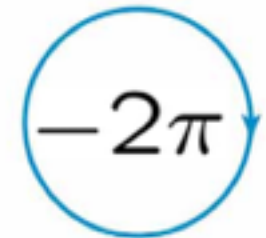


# Curvature: Some Intuition

## Turning number theorem

For a closed curve, the integral of curvature is an integer multiple of  $2\pi$

$$\int_{\Omega} \kappa ds = 2\pi k$$



# Take Home Message

**In the limit of a refinement sequence**, discrete measure of length and curvature **agree** with continuous measures

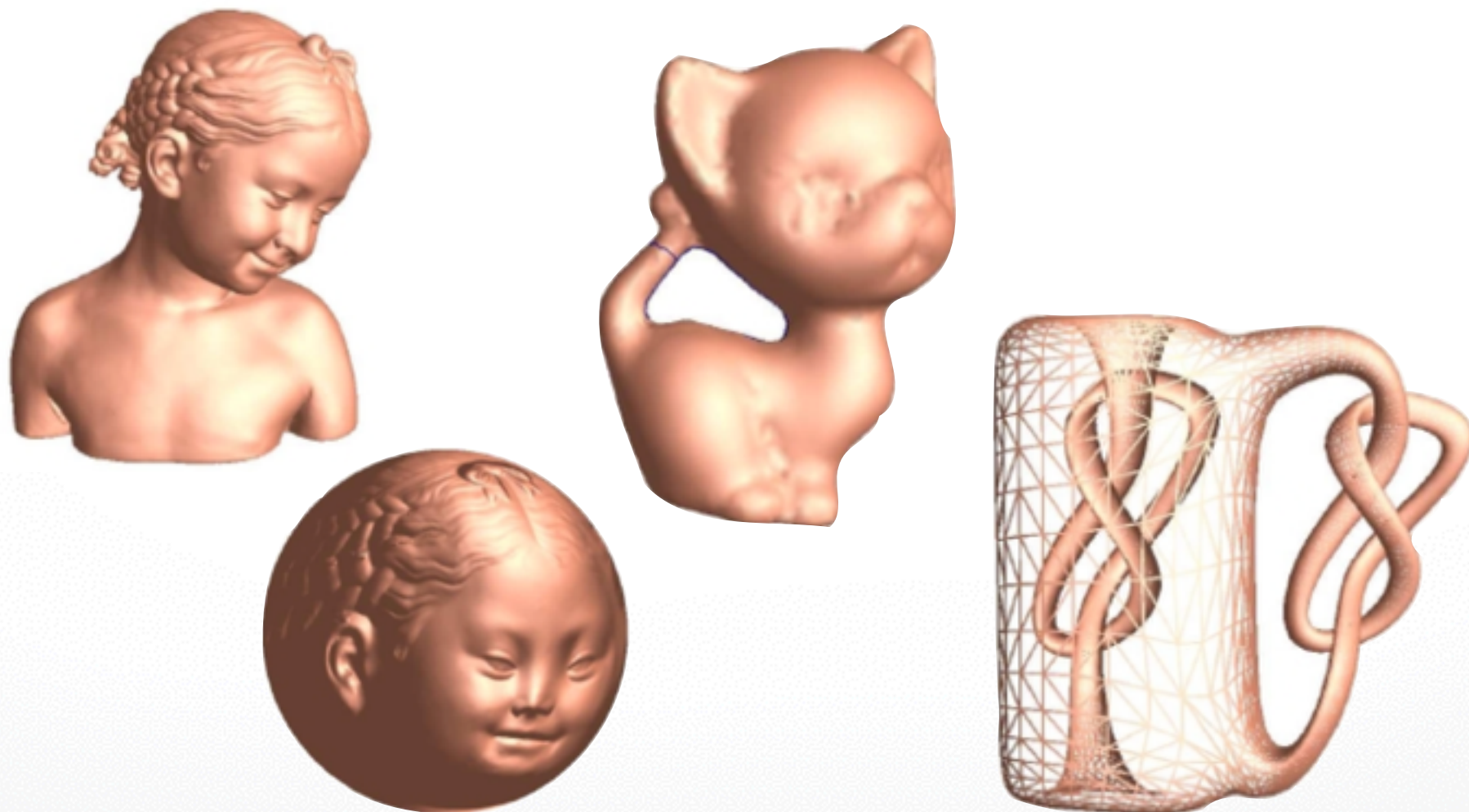
# Outline

- Parametric Curves
- **Parametric Surfaces**

# Surfaces

## What characterizes shape?

- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions



# Metric on Surfaces

## Measure Stuff

- angle, length, area
  - requires an inner product
- we have:
  - Euclidean inner product in domain
- we want to turn this into:
  - inner product on surface

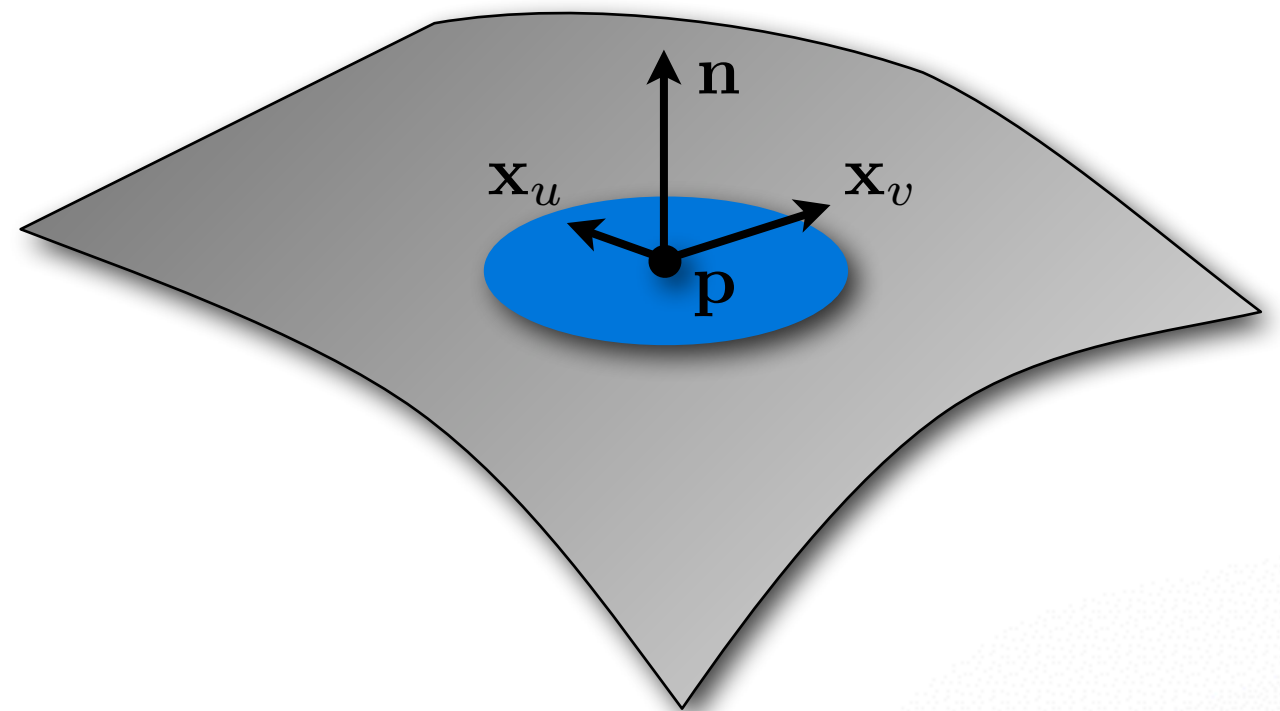
# Parametric Surfaces

## Continuous surface

$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

## Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0} \quad \text{normal exists}$$

# Angles on Surface

**Curve  $[u(t), v(t)]$  in uv-plane defines curve on the surface  $\mathbf{x}(u, v)$**

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

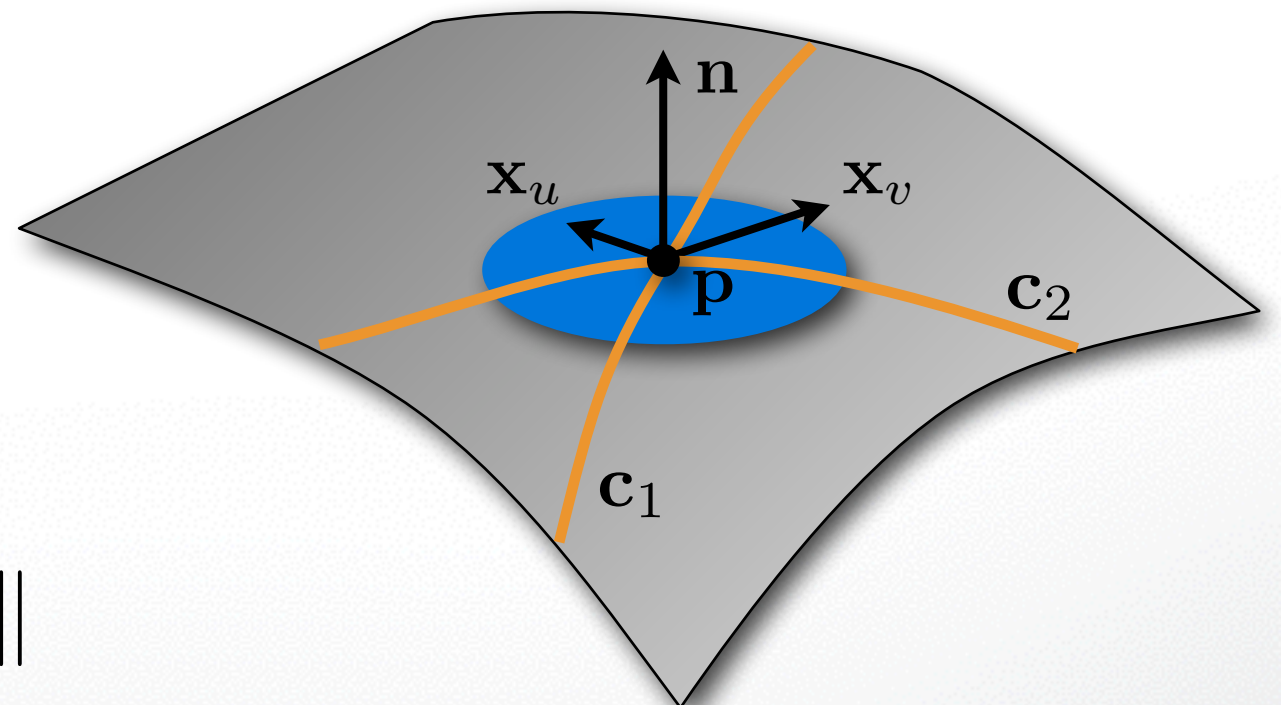
**Two curves  $\mathbf{c}_1$  and  $\mathbf{c}_2$  intersecting at  $\mathbf{p}$**

- angle of intersection?
- two tangents  $\mathbf{t}_1$  and  $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos \theta \|\mathbf{t}_1\| \|\mathbf{t}_2\|$$



# Angles on Surface

**Curve  $[u(t), v(t)]$  in uv-plane defines curve on the surface  $\mathbf{x}(u, v)$**

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

**Two curves  $\mathbf{c}_1$  and  $\mathbf{c}_2$  intersecting at  $\mathbf{p}$**

$$\begin{aligned}\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\ &= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\ &= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_v^T \mathbf{x}_u & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}\end{aligned}$$

# First Fundamental Form

## First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

## Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

# First Fundamental Form

First fundamental form **I** allows to measure  
(w.r.t. surface metric)

Angles  $\mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$

Length 
$$\begin{aligned} ds^2 &= \langle (du, dv), (du, dv) \rangle \\ &= E du^2 + 2F du dv + G dv^2 \end{aligned}$$
 **squared  
infinitesimal  
length**

Area 
$$\begin{aligned} dA &= \|\mathbf{x}_u \times \mathbf{x}_v\| du dv \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} du dv \\ &= \sqrt{EG - F^2} du dv \end{aligned}$$

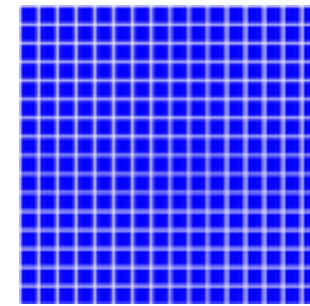
**infinitesimal  
Area**

**cross product → determinant with unit vectors → area**

# Sphere Example

## Spherical parameterization

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u, v) \in [0, 2\pi) \times [0, \pi)$$



## Tangent vectors

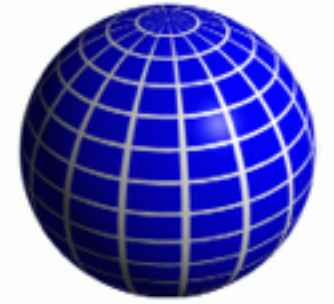
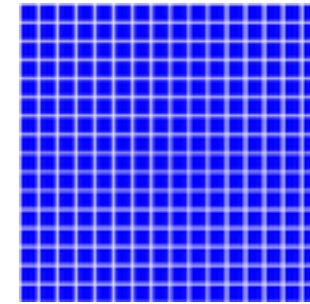
$$\mathbf{x}_u(u, v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u, v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

## First fundamental Form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

# Sphere Example

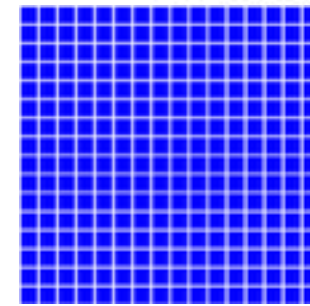
Length of equator  $\mathbf{x}(t, \pi/2)$



$$\begin{aligned}\int_0^{2\pi} 1 \, ds &= \int_0^{2\pi} \sqrt{E (u_t)^2 + 2F u_t v_t + G (v_t)^2} \, dt \\ &= \int_0^{2\pi} \sin v \, dt \\ &= 2\pi \sin v = 2\pi\end{aligned}$$

# Sphere Example

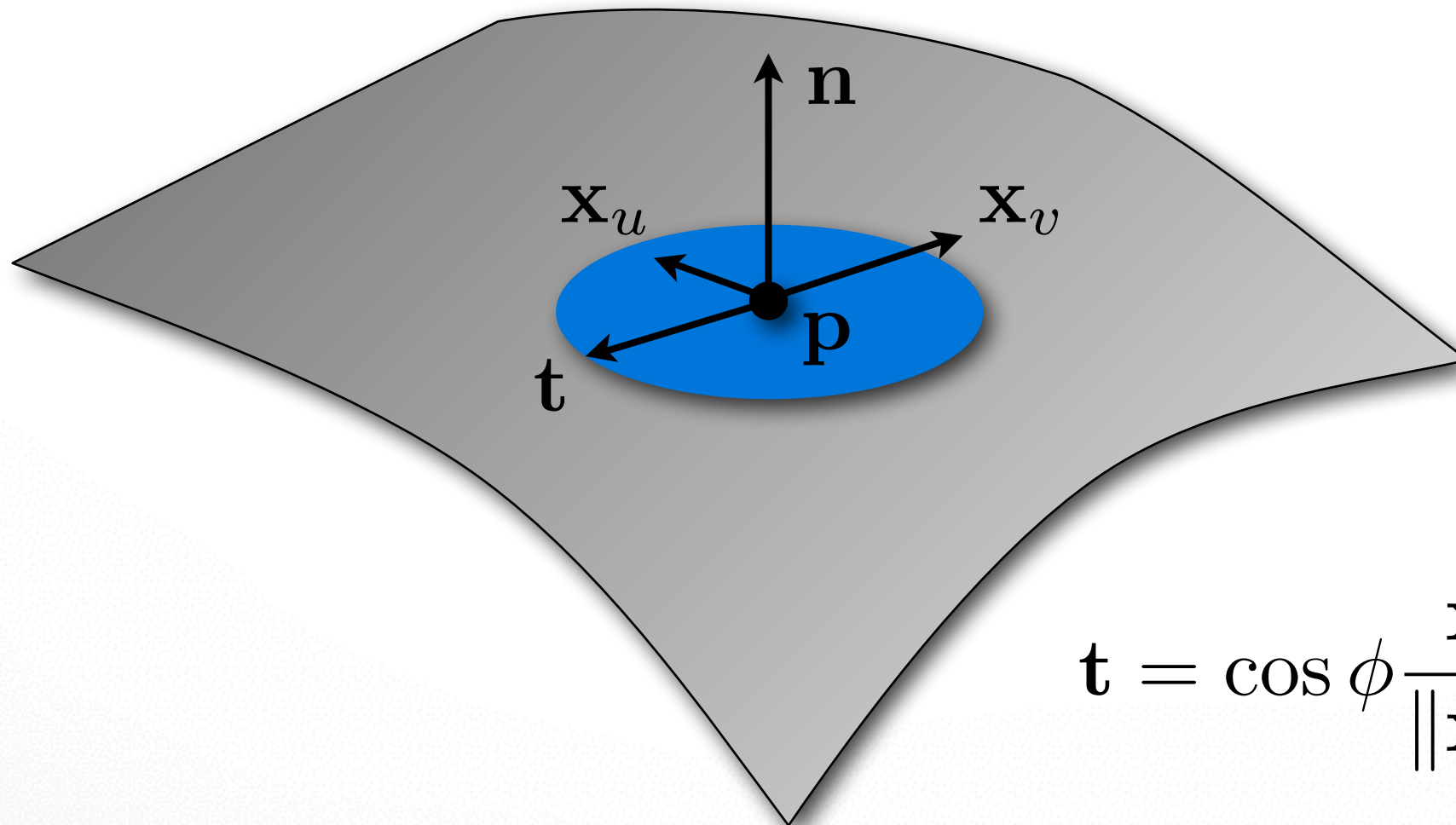
Area of a sphere



$$\begin{aligned}\int_0^\pi \int_0^{2\pi} 1 \, dA &= \int_0^\pi \int_0^{2\pi} \sqrt{EG - F^2} \, du \, dv \\ &= \int_0^\pi \int_0^{2\pi} \sin v \, du \, dv \\ &= 4\pi\end{aligned}$$

# Normal Curvature

Tangent vector  $\mathbf{t}$  ...



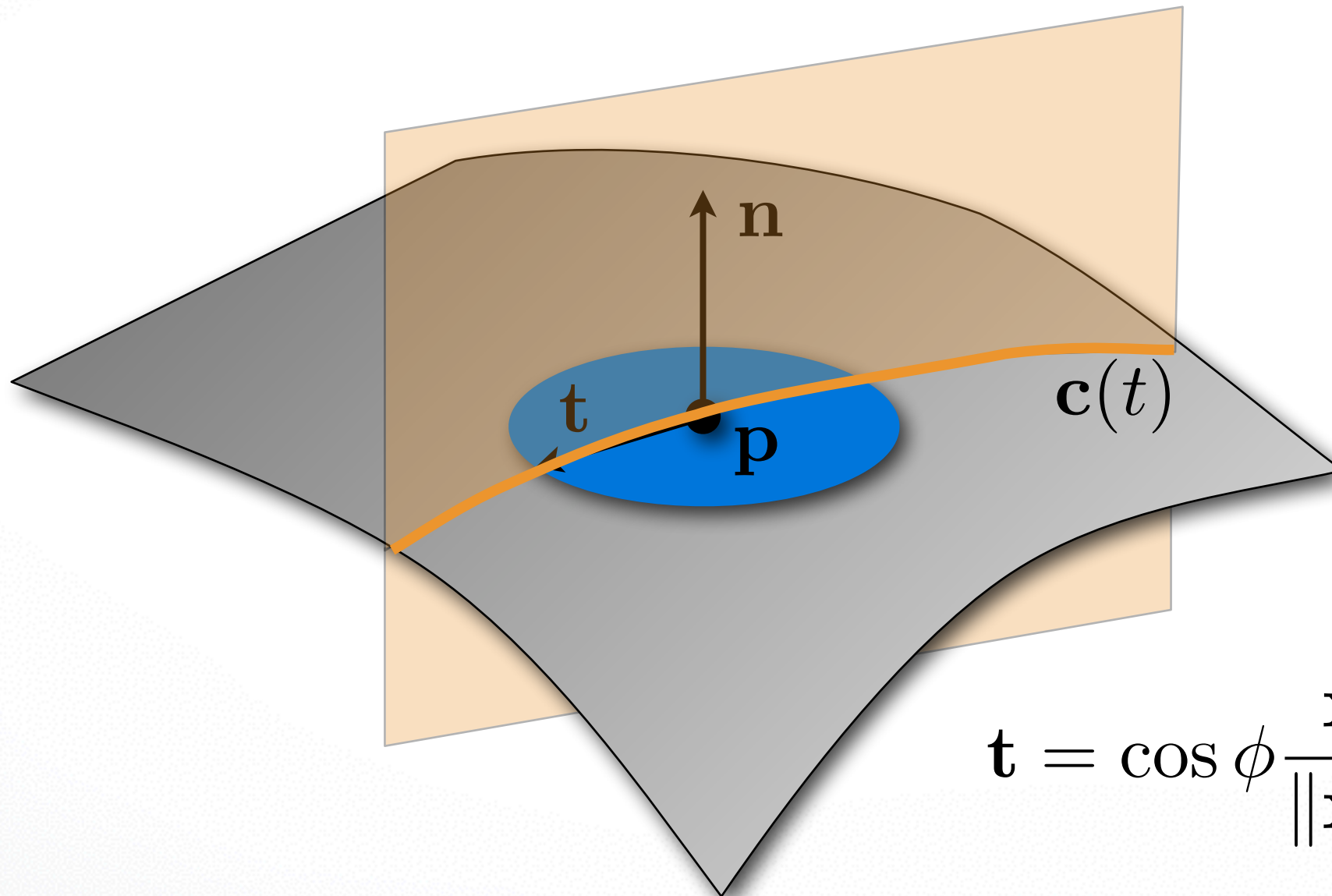
$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

unit vector

# Normal Curvature

... defines intersection plane, yielding curve  $\mathbf{c}(t)$

normal curve



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

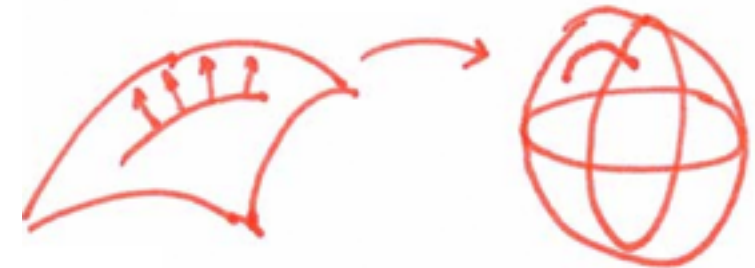
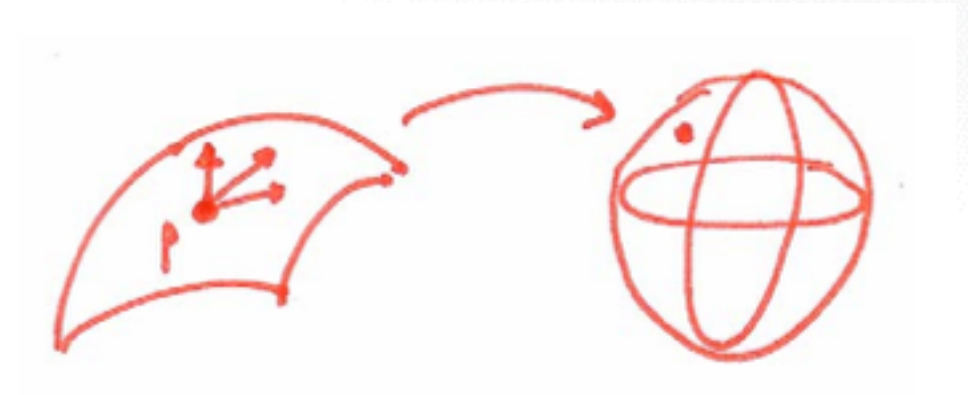
# Geometry of the Normal

## Gauss map

- normal at point

$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p) \quad N : S \rightarrow \mathbb{S}^2$$

- consider curve in surface again
  - study its curvature at p
  - normal “tilts” along curve



# Normal Curvature

Normal curvature  $\kappa_n(t)$  is defined as curvature of the normal curve  $\mathbf{c}(t)$  at point  $\mathbf{p}(t) = \mathbf{x}(u, v)$

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \quad \begin{array}{l} \mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} = (a, b) \end{array}$$

# Surface Curvature(s)

## Principal curvatures

- Maximum curvature  $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
- Minimum curvature  $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler theorem  $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Corresponding principal directions  $\mathbf{e}_1$  ,  $\mathbf{e}_2$  are orthogonal



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## Special curvatures

- Mean curvature  $H = \frac{\kappa_1 + \kappa_2}{2}$  **extrinsic**
- Gaussian curvature  $K = \kappa_1 \cdot \kappa_2$  **intrinsic (only first FF)**

# Invariants

## Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$
$$\operatorname{tr} dN_p = \kappa_1 + \kappa_2 = H$$

- eigenvalues and orthovectors

$$dN_p(e_1) = \kappa_1 e_1 \quad dN_p(e_2) = \kappa_2 e_2$$

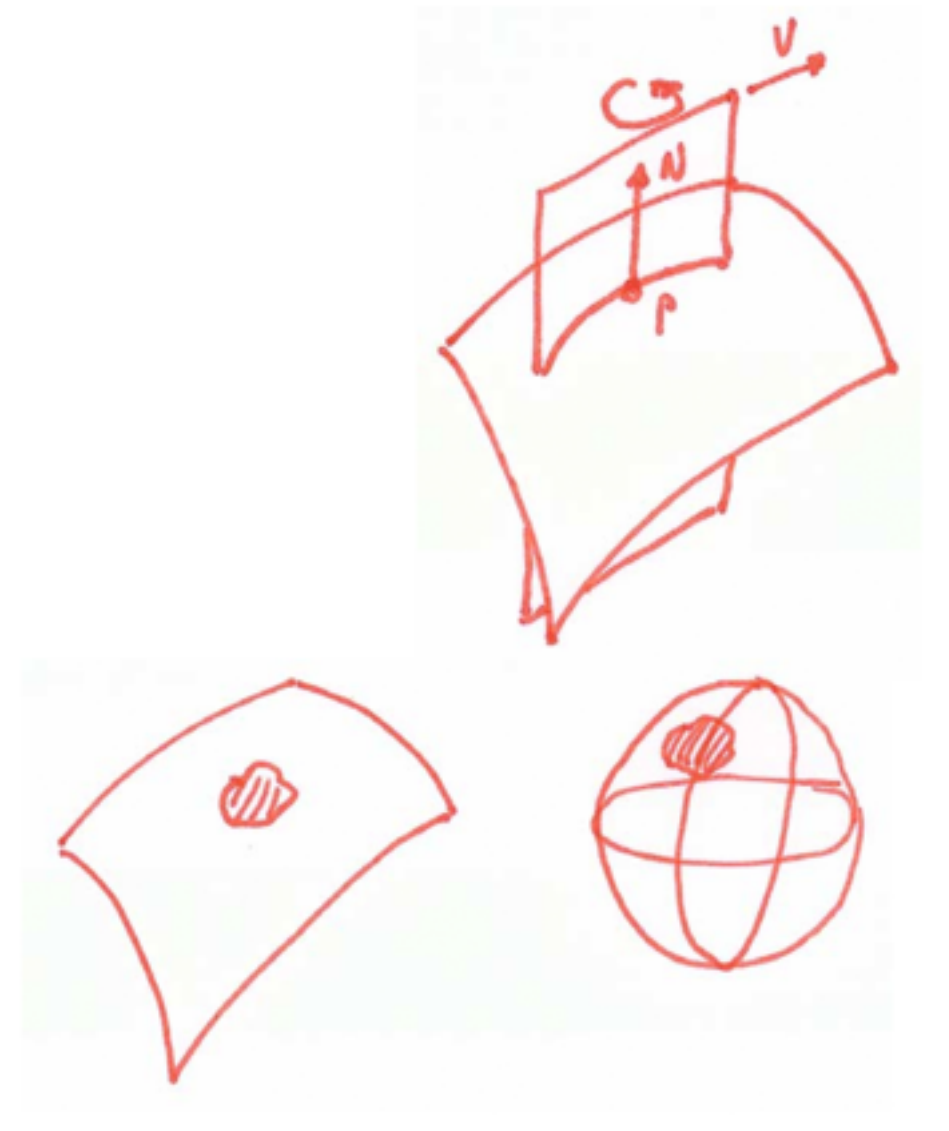
$$II_p|_{\mathcal{S} \subset T_p S} \begin{cases} \text{max} \rightarrow \kappa_1 \\ \text{min} \rightarrow \kappa_2 \end{cases}$$

# Mean Curvature

## Integral representations

$$H_p/2 = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) d\theta$$

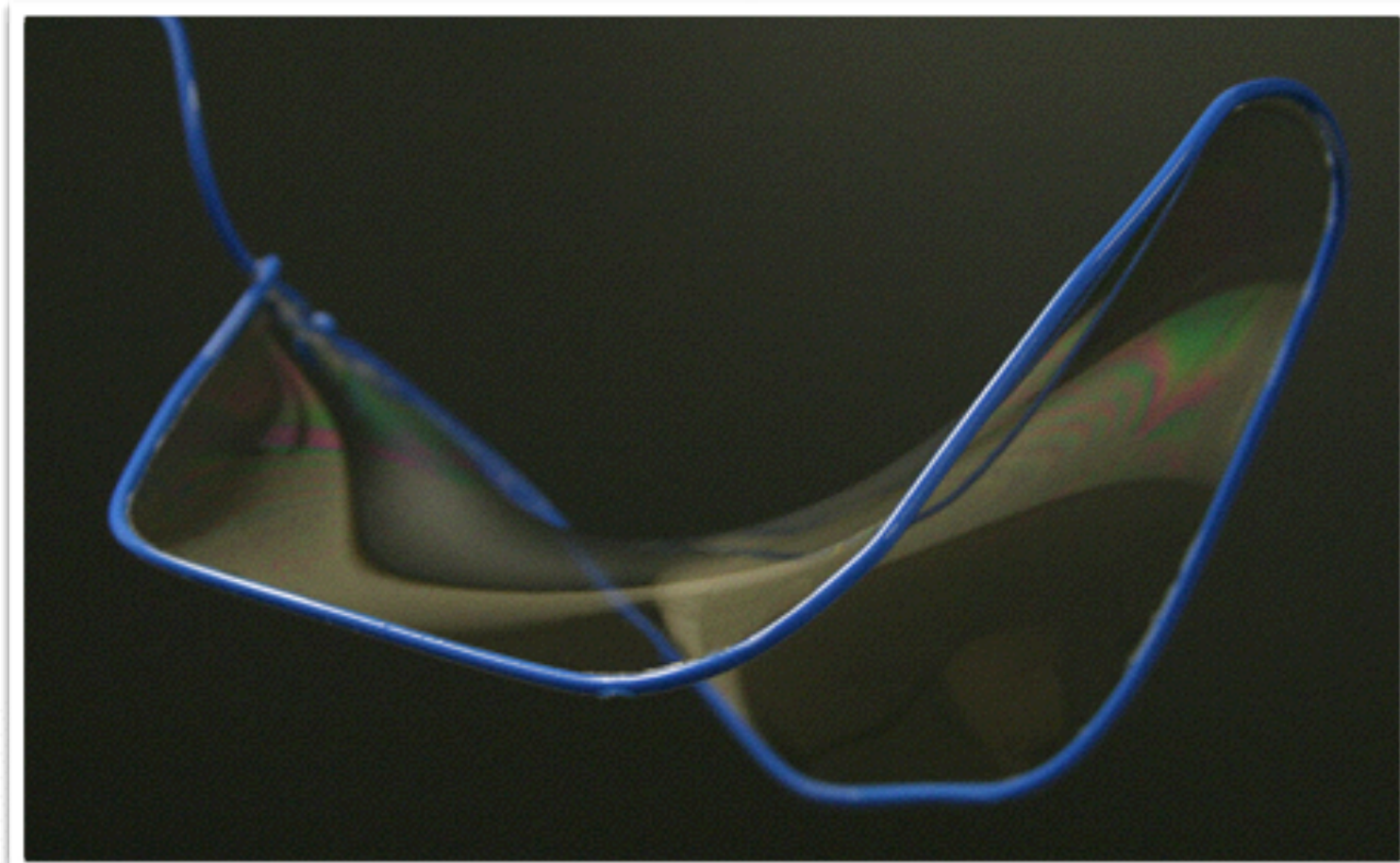
$$K_p = \lim_{A \rightarrow 0} \frac{A_G}{A}$$



# Curvature of Surfaces

Mean curvature  $H = \frac{\kappa_1 + \kappa_2}{2}$

- $H = 0$  everywhere  $\rightarrow$  minimal surface



soap film

# Curvature of Surfaces

Mean curvature  $H = \frac{\kappa_1 + \kappa_2}{2}$

- $H = 0$  everywhere  $\rightarrow$  minimal surface



Green Void, Sydney  
Architects: Lava



# Curvature of Surfaces

**Gaussian curvature**  $K = \kappa_1 \cdot \kappa_2$

- $K = 0$  everywhere  $\rightarrow$  developable surface

**surface that can be flattened to a plane without distortion (stretching or compression)**



Disney, Concert Hall, L.A.  
Architects: Gehry Partners



Timber Fabric  
IBOIS, EPFL

# Shape Operator

## Derivative of Gauss map

- second fundamental form

$$II_p(v) = \langle dN_p(v), v \rangle$$

- local coordinates

$$II_p = - \begin{pmatrix} \langle N, S_{uu} \rangle & \langle N, S_{uv} \rangle \\ \langle N, S_{vu} \rangle & \langle N, S_{vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

# Intrinsic Geometry

## Properties of the surface that only depend on the first fundamental form

- length
- angles
- Gaussian curvature (Theorema Egregium)  
**remarkable theorem (Gauss)**

$$K = \lim_{r \rightarrow 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

**Gaussian curvature of a surface is invariant under local isometry**

# Classification

**Point  $\mathbf{x}$  on the surface is called**

- elliptic, if  $K > 0$
- hyperbolic, if  $K < 0$
- parabolic, if  $K = 0$
- umbilic, if  $\kappa_1 = \kappa_2$  **or isotropic**

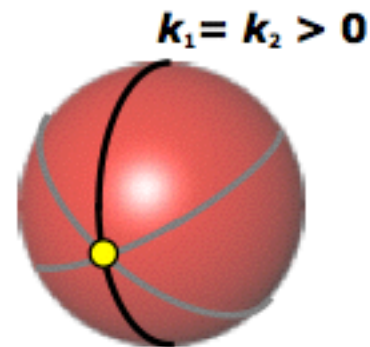
**Gaussian curvature  $K$**

# Classification

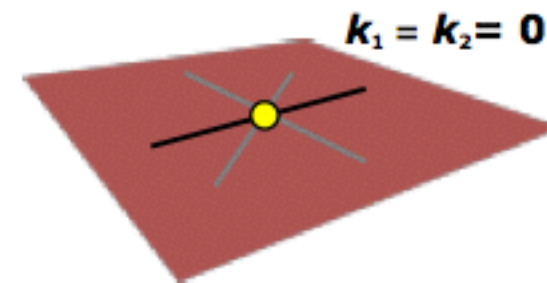
Point  $x$  on the surface is called

## Isotropic

Equal in all directions



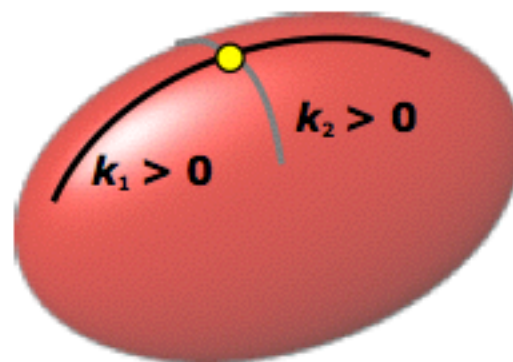
spherical



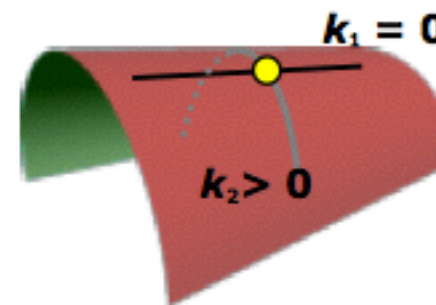
planar

## Anisotropic

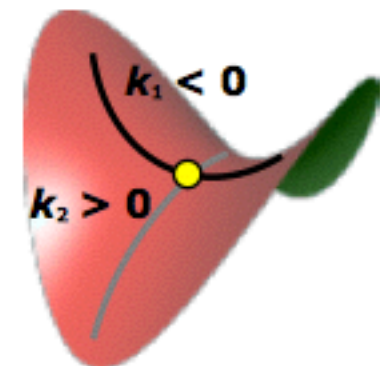
Distinct principal directions



elliptic  
 $K > 0$



parabolic  
 $K = 0$   
developable



hyperbolic  
 $K < 0$

# Gauss-Bonnet Theorem

For any closed manifold surface with Euler characteristic  $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

$$\int K(\text{👉}) = \int K(\text{🐮}) = \int K(\text{🌐}) = 4\pi$$

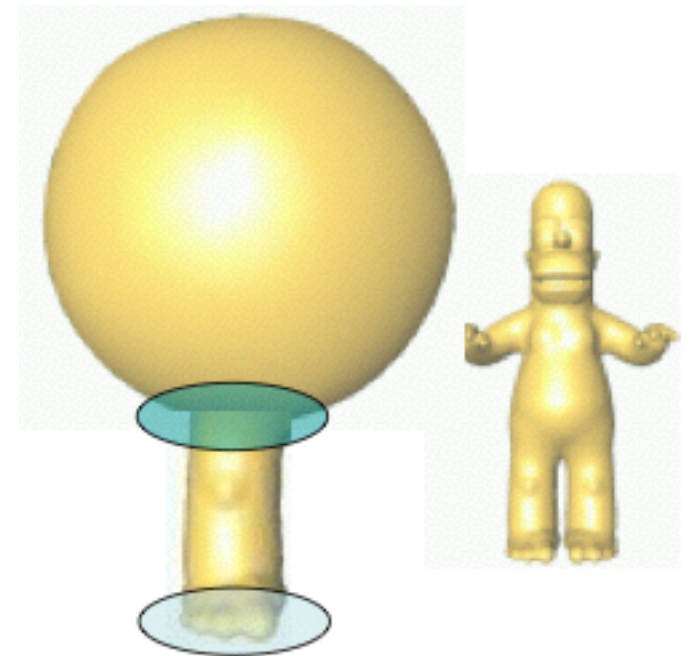
# Gauss-Bonnet Theorem

## Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



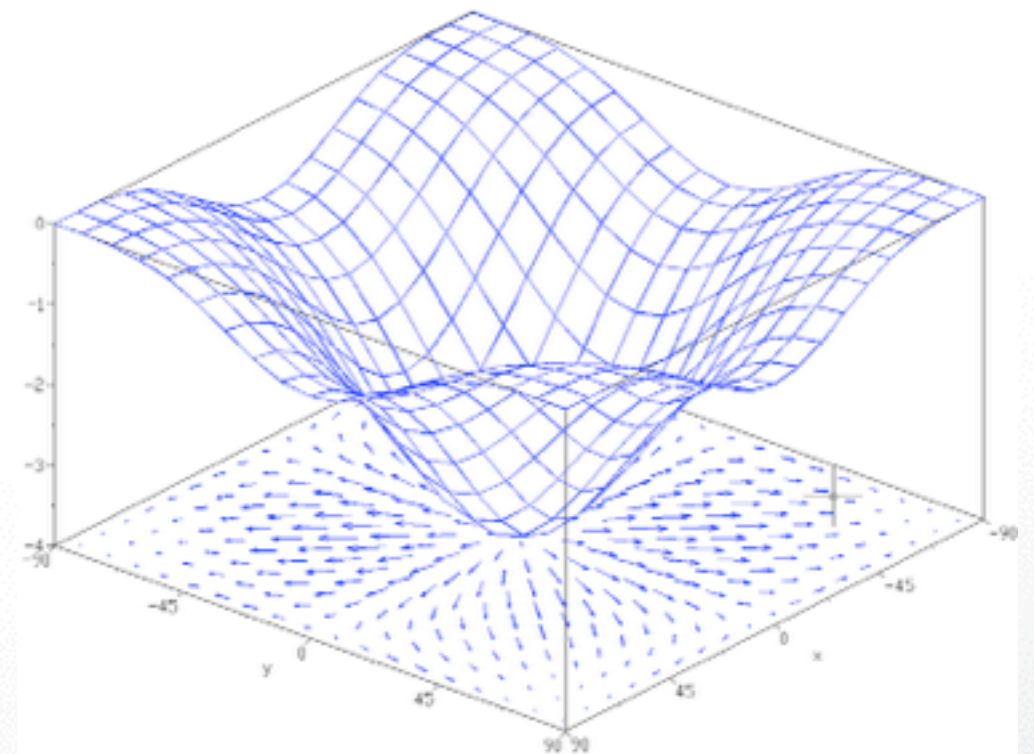
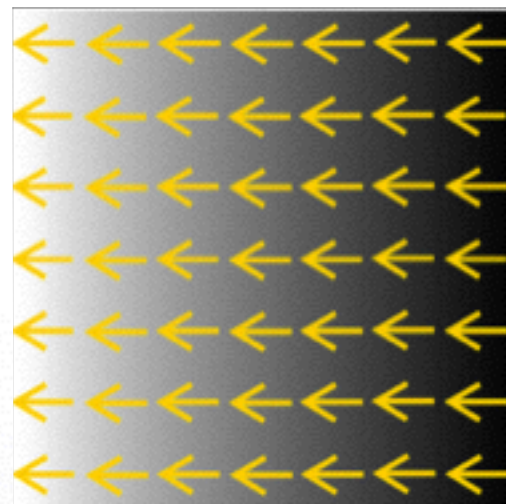
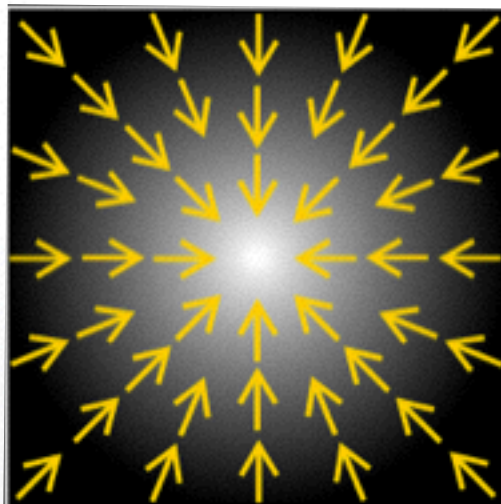
when sphere is deformed, new  
**positive and negative curvature cancel out**

# Differential Operators

## Gradient

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend



# Differential Operators

## Divergence

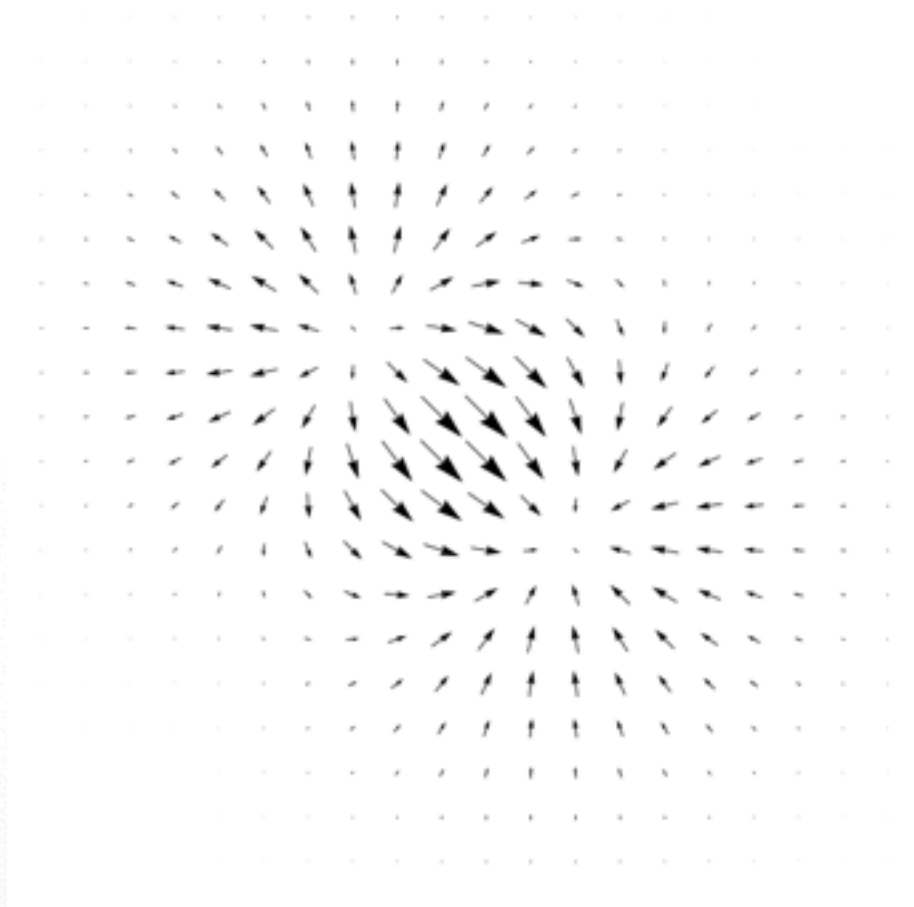
$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: incompressible fluid
  - velocity field is divergence-free

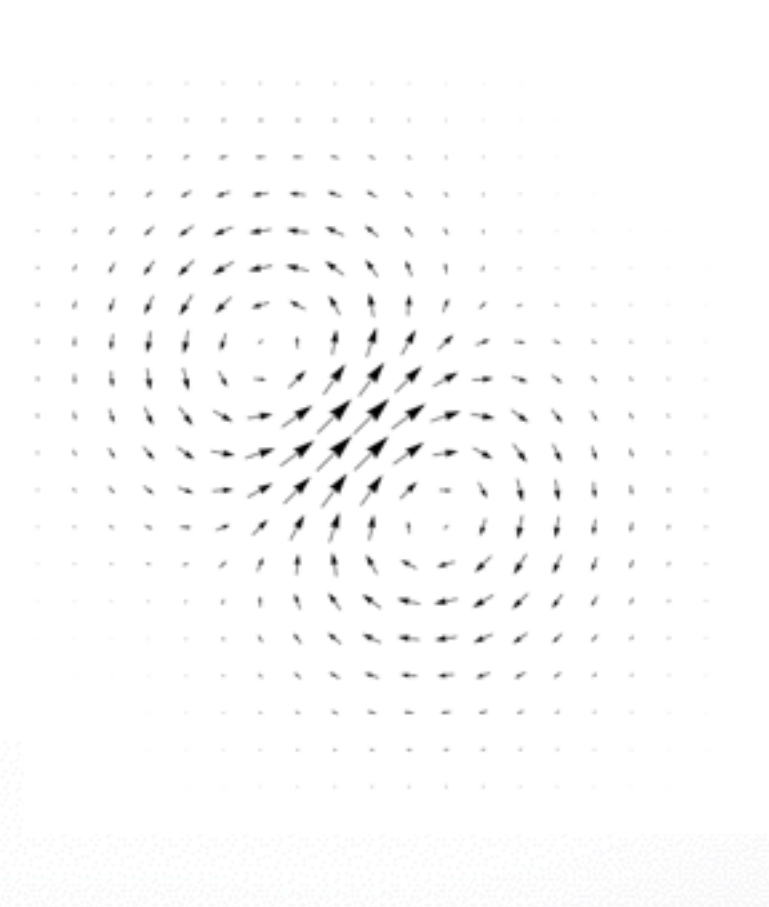
# Differential Operators

## Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

# Laplace Operator

The diagram illustrates the Laplace operator formula with the following components and annotations:

- Laplace operator**: Points to the  $\Delta$  symbol.
- function in Euclidean space**: Points to the  $f$  symbol.
- divergence operator**: Points to the  $\text{div}$  symbol.
- gradient operator**: Points to the  $\nabla$  symbol.
- 2nd partial derivatives**: Points to the  $\frac{\partial^2 f}{\partial x_i^2}$  term.
- Cartesian coordinates**: Points to the  $x_i$  in the denominator.

$$\Delta f = \text{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

# Laplace-Beltrami Operator

## Extension of Laplace to functions on manifolds

Laplace-  
Beltrami

gradient  
operator

...of the surface

$$\Delta_S f = \operatorname{div}_S \nabla_S f$$

function on  
manifold  $S$

divergence  
operator

**Laplace on the surface**

# Laplace-Beltrami Operator

The diagram illustrates the Laplace-Beltrami operator equation on a manifold  $\mathcal{S}$ . The equation is  $\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H \mathbf{n}$ . Arrows point from descriptive labels to the corresponding parts of the equation: 'Laplace-Beltrami' points to  $\Delta_{\mathcal{S}}$ , 'function on manifold  $\mathcal{S}$ ' points to  $\mathbf{x}$ , 'gradient operator' points to  $\nabla_{\mathcal{S}}$ , 'divergence operator' points to  $\operatorname{div}_{\mathcal{S}}$ , 'mean curvature' points to  $H$ , and 'surface normal' points to  $\mathbf{n}$ .

Laplace-Beltrami

gradient operator

mean curvature

function on manifold  $\mathcal{S}$

divergence operator

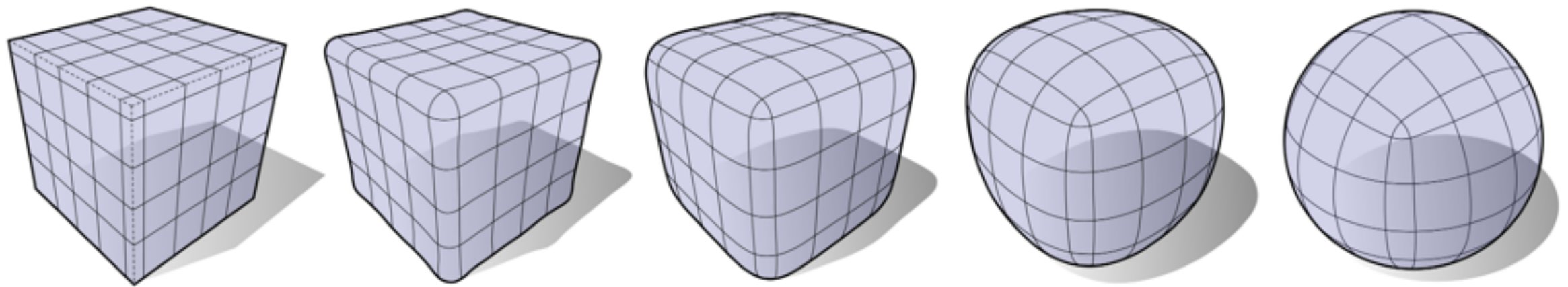
surface normal

$$\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H \mathbf{n}$$

# Literature

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# Next Time



**Discrete** Differential Geometry

<http://cs599.hao-li.com>

# Thanks!

