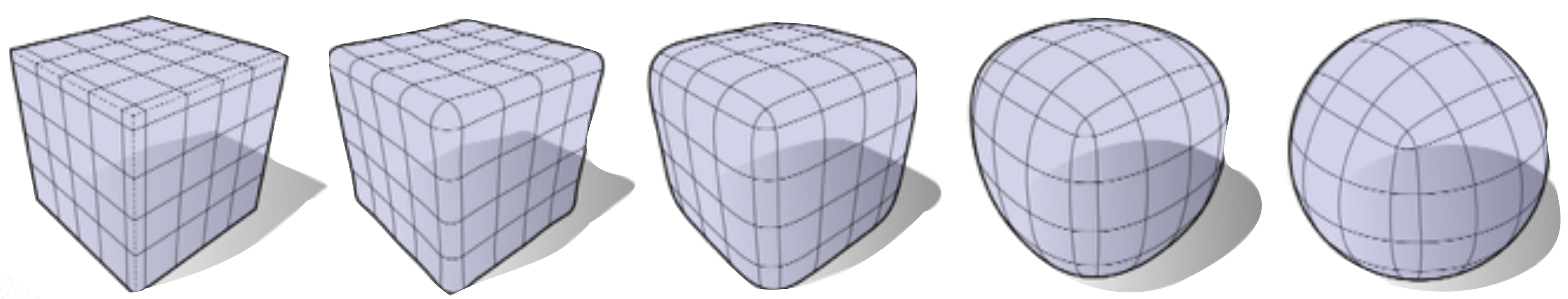
3.1 Classic Differential Geometry



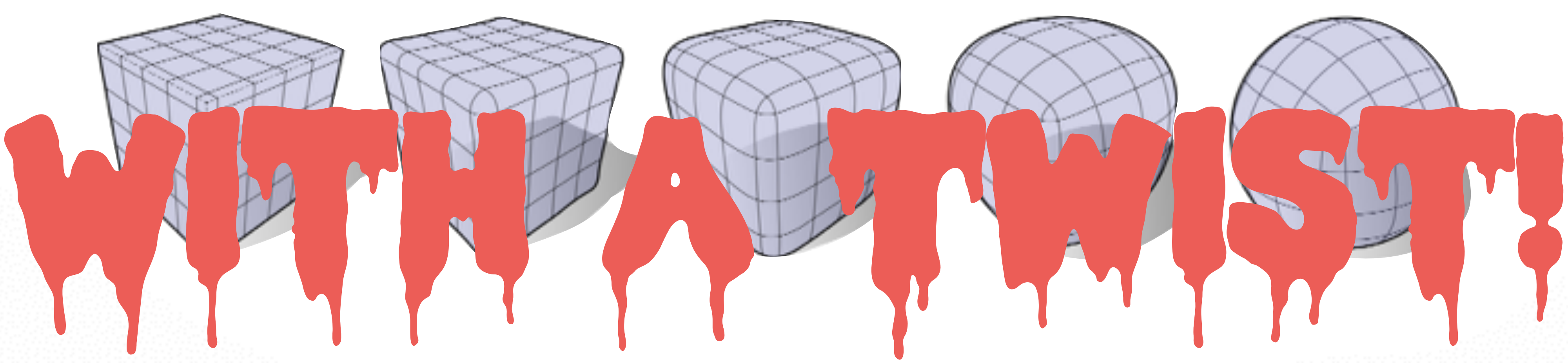
Hao Li

http://cs599.hao-li.com

Spring 2014

CSCI 599: **Digital Geometry Processing**

3.1 Classic Differential Geometry



Hao Li

http://cs599.hao-li.com

Administrative

- **Exercise handouts:** 11:59 PM every 2nd Wednesday
- My first office hours later from 2pm to 4pm
- This week only lecture.

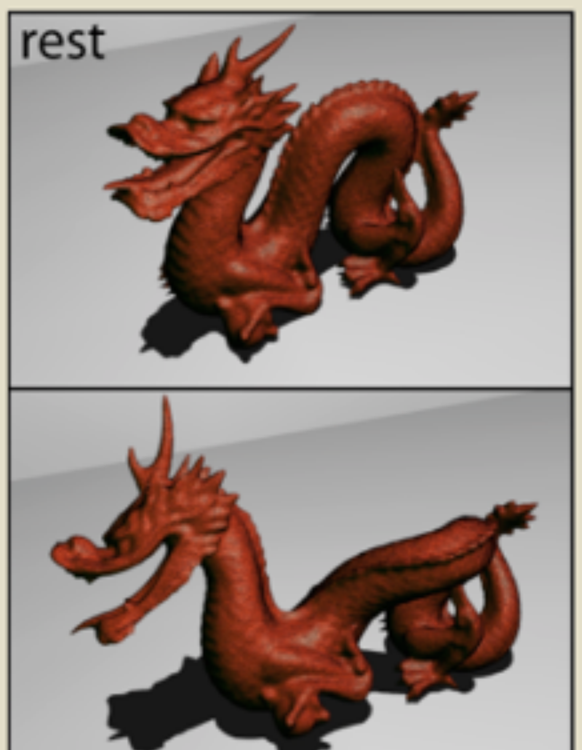
Some Updates: **run.usc.edu/vega**

Another awesome free library with half-edge data-structure

By Prof. Jernej Barbic

Vega FEM

MAIN DOWNLOAD/FAQ SCREENSHOTS ABOUT



JURIJ VEGA (1754-1802)

USC Viterbi School of Engineering

VEGA FEM LIBRARY

NEW: Vega FEM 2.0 released on Oct 8, 2013. New features described below.

Vega is a computationally efficient and stable C/C++ physics library for three-dimensional deformable object simulation. It is designed to model large deformations, including geometric and material nonlinearities, and can also efficiently simulate linear systems. Vega is open-source and free. It is released under the 3-clause BSD license, which means that it can be used freely both in academic research and in commercial applications.

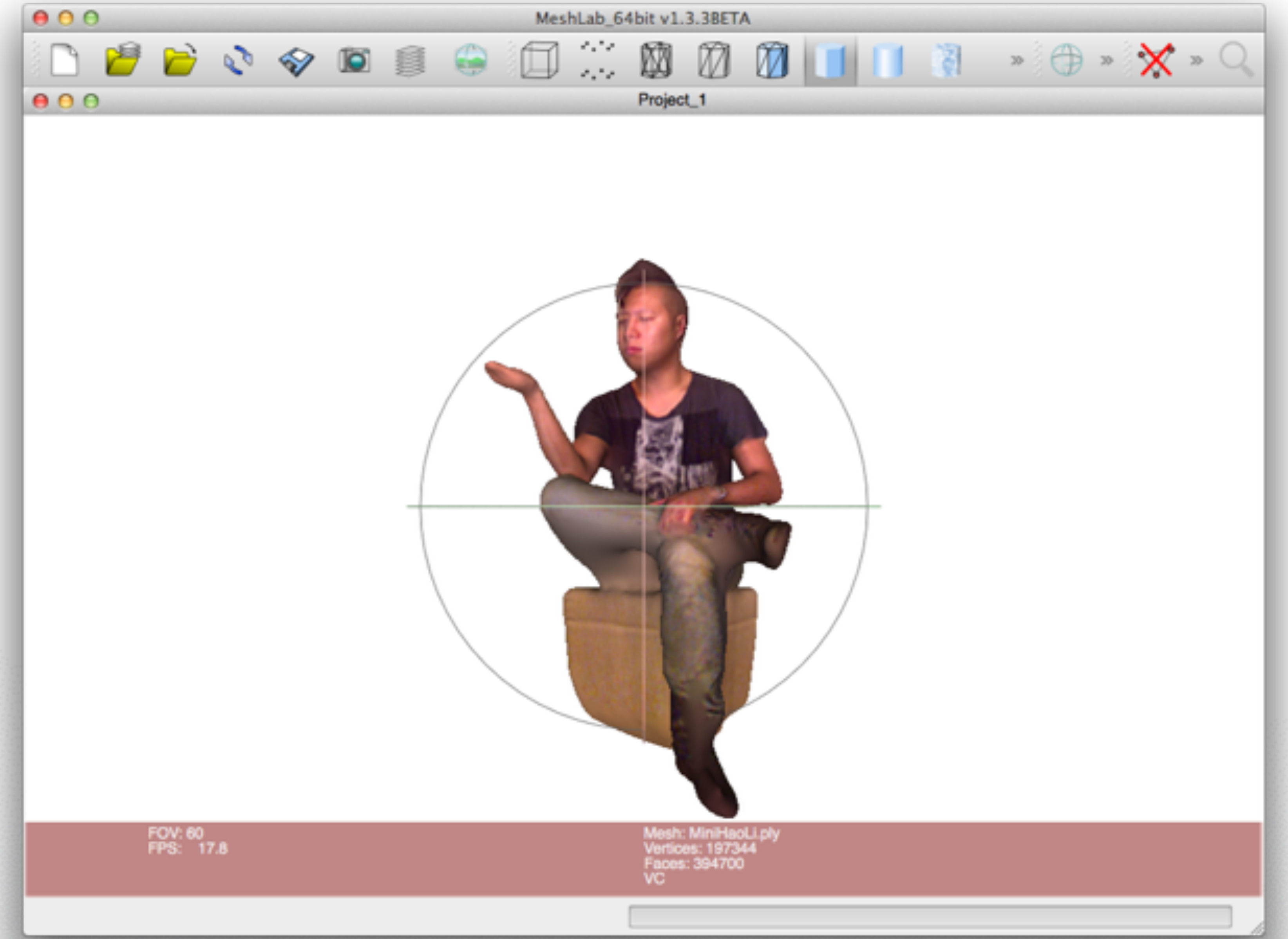
Vega implements several widely used methods for simulation of large deformations of 3D **solid** deformable objects:

- co-rotational linear FEM elasticity [MG04]; it can also compute the exact tangent stiffness matrix [Bar12] (similar to [CPSS10]),
- linear FEM elasticity [Sha90],
- invertible isotropic nonlinear FEM models [ITF04, TSIF05],

FYI

MeshLab

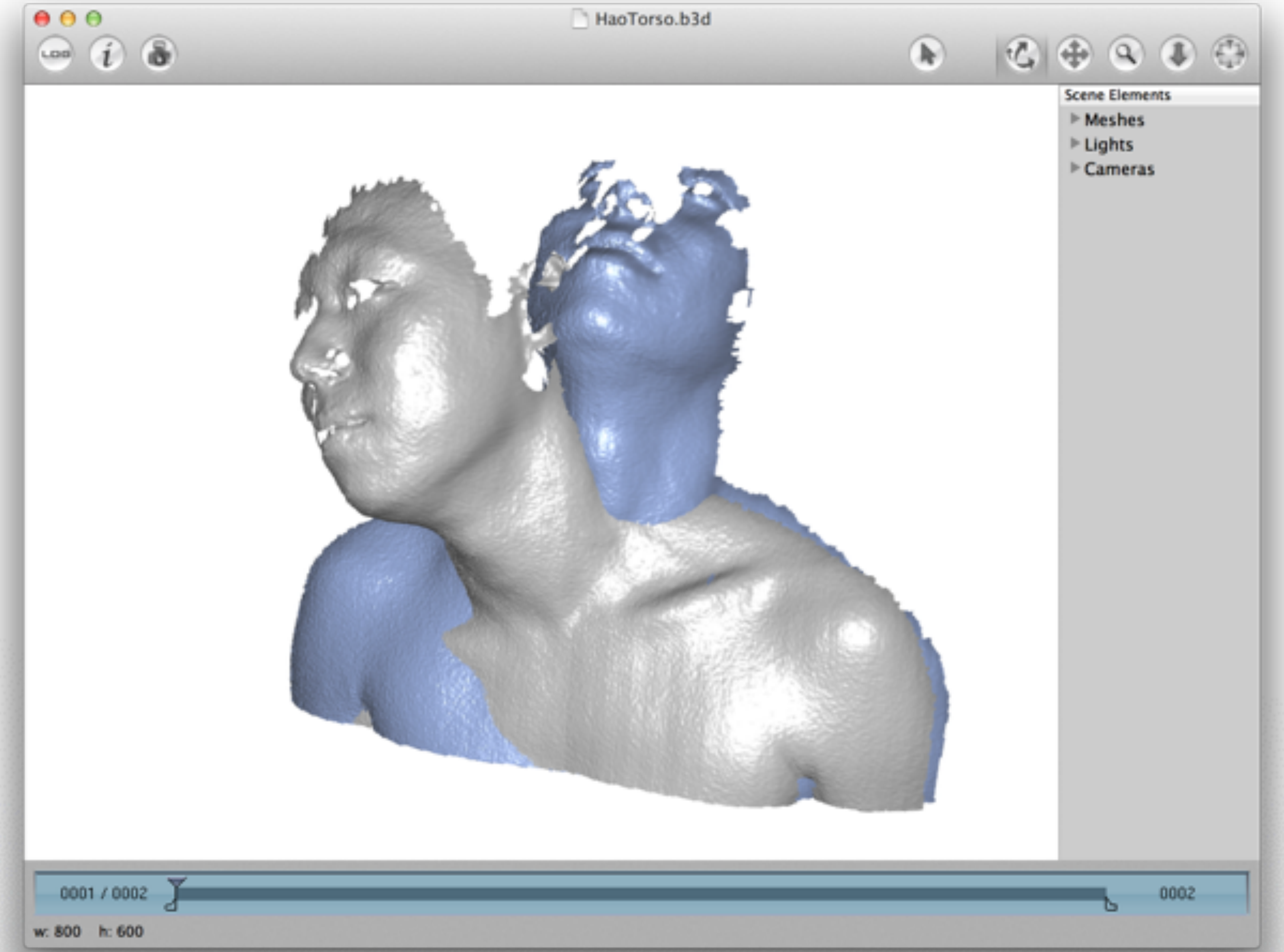
Popular Mesh Processing Software (meshlab.sourceforge.net)



FYI

BeNTO3D

Mesh Processing Framework for Mac (www.bento3d.com)



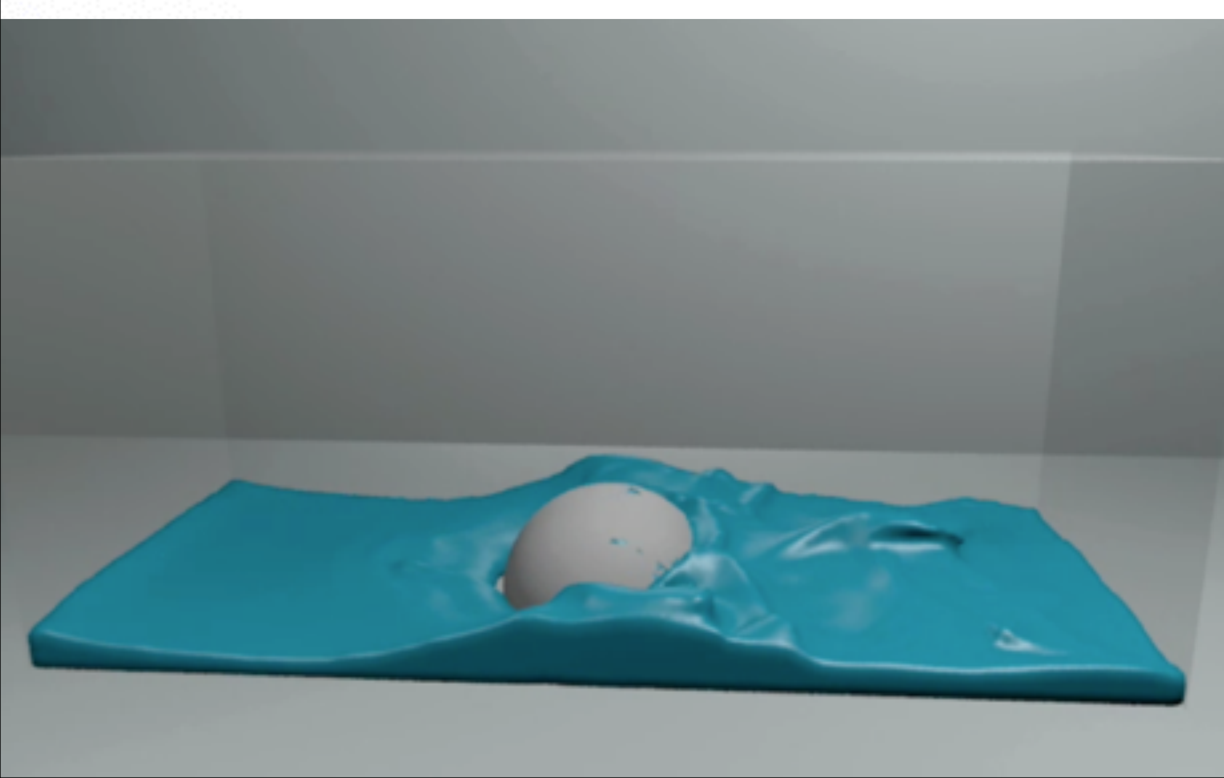
Last Time

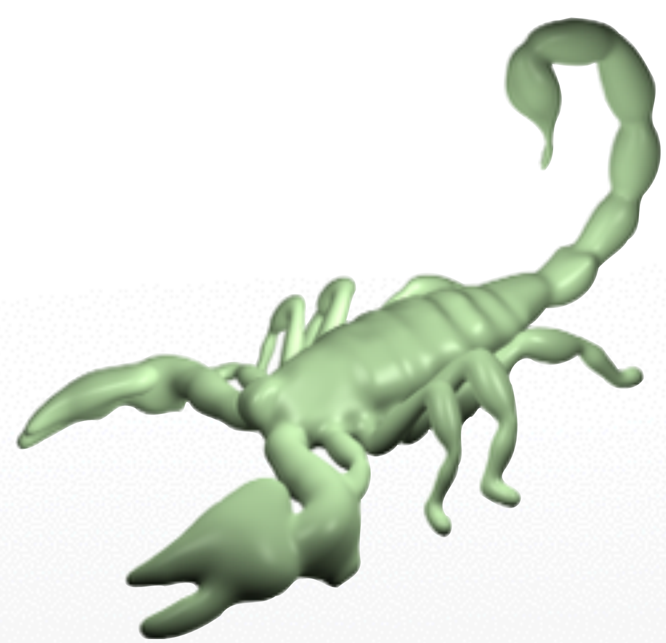
Discrete Representations

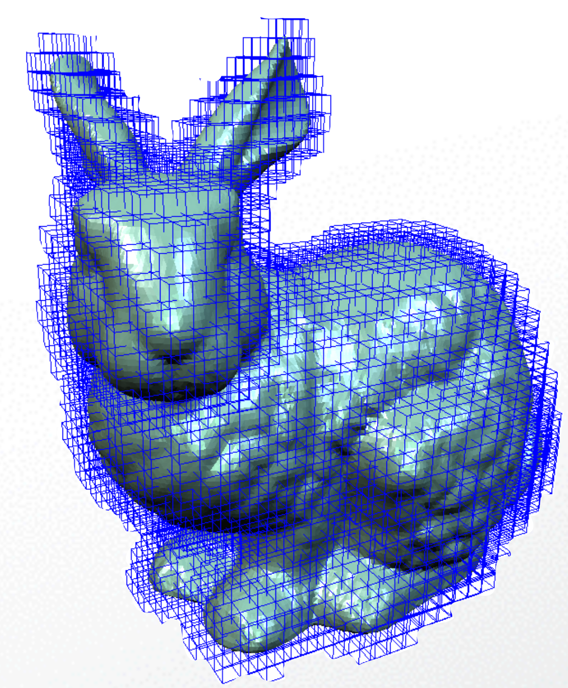
- Explicit (parametric, polygonal meshes)
- Implicit Surfaces (SDF, grid representation)
- Conversions
  - E→I: Closest Point, SDF, Fast Marching
  - I→E: Marching Cubes Algorithm

Geometry

Topology



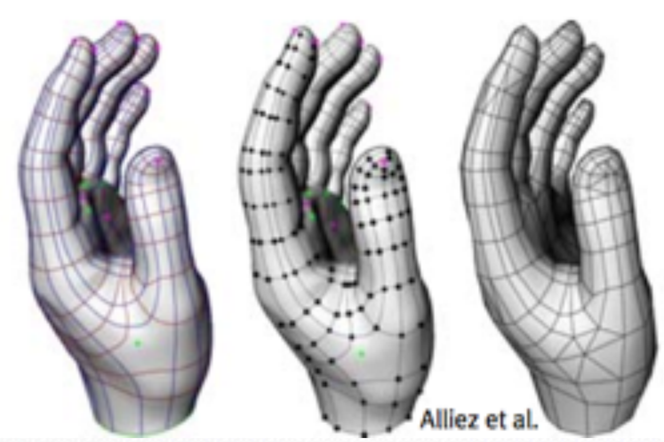


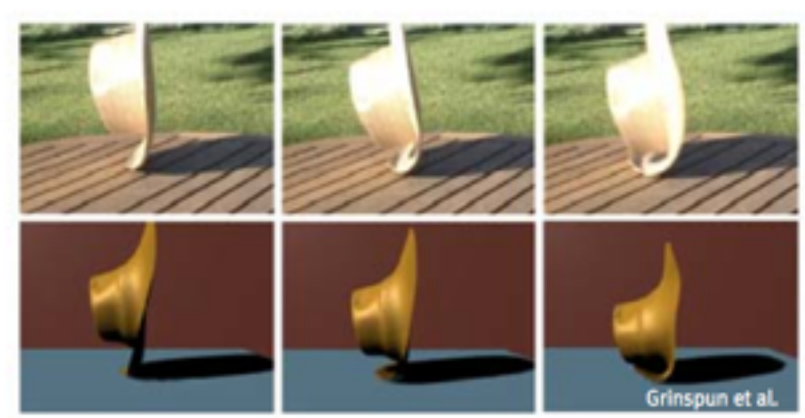


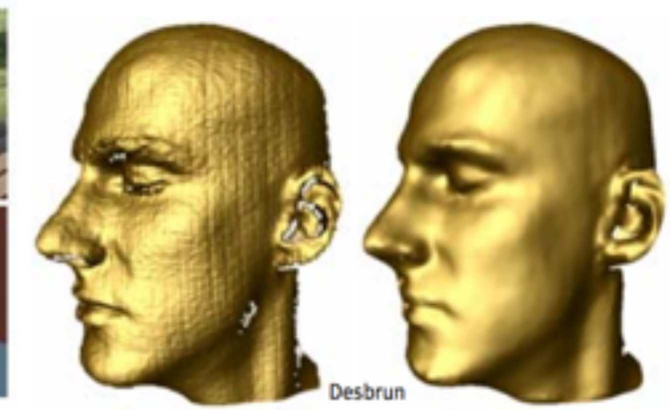
Differential Geometry

Why do we care?

- Geometry of surfaces
- Mothertongue of physical theories
- Computation: processing / simulation



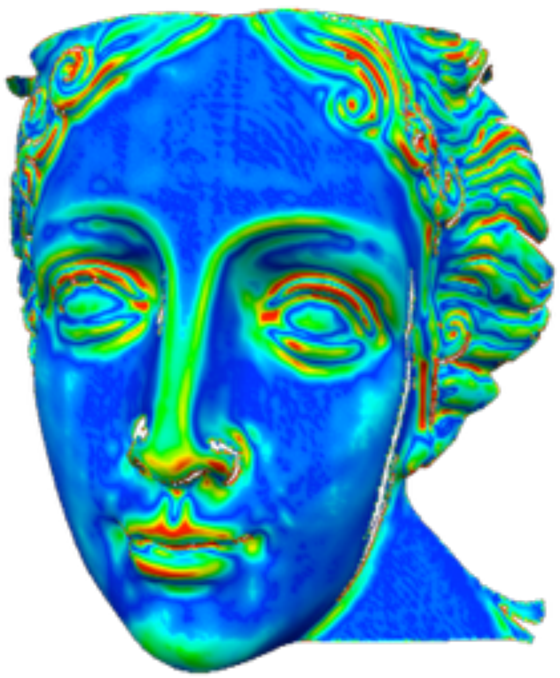


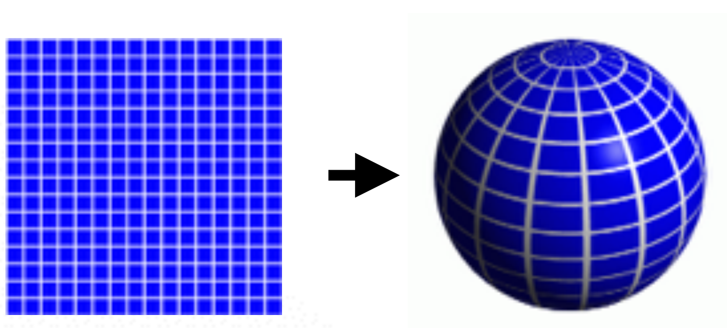


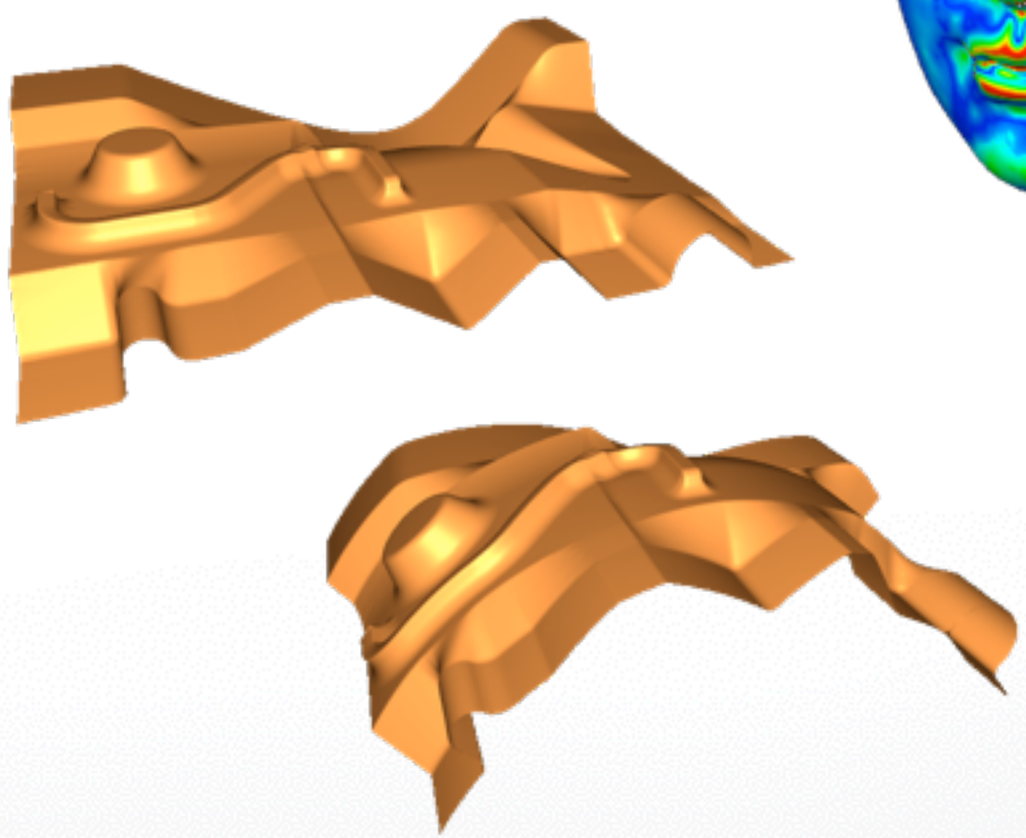
Motivation

We need differential geometry to compute

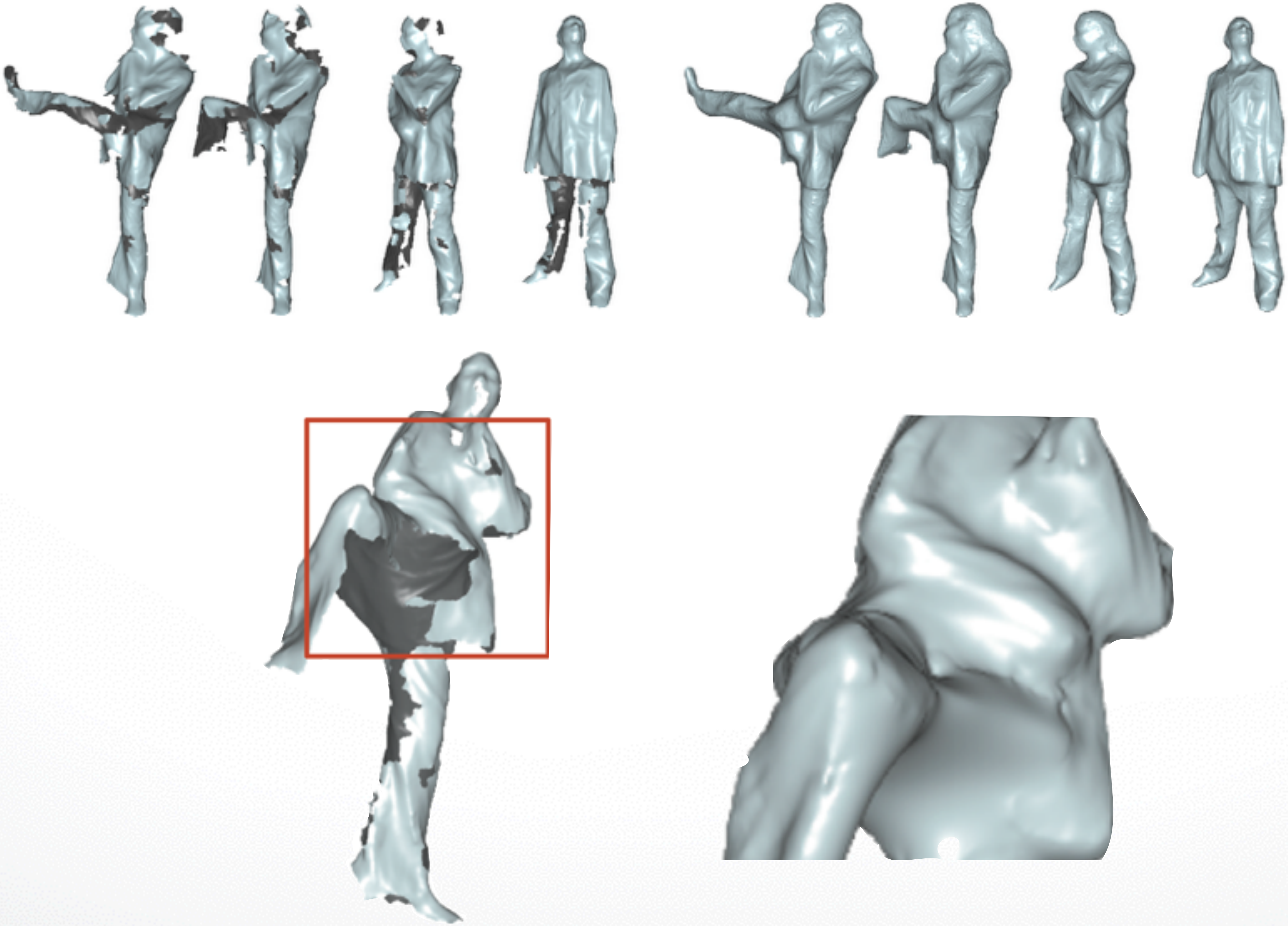
- surface curvature
- paramaterization distortion
- deformation energies



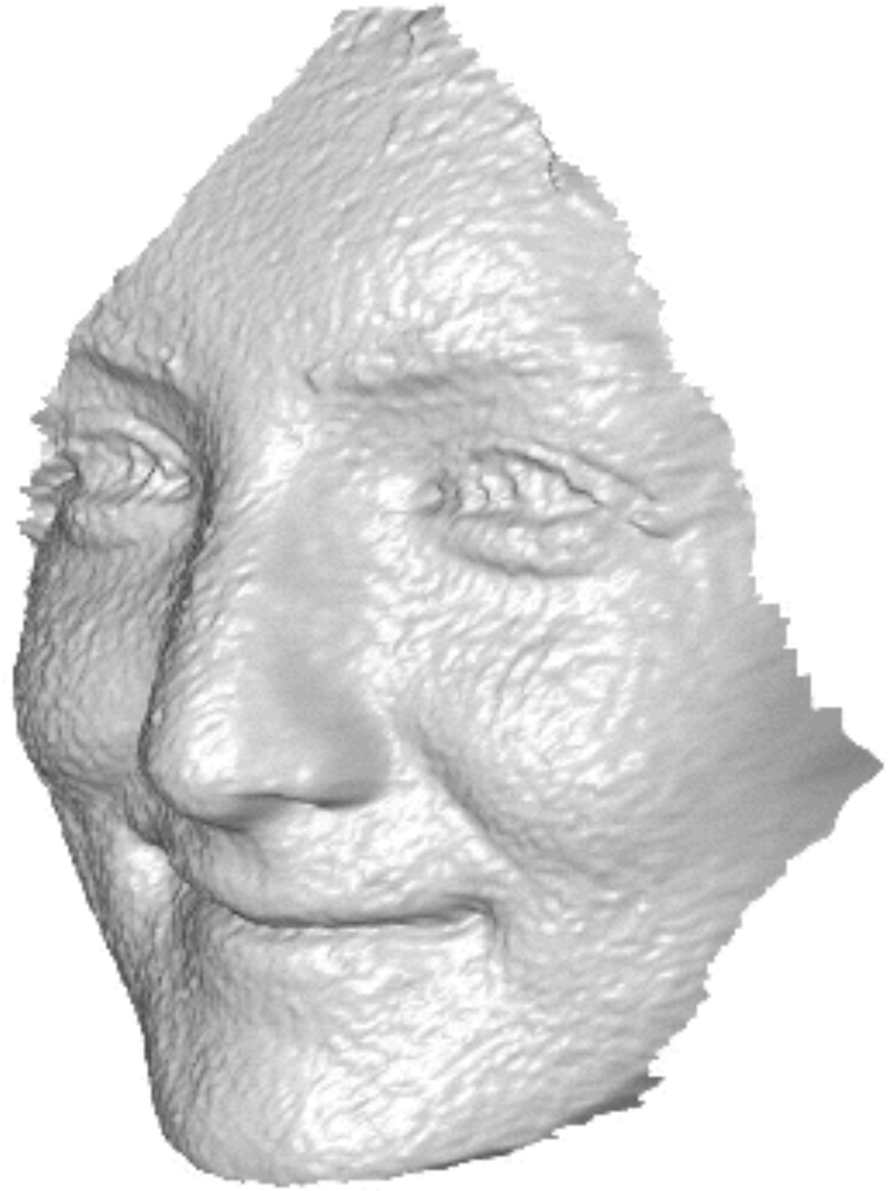




Applications: 3D Reconstruction



Applications: Head Modeling

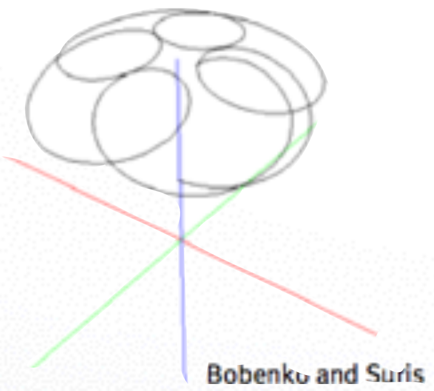


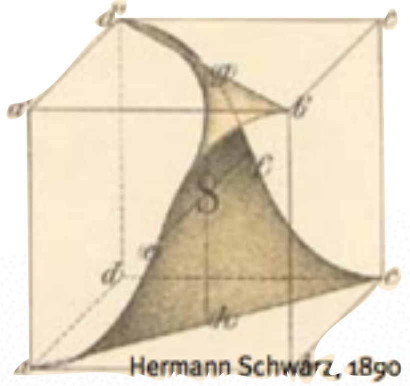
Applications: Facial Animation

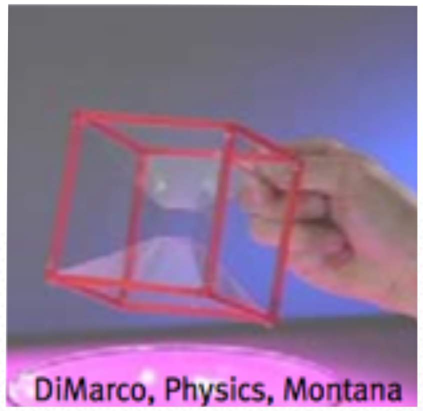
Motivation

Geometry is the key

- studied for centuries (Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether…)
- mostly differential geometry
  - differential and integral calculus
- invariants and symmetries







Getting Started

How to apply DiffGeo ideas?

- surfaces as a collecition of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - **discrete** vs. **discretized**

Let’s look at that first

Getting Started

What characterizes structure(s)?

- What is shape?
  - Euclidean Invariance
- What is physics?
  - Conservation/Balance Laws
- What can we measure?
  - area, curvature, mass, flux, circulation

Getting Started

Invariant descriptors

- quantities invariant under a set of transformations

Intrinsic descriptor

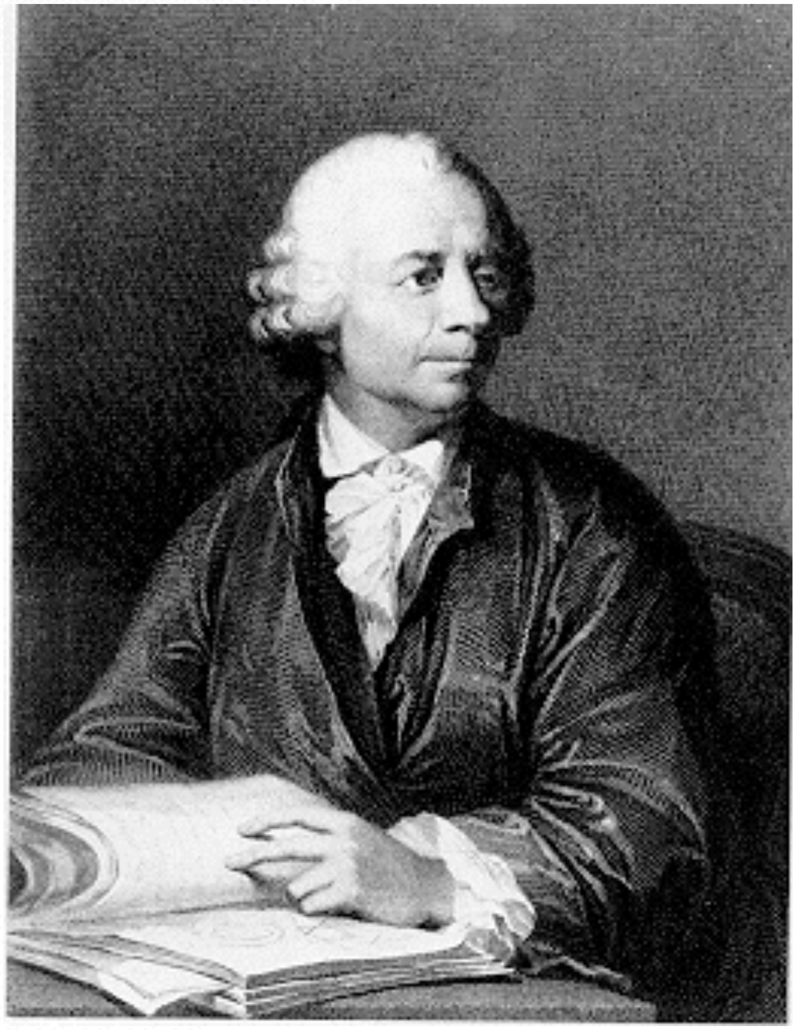
- quantities which do notd depend on a coordinate frame

Outline

- **Parametric Curves**
- Parametric Surfaces

Formalism & Intuition

Differential Geometry

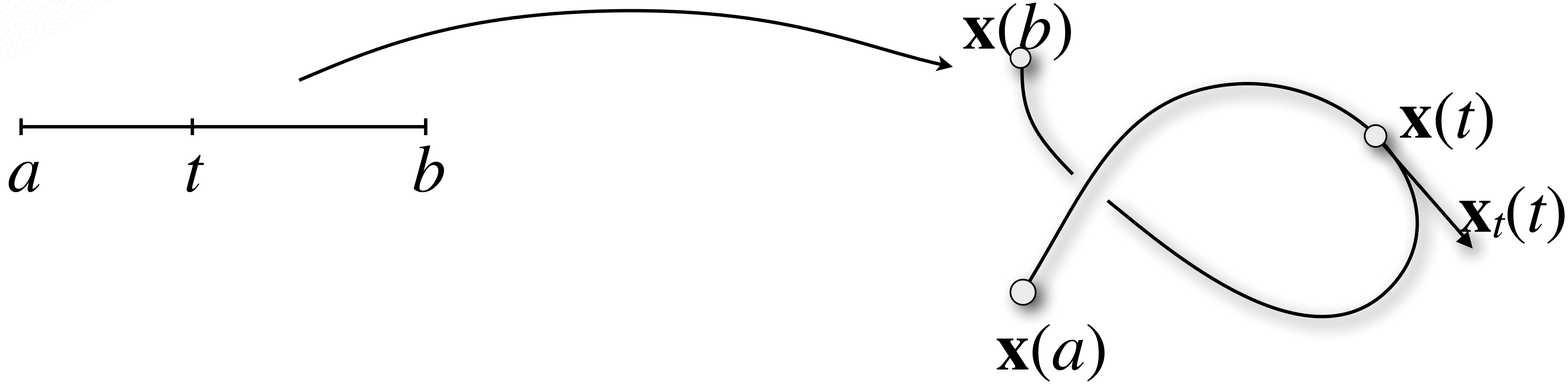


Leonard Euler (1707-1783)

Carl Friedrich Gauss (1777-1855)

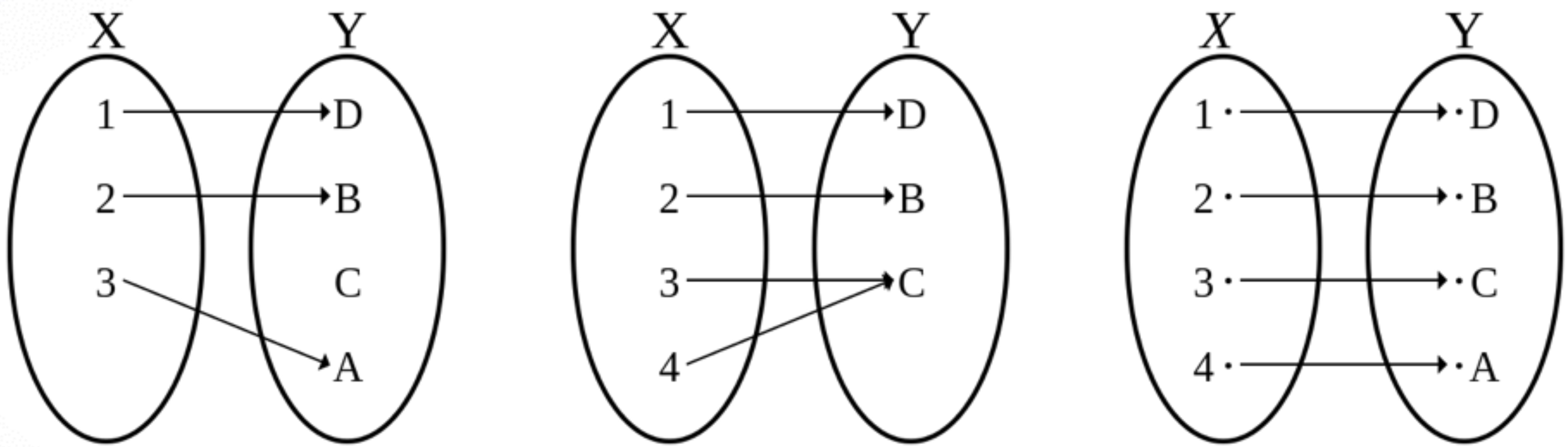
Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)\,/\mathrm{d}t \\ \mathrm{d}y(t)\,/\mathrm{d}t \\ \mathrm{d}z(t)\,/\mathrm{d}t \end{pmatrix}$$

Recall: **Mappings**



Injective

Surjective

Bijective

NO SELF-INTERSECTIONS

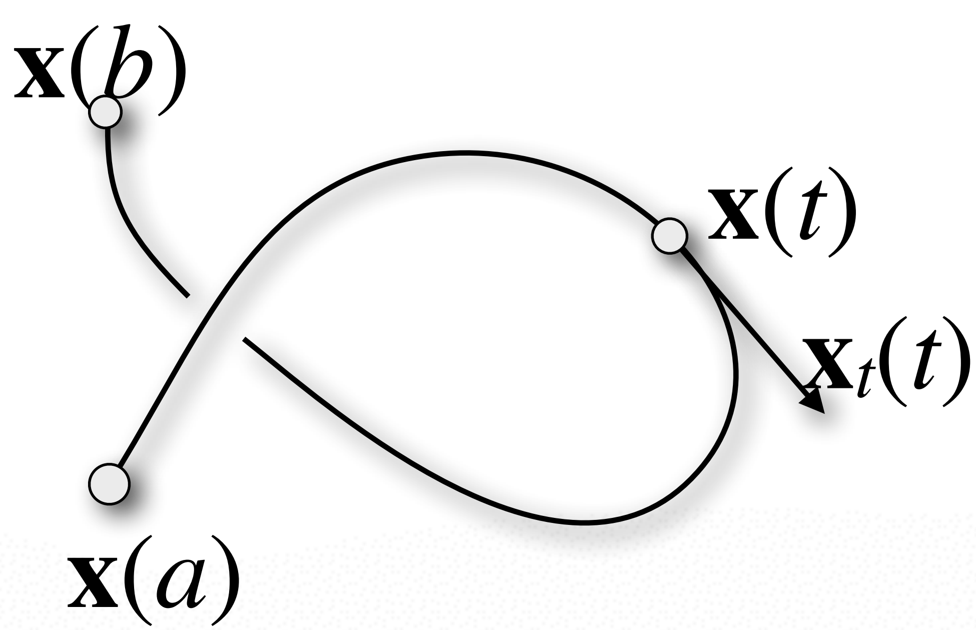
SELF-INTERSECTIONS

AMBIGUOUS PARAMETERIZATION

Parametric Curves

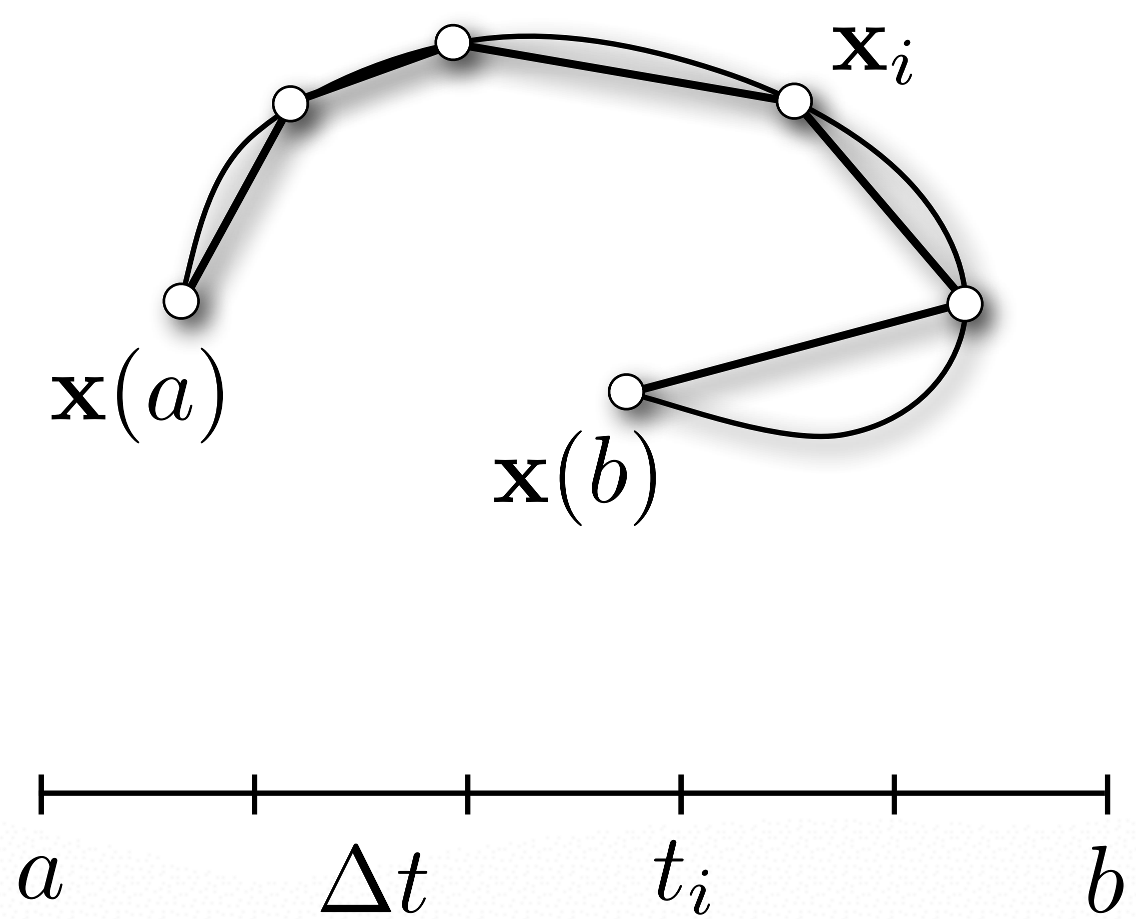
A parametric curve $\mathbf{x}(t)$ is

- simple: $\mathbf{x}(t)$ is injective (no self-intersections)
- differentiable: $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
- regular: $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$



Length of a Curve

Let $t_i = a + i\Delta t$ **and** $\mathbf{x}_i = \mathbf{x}(t_i)$



Length of a Curve

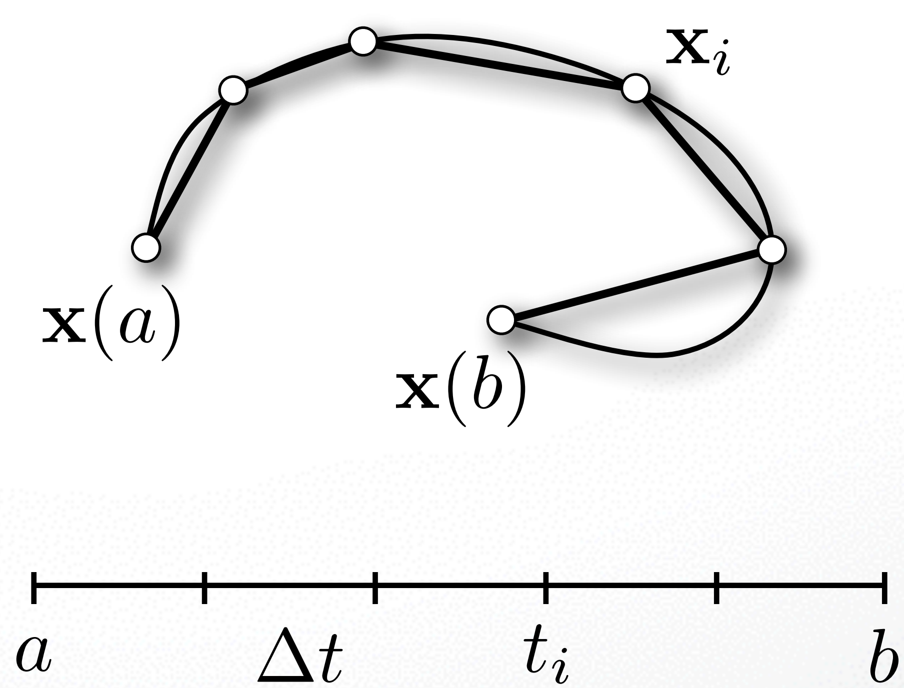
Polyline chord length

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t\,, \qquad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

norm change

Curve arc length ($\Delta t \to 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\|\, \mathrm{d}t$$



length =

integration of infinitesimal change

× norm of speed

Re-Parameterization

Mapping of parameter domain

$$u : [a, b] \to [c, d]$$

Re-parameterization w.r.t. $u(t)$

$$[c, d] \to \mathbb{R}^3\,, \quad t \mapsto \mathbf{x}(u(t))$$

Derivative (chain rule)

$$\frac{\mathrm{d}\mathbf{x}(u(t))}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{x}_u(u(t))\; u_t(t)$$

Re-Parameterization

Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] \quad , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

$$\Rightarrow \quad \mathbf{g}(\phi(t)) \ = \ \mathbf{f}(t)$$

Arc Length Parameterization

Mapping of parameter domain:

$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\|\,\mathrm{d}t$$

Parameter s for $\mathbf{x}(s)$ equals length from $\mathbf{x}(a)$ to $\mathbf{x}(s)$

$$\mathbf{x}(s) \;=\; \mathbf{x}(s(t)) \qquad\qquad \mathrm{d}s \;=\; \|\mathbf{x}_t\|\,\mathrm{d}t$$

same infinitesimal change

Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1\,, \quad \mathbf{x}_s(s)\cdot\mathbf{x}_{ss}(s) = 0$$

defines orthonormal frame

The Frenet Frame

Taylor expansion

$$\mathbf{x}(t+h) \;=\; \mathbf{x}(t) \;+\; \mathbf{x}_t(t)\,h \;+\; \frac{1}{2}\mathbf{x}_{tt}(t)\,h^2 \;+\; \frac{1}{6}\mathbf{x}_{ttt}(t)\,h^3 \;+\; \ldots$$

for convergence analysis and approximations

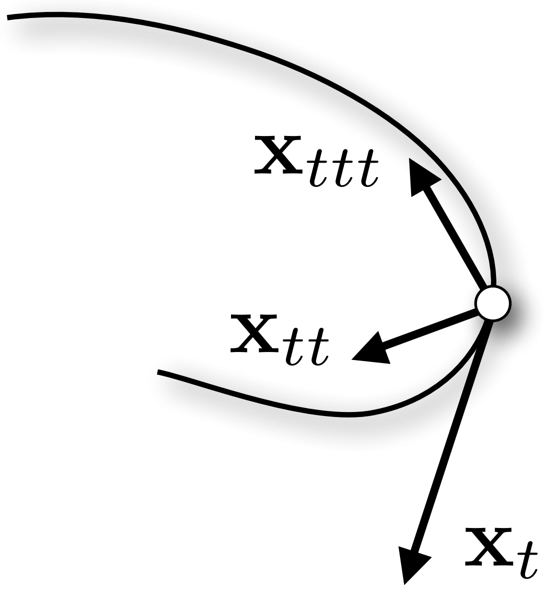
Define local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ (Frenet frame)

$$\mathbf{t} \;=\; \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|} \qquad \mathbf{n} \;=\; \mathbf{b} \times \mathbf{t} \qquad \mathbf{b} \;=\; \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

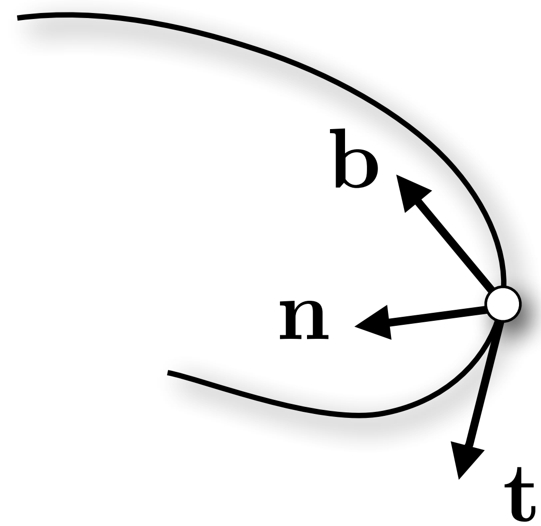
tangent   main normal   binormal

The Frenet Frame

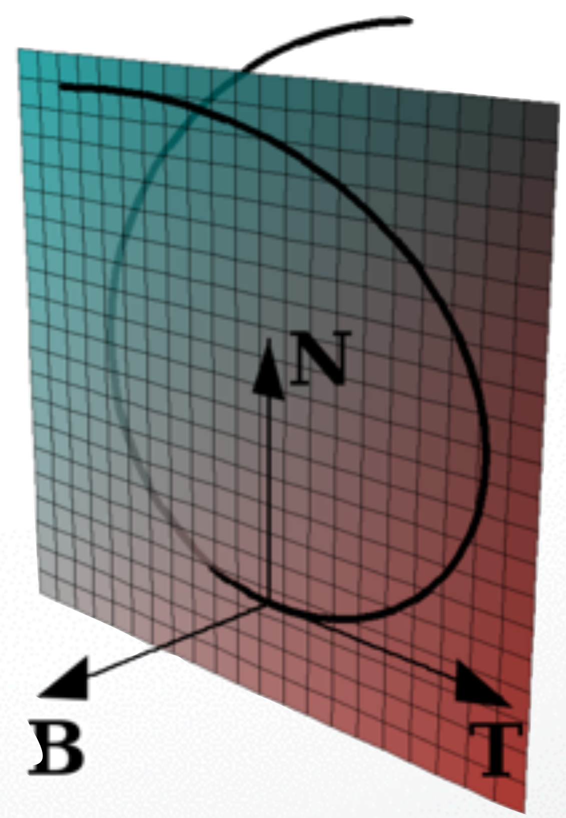
Orthonormalization of local frame



local affine frame



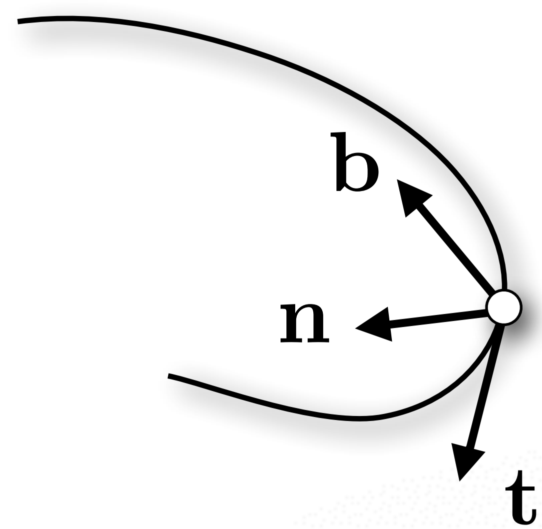
Frenet frame



The Frenet Frame

Frenet-Serret: Derivatives w.r.t. arc length s

$$
\begin{aligned}
\mathbf{t}_s &= \quad\quad\quad +\kappa\mathbf{n} \\
\mathbf{n}_s &= -\kappa\mathbf{t} \quad\quad\quad +\tau\mathbf{b} \\
\mathbf{b}_s &= \quad\quad\quad -\tau\mathbf{n}
\end{aligned}
$$



Curvature (deviation from straight line)

$$\kappa = \|\mathbf{x}_{ss}\|$$

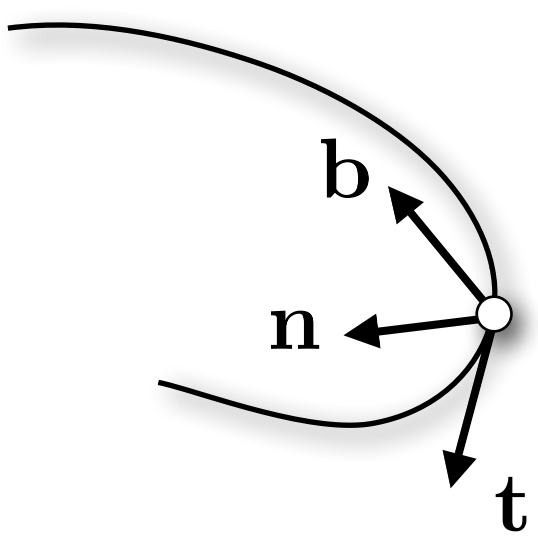
Torsion (deviation from planarity)

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature and Torsion

Planes defined by $\mathbf{x}$ and two vectors:

- osculating plane: vectors $\mathbf{t}$ and $\mathbf{n}$
- normal plane: vectors $\mathbf{n}$ and $\mathbf{b}$
- rectifying plane: vectors $\mathbf{t}$ and $\mathbf{b}$



Osculating circle

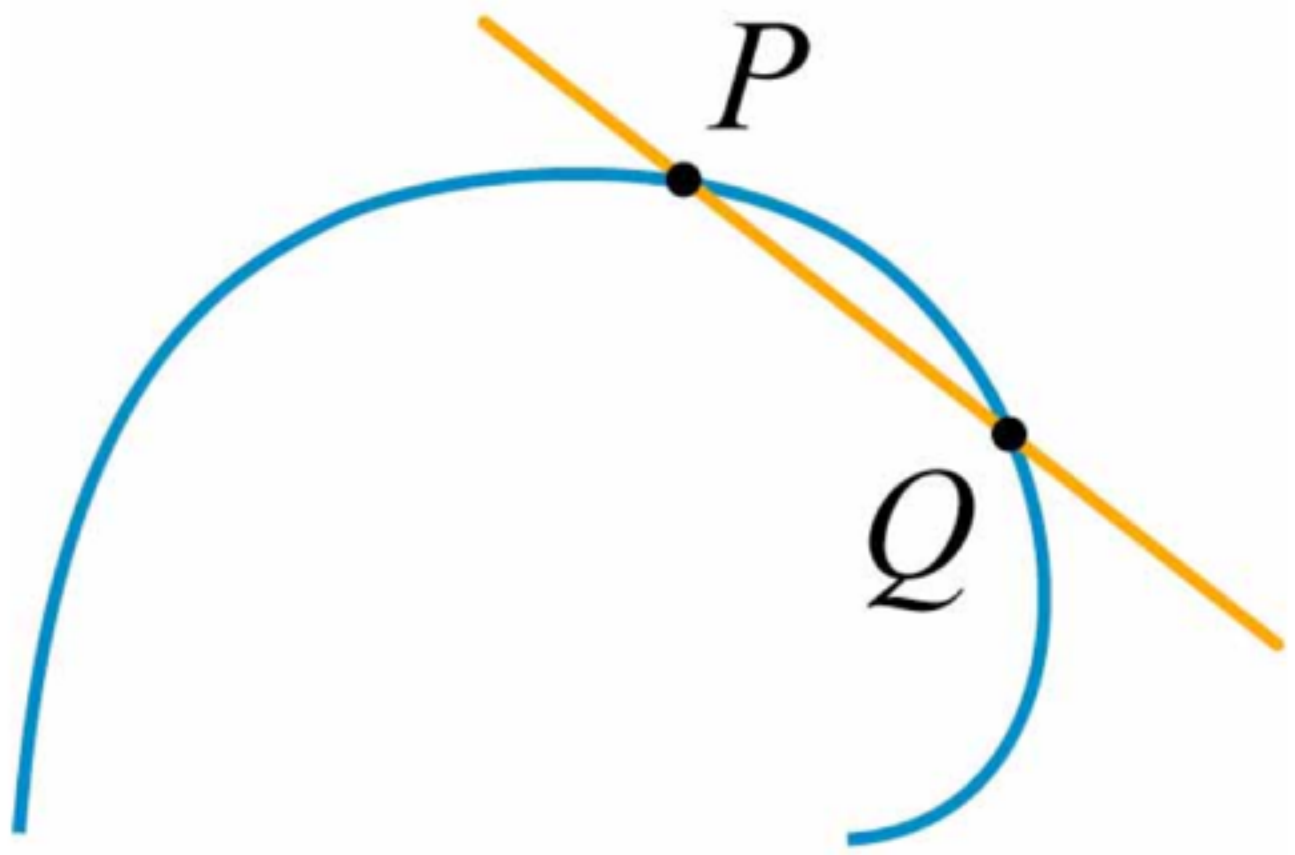
- second order contact with curve
- center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
- radius $1/\kappa$

Curvature and Torsion

- **Curvature**: Deviation from straight line
- **Torsion**: Deviation from planarity
- Independent of parameterization
  - **intrinsic** properties of the curve
- Euclidean invariants
  - **invariant** under rigid motion
- Define curve **uniquely** up to a rigid motion

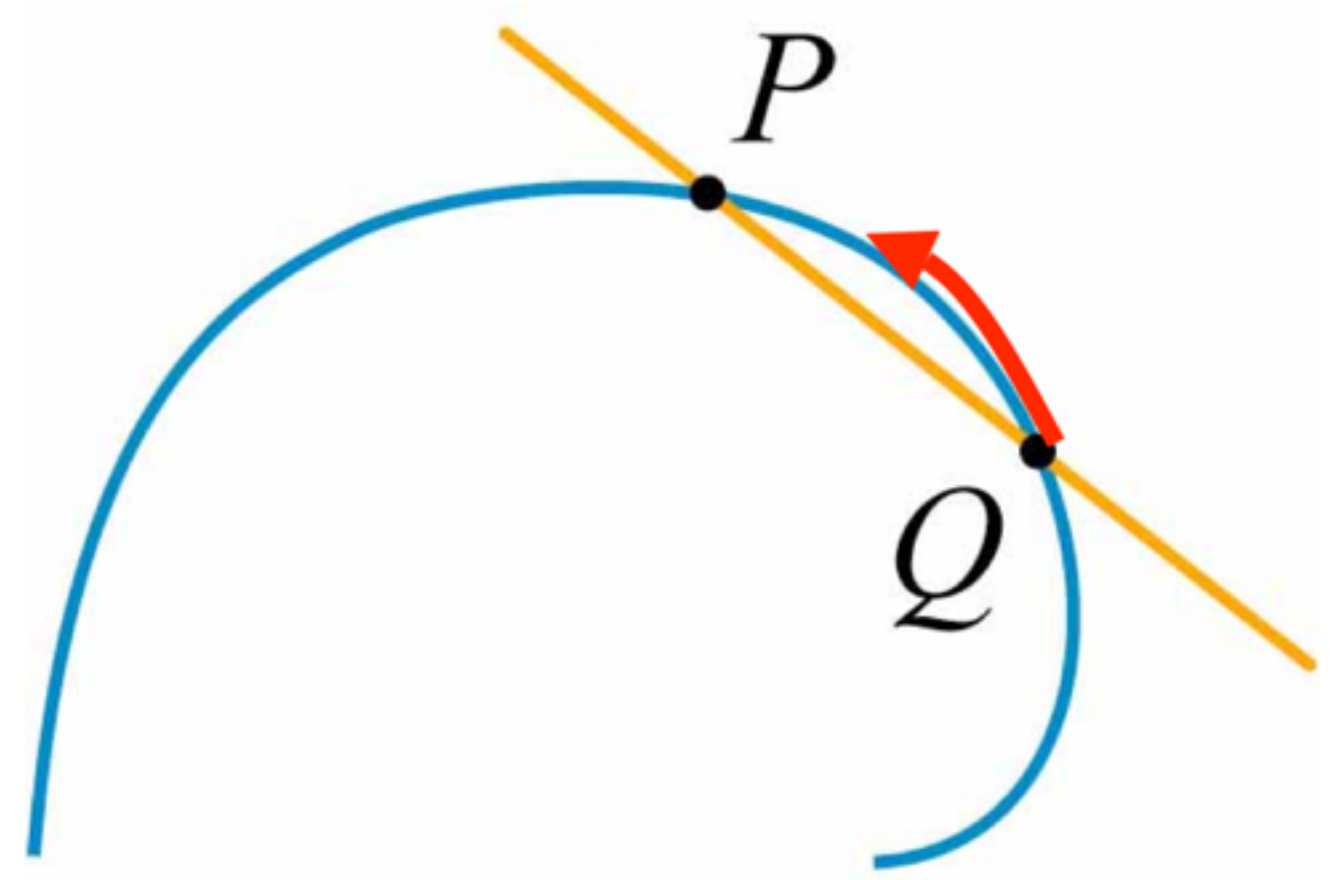
Curvature: Some Intuition

A line through two points on the curve (Secant)



Curvature: Some Intuition

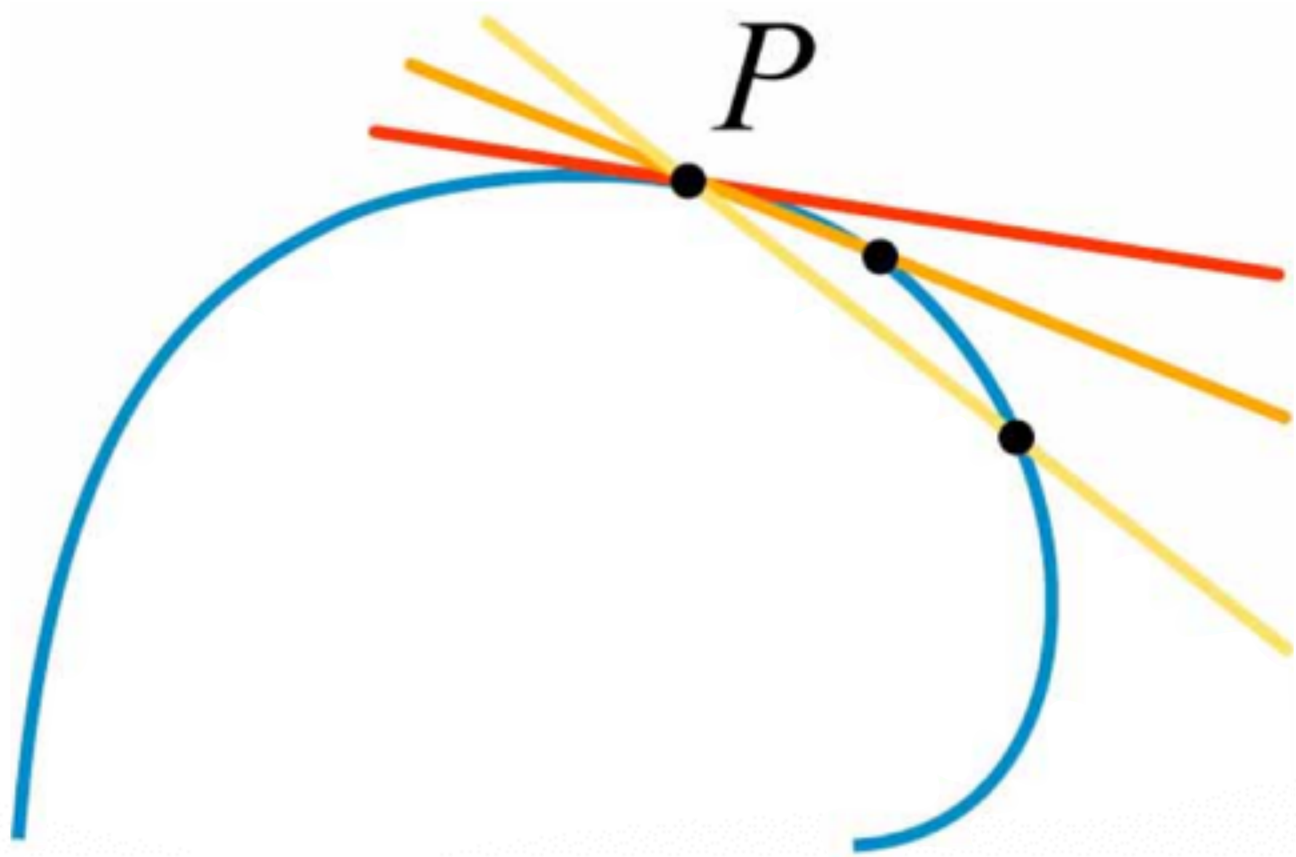
A line through two points on the curve (Secant)



Curvature: Some Intuition

Tangent, the first approximation

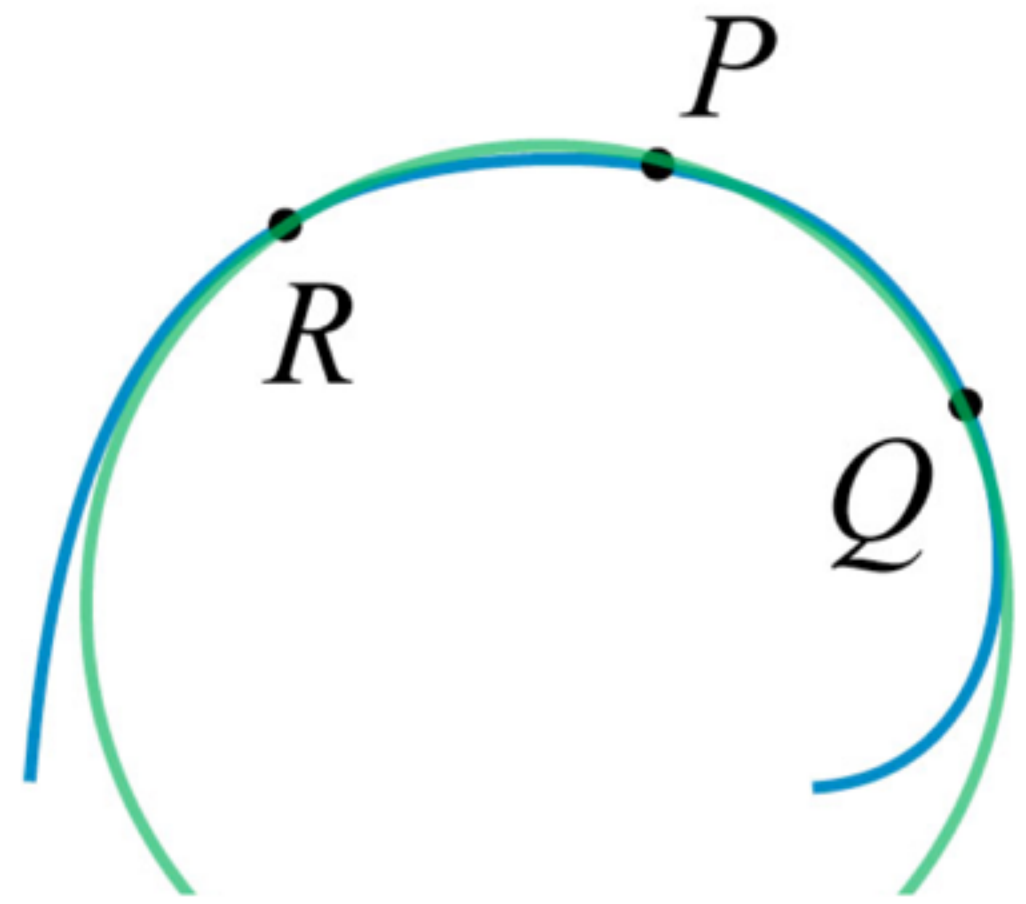
limiting secant as the two points come together



Curvature: Some Intuition

Circle of curvature

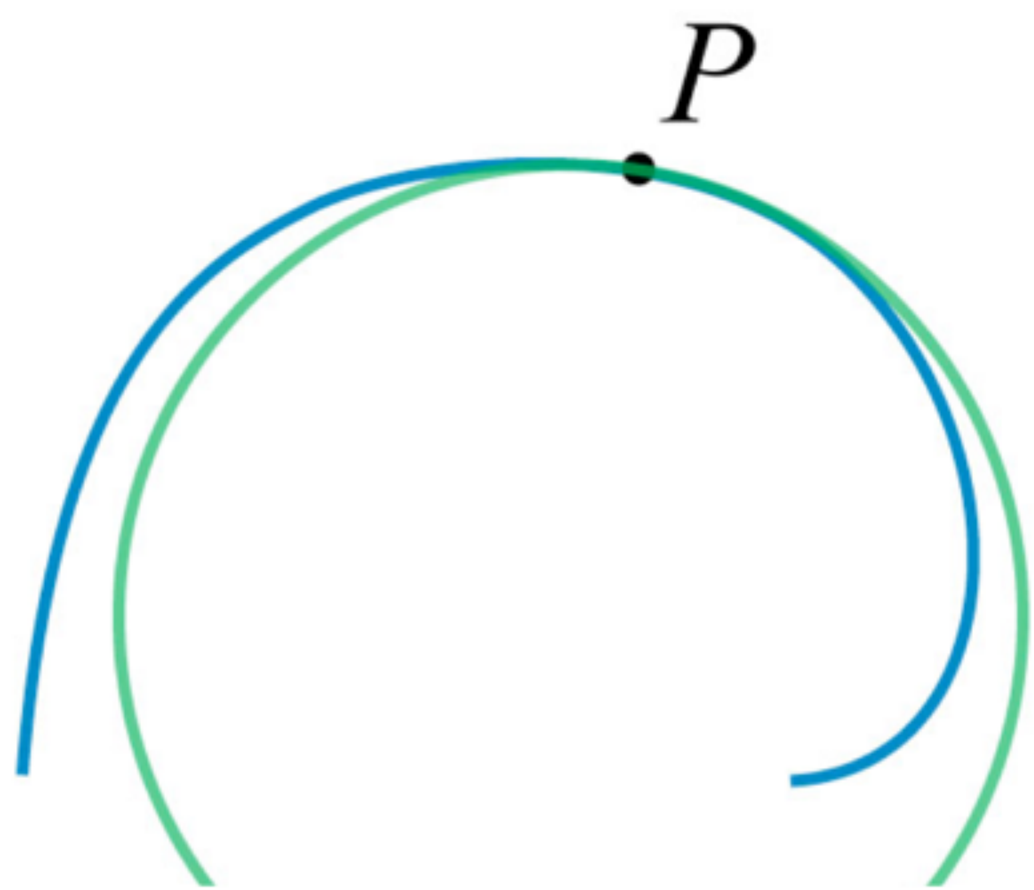
Consider the circle passing through 3 pints of the curve



Curvature: Some Intuition

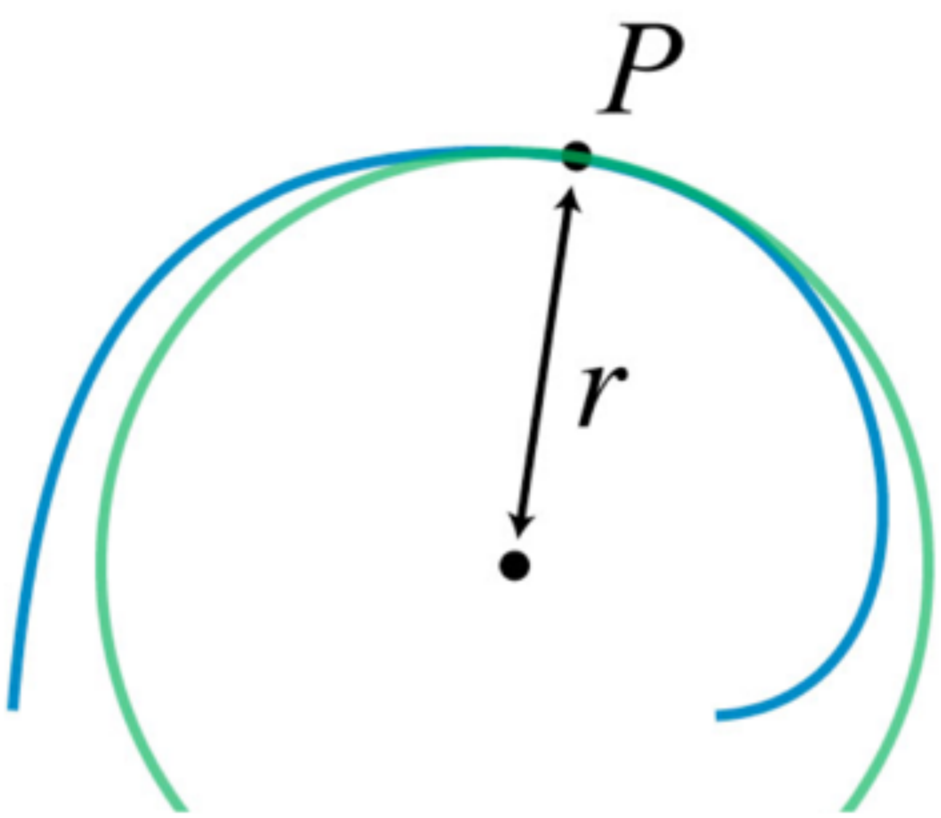
Circle of curvature

The limiting circle as three points come together



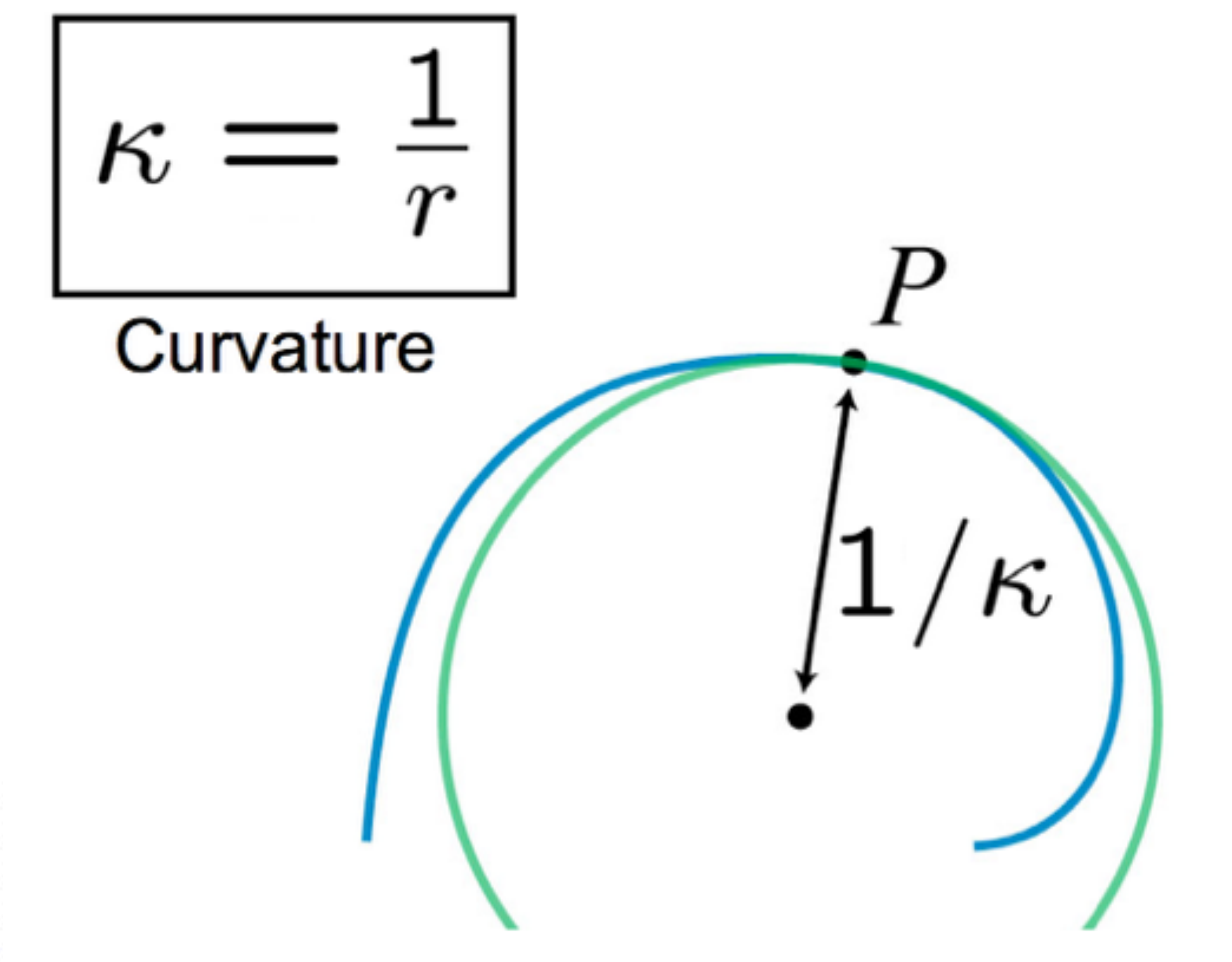
Curvature: Some Intuition

Radius of curvature *r*



Curvature: Some Intuition

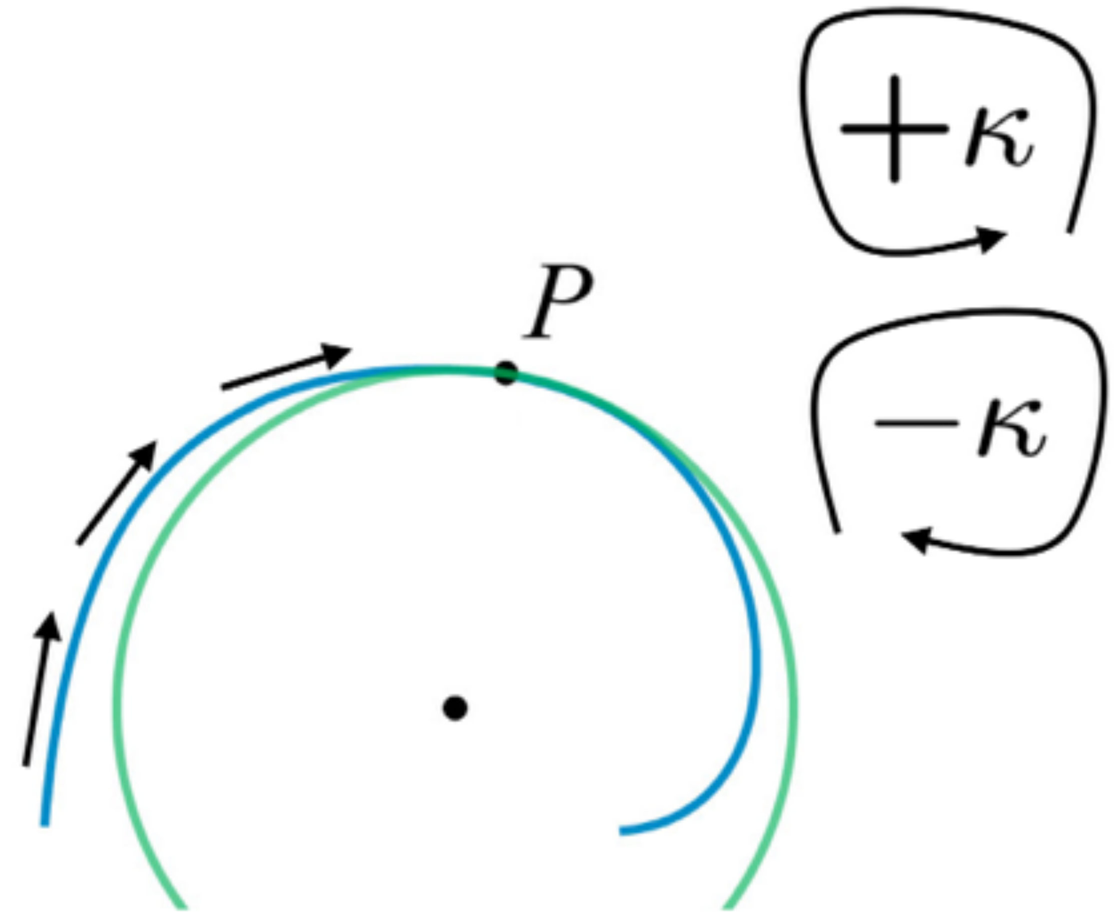
Radius of curvature *r*



Curvature: Some Intuition

Signed curvature

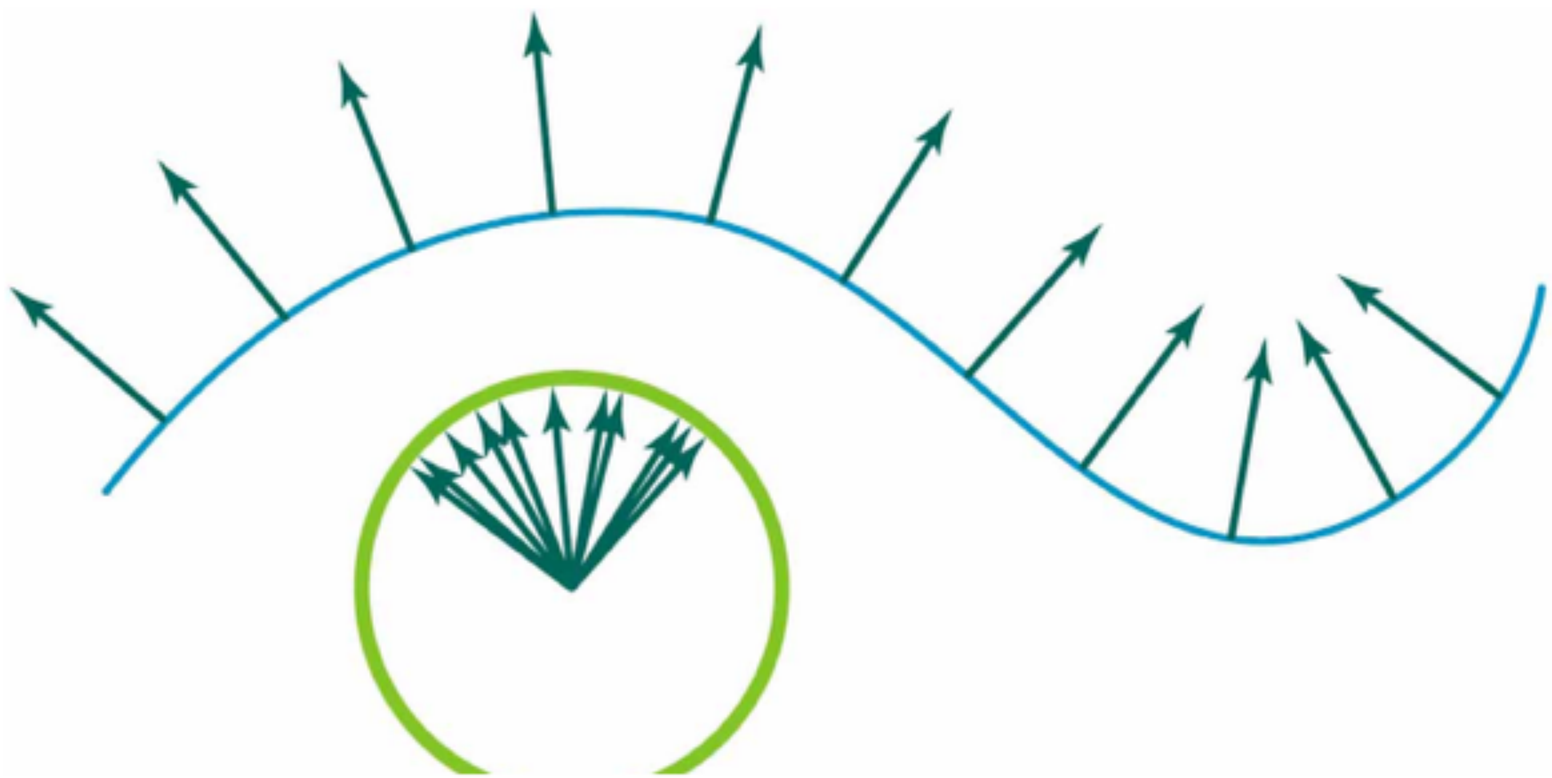
Sense of traversal along curve



Curvature: Some Intuition

Gauß map $\hat{\mathbf{n}}(\mathbf{x})$

Point on curve maps to point on unit circle



Curvature: Some Intuition

Shape operator (Weingarten map)

Change in normal as we slide along curve

negative directional derivative D of Gauß map

$$\mathbf{S}(\mathbf{v}) = -D_{\mathbf{v}}\hat{\mathbf{n}}$$

describes directional curvature

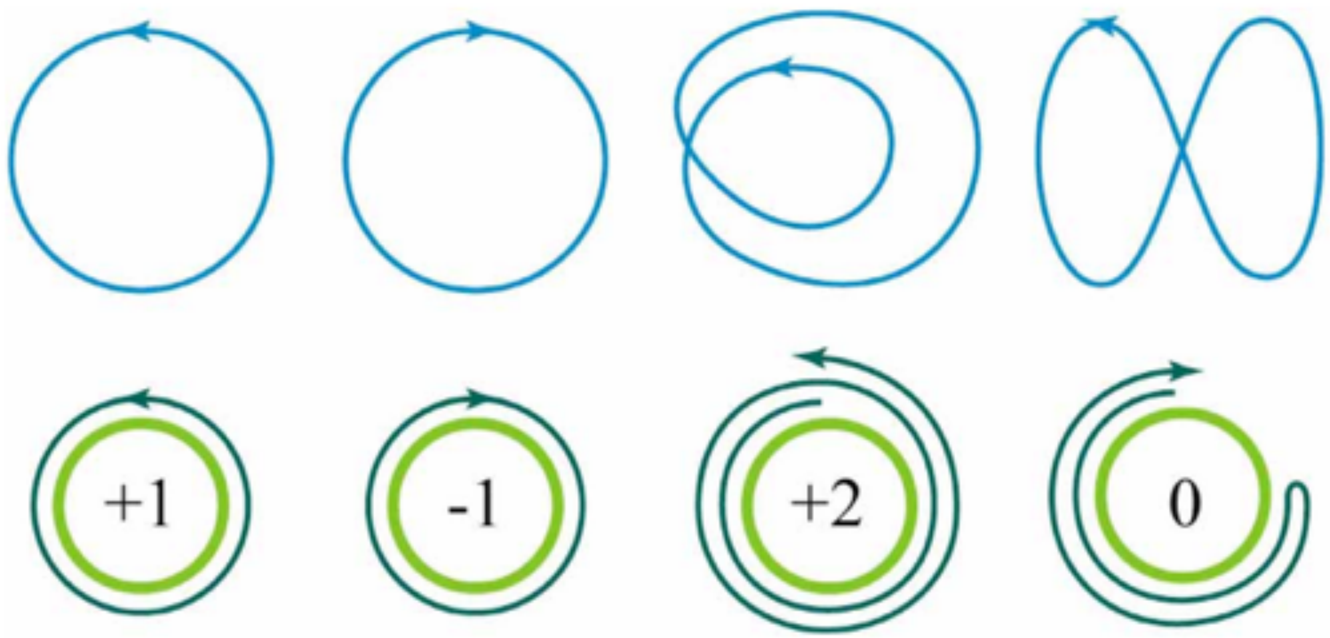
using normals as degrees of freedom

→ accuracy/convergence/implementation (discretization)

Curvature: Some Intuition

Turning number, *k*

Number of orbits in Gaussian image

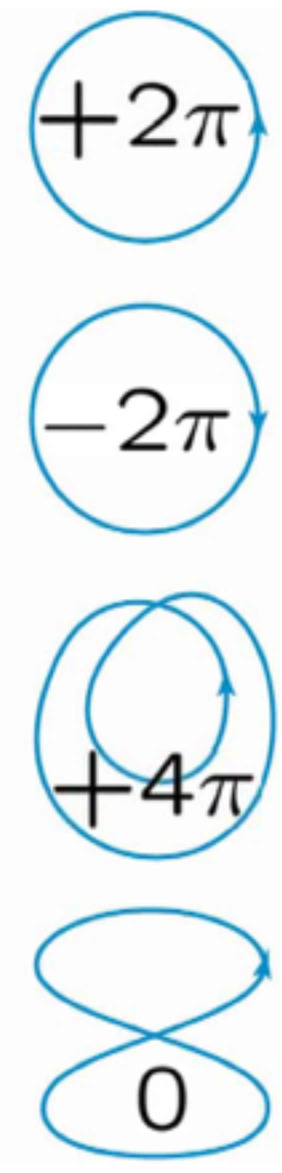


Curvature: Some Intuition

Turning number theorem

For a closed curve, the integral of curvature is an integer multiple of 2π

$$\boxed{\int_{\Omega} \kappa ds = 2\pi k}$$



Take Home Message

In the limit of a refinement sequence, discrete measure of length and curvature **agree** with continuous measures

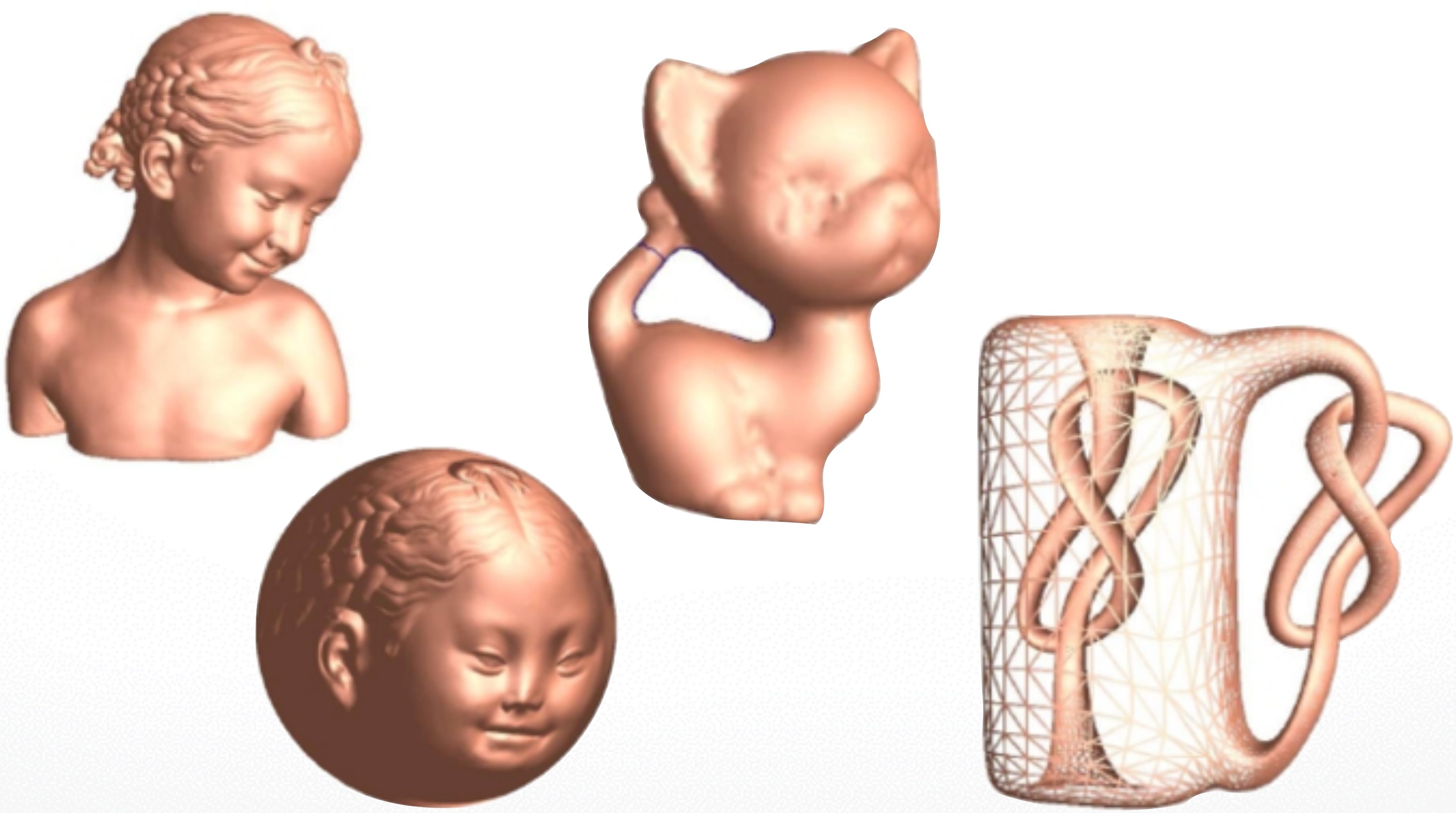
Outline

- Parametric Curves
- **Parametric Surfaces**

Surfaces

What characterizes shape?

- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions



Metric on Surfaces

Measure Stuff

- angle, length, area
  - requires an inner product
- we have:
  - Euclidean inner product in domain
- we want to turn this into:
  - inner product on surface

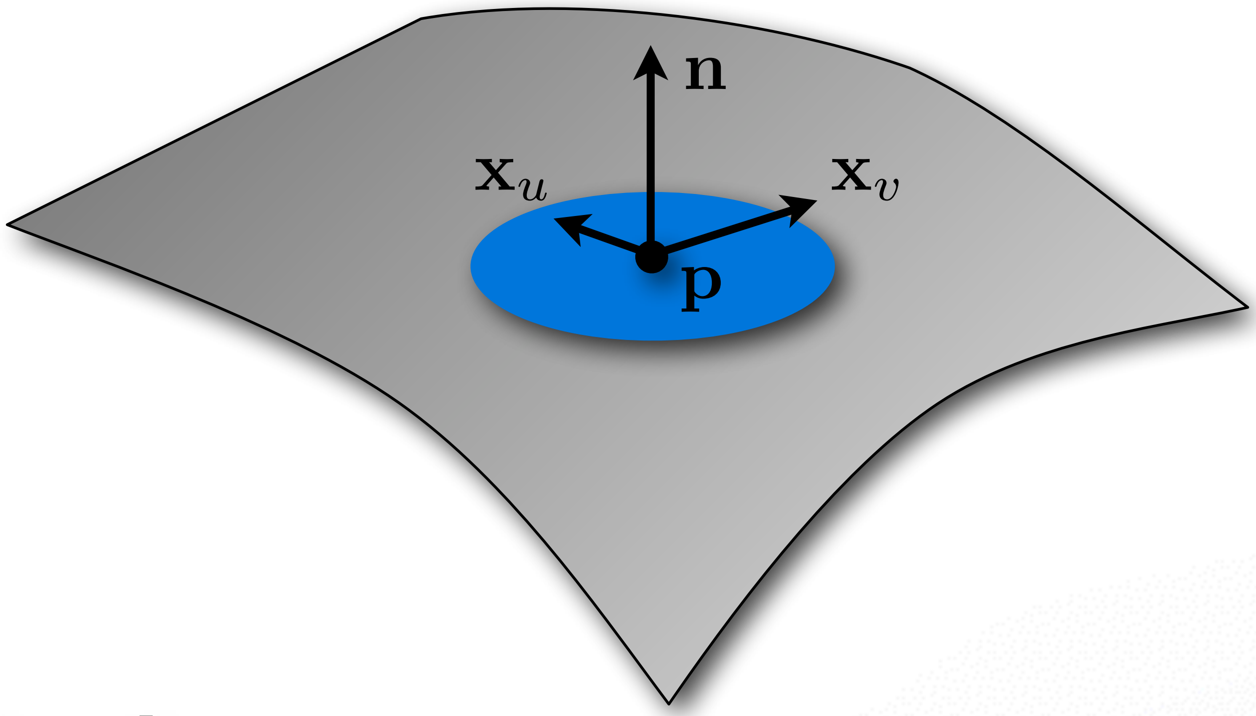
Parametric Surfaces

Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$ normal exists

Angles on Surface

Curve $[u(t), v(t)]$ in uv-plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

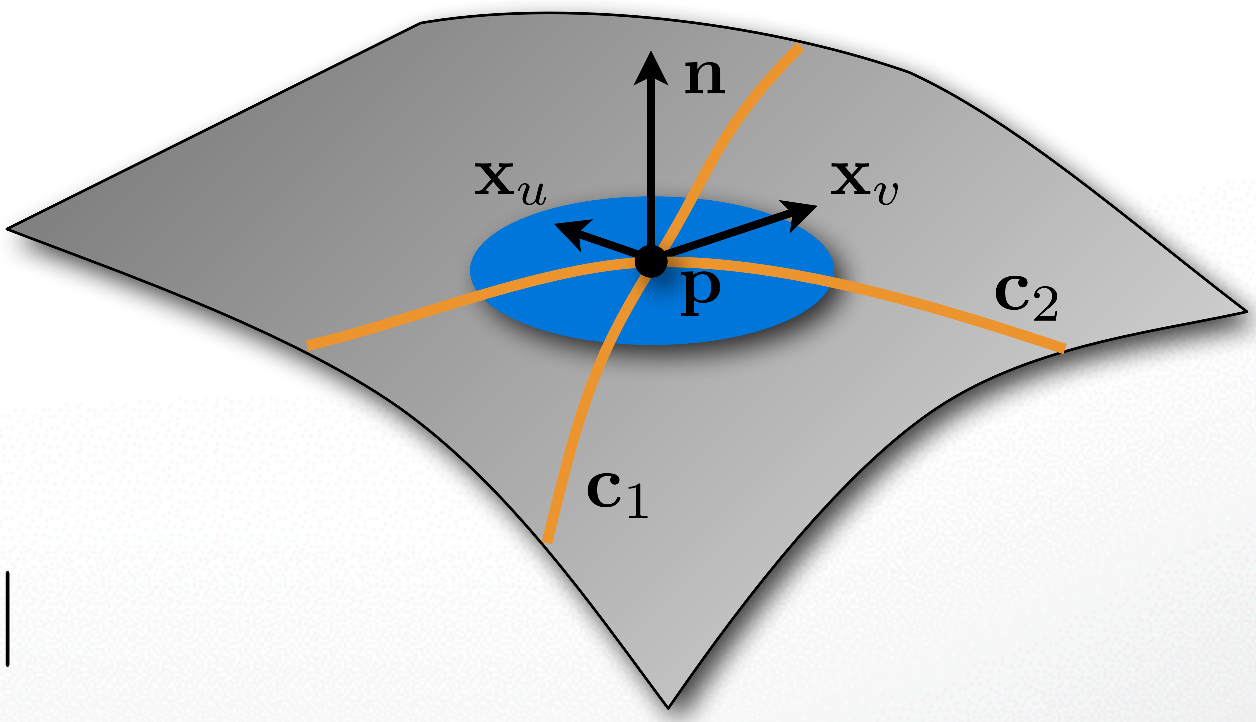
Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

- angle of intersection?
- two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



Angles on Surface

Curve $[u(t), v(t)]$ in uv-plane defines curve on the surface $\mathbf{x}(u, v)$

$$\mathbf{c}(t) = \mathbf{x}(u(t)\,, v(t))$$

Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{p}$

$$\begin{aligned}
\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\
&= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1)\, \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\
&= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}
\end{aligned}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure (w.r.t. surface metric)

Angles $\mathbf{t}_1^\top \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle$

Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

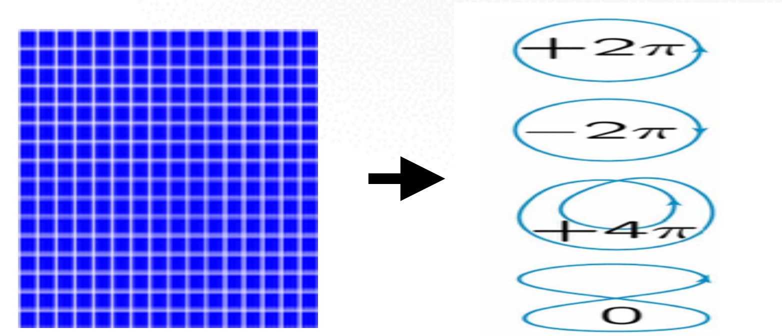
squared infinitesimal length

Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

infinitesimal Area

cross product → determinant with unit vectors → area

Sphere Example



Spherical parameterization

$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Tangent vectors

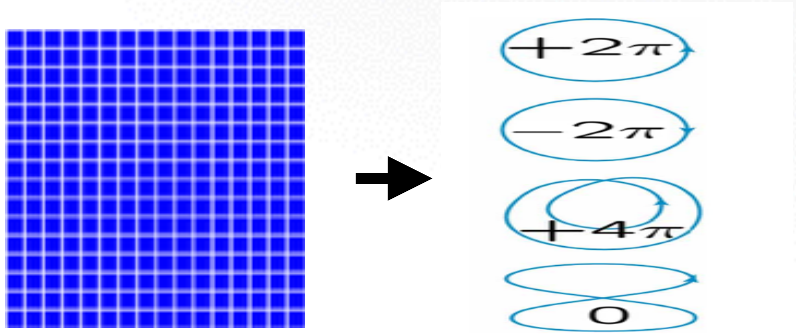
$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \qquad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

First fundamental Form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Sphere Example

Length of equator $\mathbf{x}(t, \pi/2)$



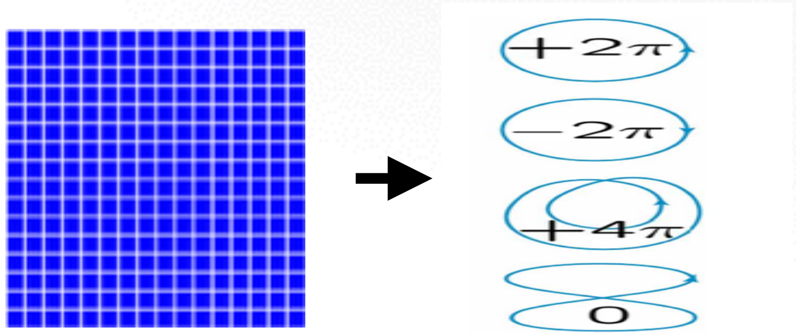
$$\int_0^{2\pi} 1\,\mathrm{d}s \;=\; \int_0^{2\pi} \sqrt{E\,(u_t)^2 + 2F u_t v_t + G\,(v_t)^2}\,\mathrm{d}t$$

$$= \int_0^{2\pi} \sin v\,\mathrm{d}t$$

$$= 2\pi \sin v \;=\; 2\pi$$

Sphere Example

Area of a sphere



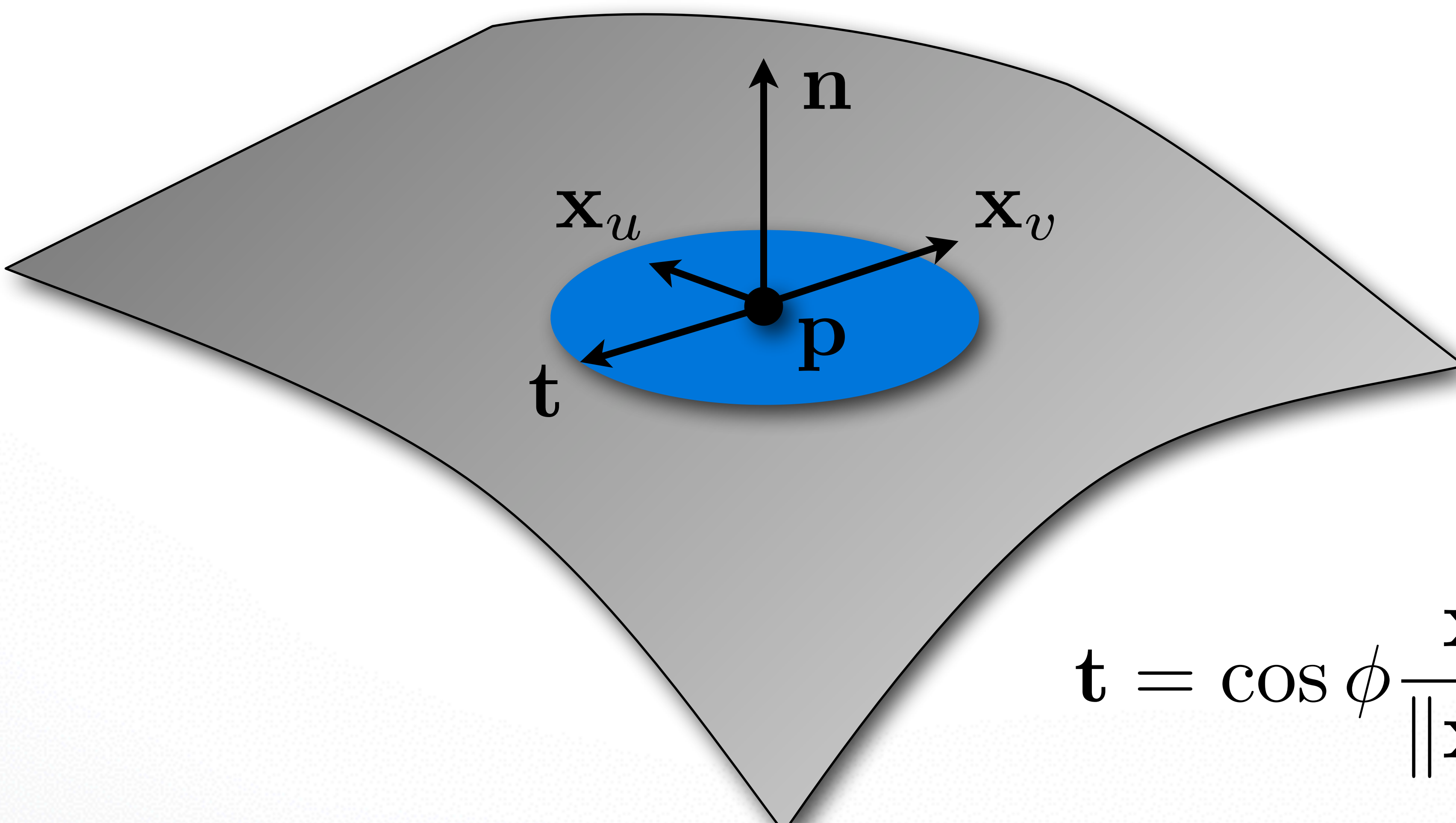
$$\int_0^{\pi}\int_0^{2\pi} 1\,\mathrm{d}A = \int_0^{\pi}\int_0^{2\pi}\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{\pi}\int_0^{2\pi}\sin v\,\mathrm{d}u\,\mathrm{d}v$$

$$= 4\pi$$

Normal Curvature

Tangent vector $\mathbf{t}$ ...



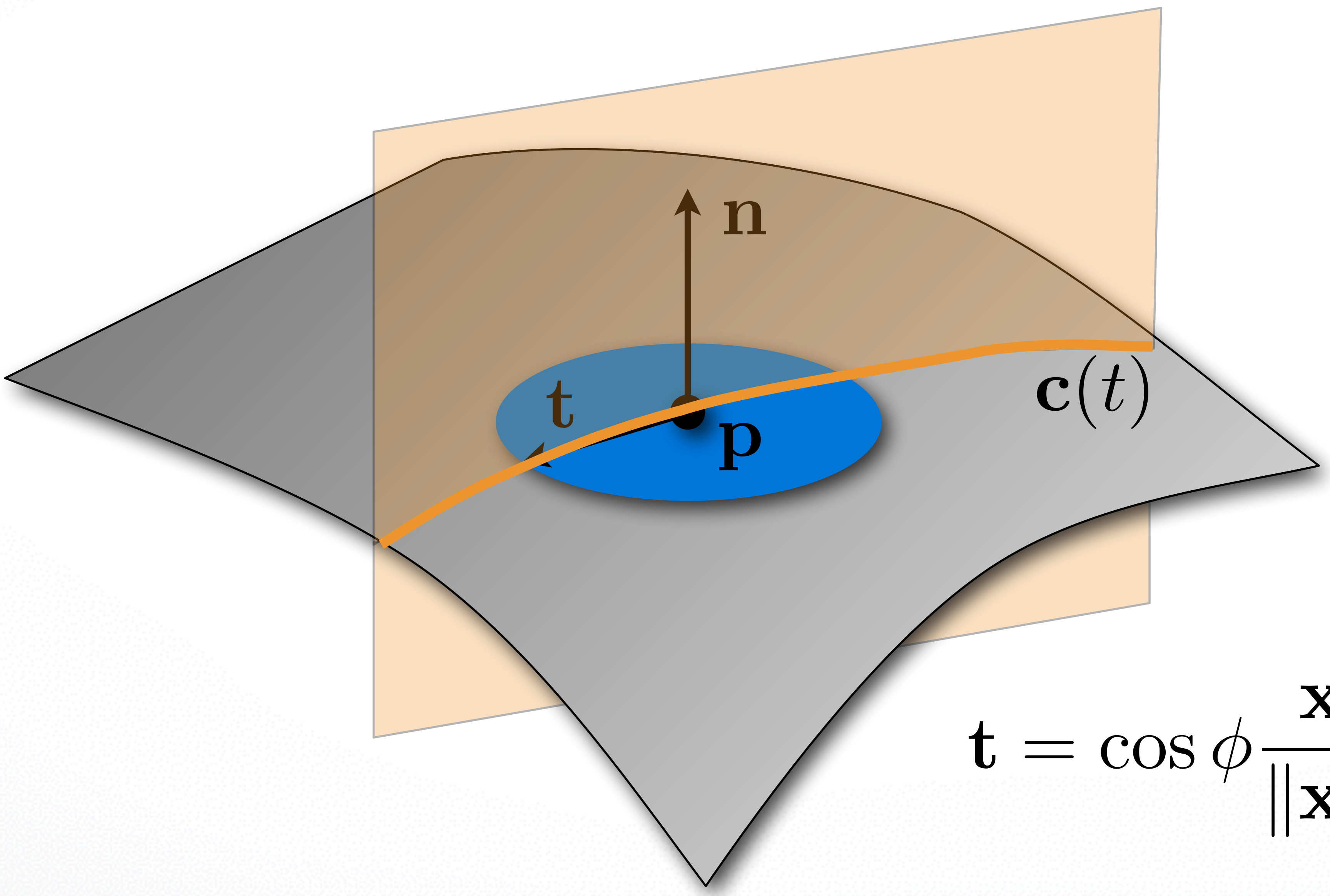
$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

unit vector

Normal Curvature

… defines intersection plane, yielding curve $\mathbf{c}(t)$

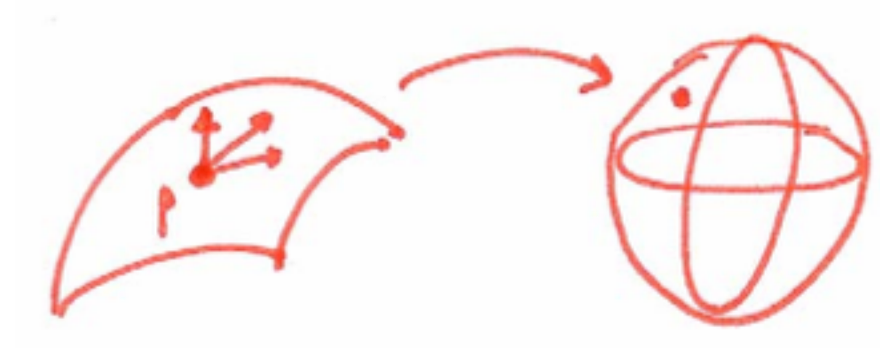
normal curve



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Geometry of the Normal

Gauss map

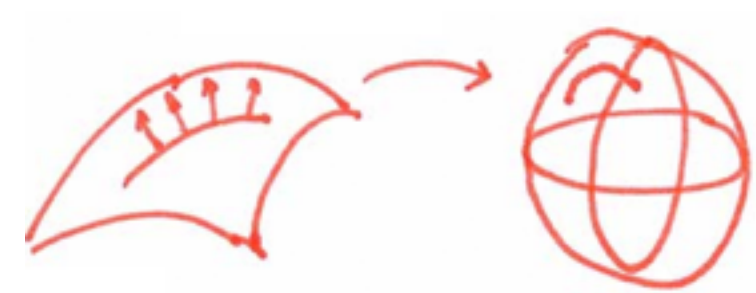


- normal at point

$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p)$$

$$N : S \to \mathbb{S}^2$$

- consider curve in surface again



  - study its curvature at p
  - normal “tilts” along curve

Normal Curvature

Normal curvature $\kappa_n(t)$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p}(t) = \mathbf{x}(u,v)$

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

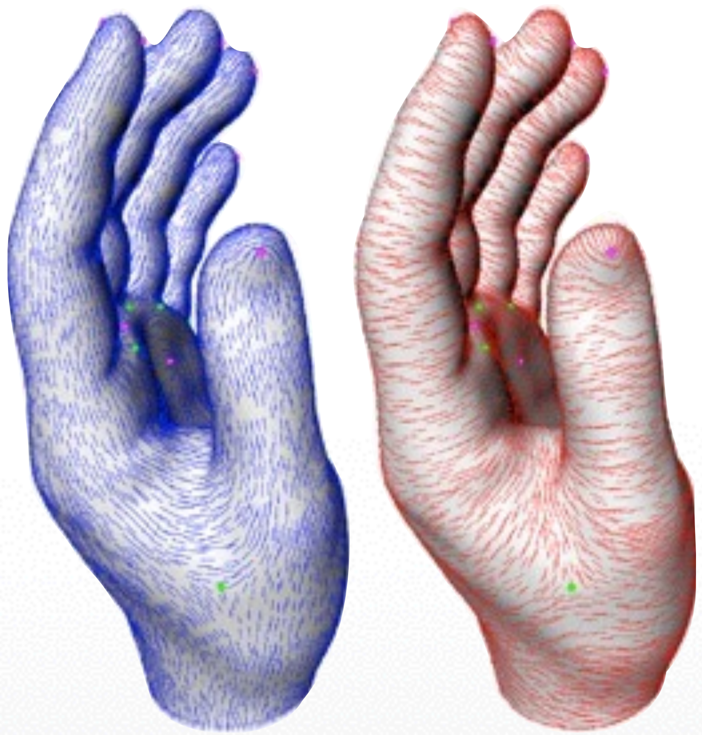
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T\,\mathbf{II}\,\bar{\mathbf{t}}}{\bar{\mathbf{t}}^T\,\mathbf{I}\,\bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

$$\begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a,b) \end{aligned}$$

Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max_\phi \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_\phi \kappa_n(\phi)$
- Euler theorem $\kappa_n(\phi) \ = \ \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Corresponding principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal



Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max_\phi \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_\phi \kappa_n(\phi)$
- Euler theorem $\kappa_n(\phi) = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$
- Corresponding principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal

Special curvatures

- Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$ **extrinsic**
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ **intrinsic (only first FF)**

Invariants

Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$
$$\operatorname{tr} dN_p = \kappa_1 + \kappa_2 = H$$

- eigenvalues and orthovectors

$$dN_p(e_1) = \kappa_1 e_1 \qquad dN_p(e_2) = \kappa_2 e_2$$

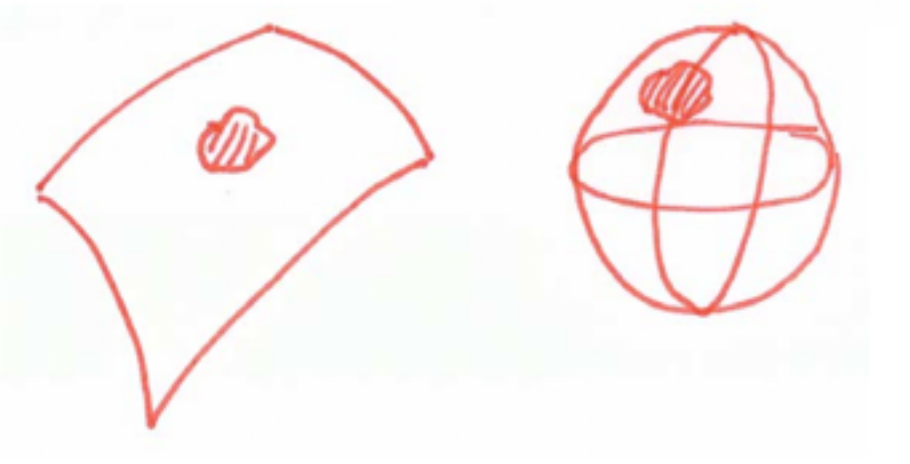
$$II_p|_{\mathbb{S} \subset T_p S} \begin{cases} \max \rightarrow \kappa_1 \\ \min \rightarrow \kappa_2 \end{cases}$$

Mean Curvature

Integral representations

$$H_p/2 = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta$$

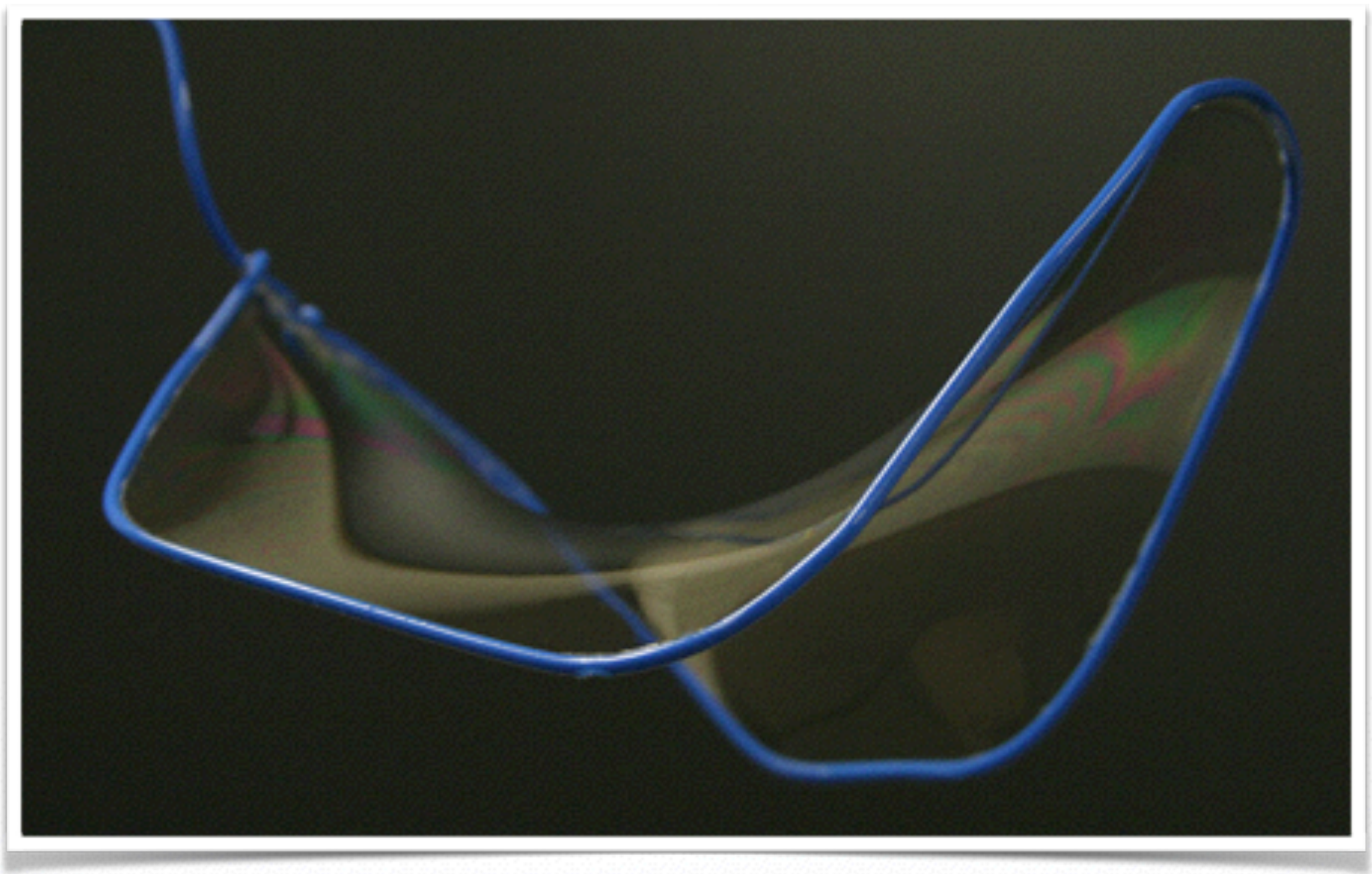
$$K_p = \lim_{A\to 0} \frac{A_G}{A}$$



Curvature of Surfaces

Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere $\rightarrow$ minimal surface

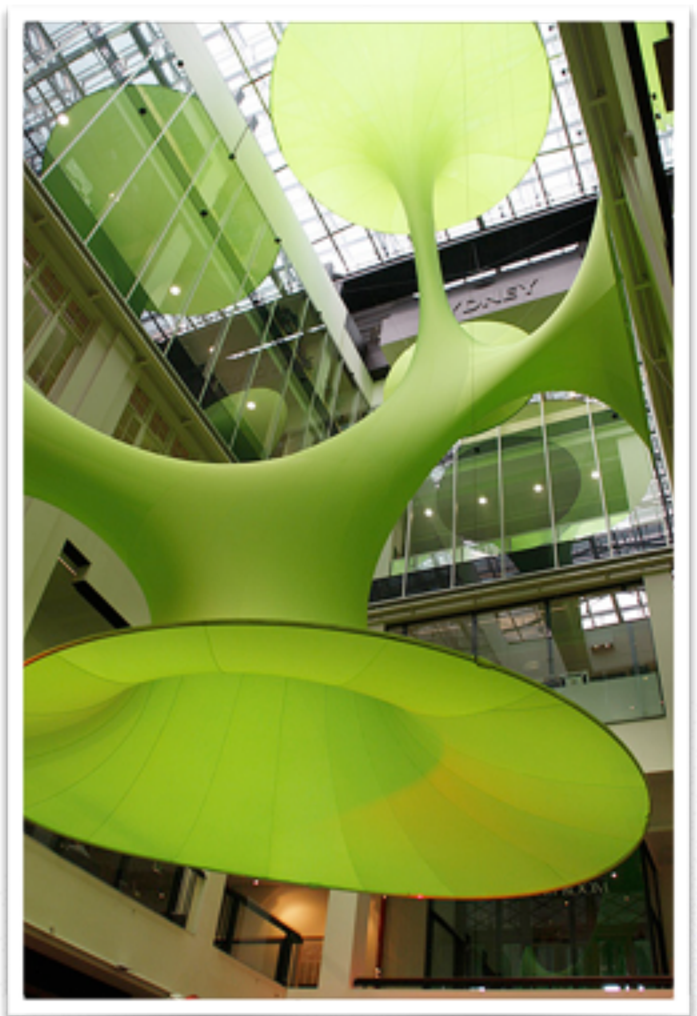


soap film

Curvature of Surfaces

Mean curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

- $H = 0$ everywhere $\rightarrow$ minimal surface



Green Void, Sydney
Architects: Lava

Curvature of Surfaces

Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

- $K = 0$ everywhere $\rightarrow$ developable surface

surface that can be flattened to a plane without distortion (stretching or compression)

Disney, Concert Hall, L.A.
Architects: Gehry Partners

Timber Fabric
IBOIS, EPFL

Shape Operator

Derivative of Gauss map

- second fundamental form

$$II_p(v) = \langle dN_p(v), v \rangle$$

- local coordinates

$$II_p = -\begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Intrinsic Geometry

Properties of the surface that only depend on the first fundamental form

- length
- angles
- Gaussian curvature (Theorema Egregium)

remarkable theorem (Gauss)

$$K = \lim_{r \to 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

Gaussian curvature of a surface is invariant under local isometry

Classification

Point $\mathbf{x}$ on the surface is called

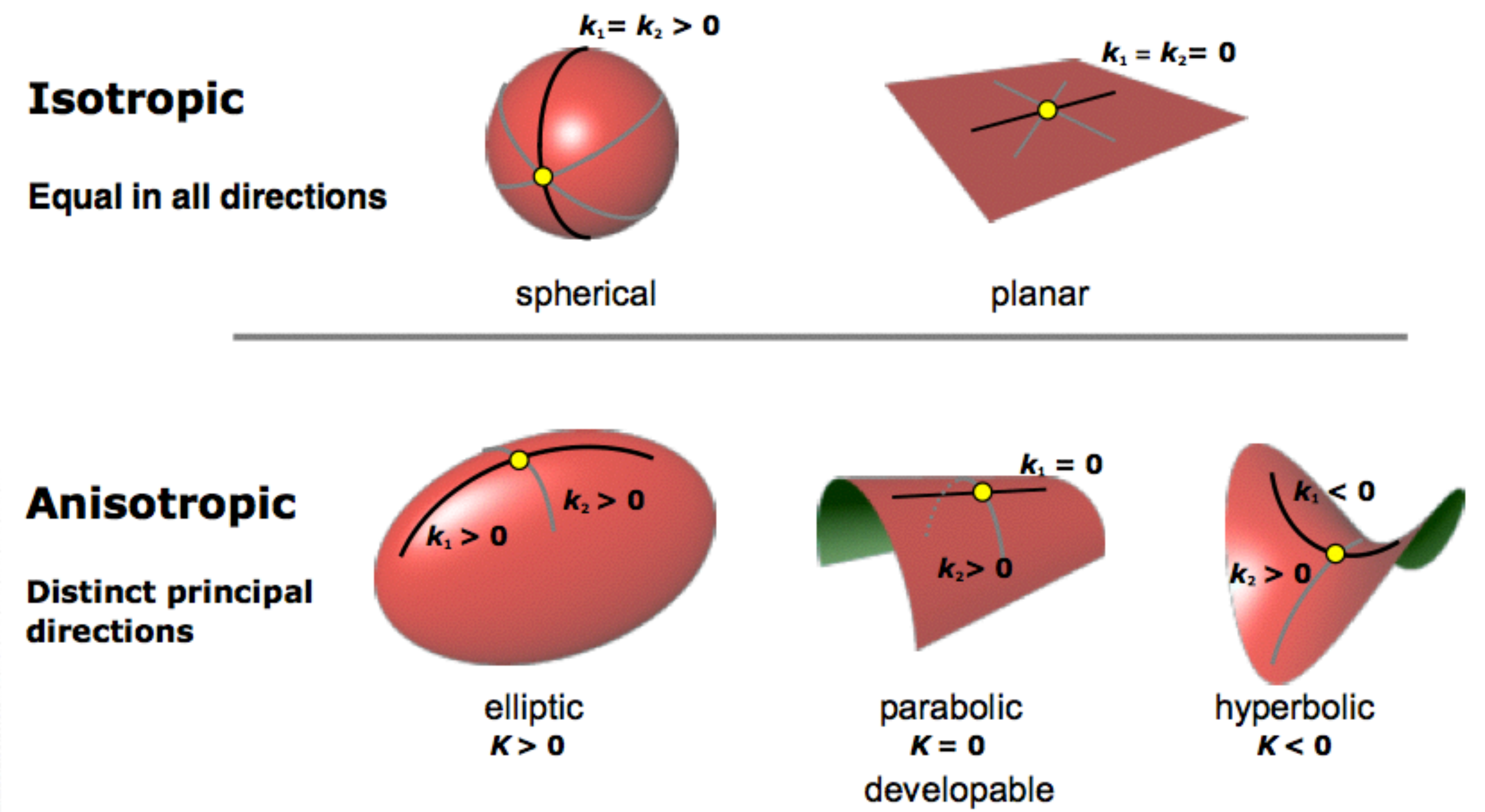
- elliptic, if $K > 0$
- hyperbolic, if $K < 0$
- parabolic, if $K = 0$
- umbilic, if $\kappa_1 = \kappa_2$

Gaussian curvature K

or isotropic

Classification

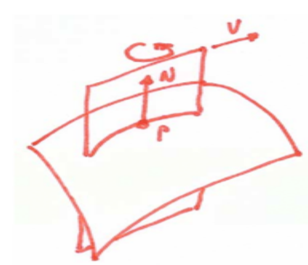
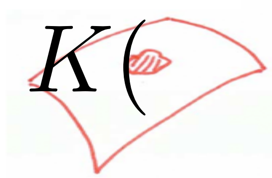
Point $\mathbf{x}$ on the surface is called



Gauss-Bonnet Theorem

For any closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($    $) = \int K($  $) = 4\pi$

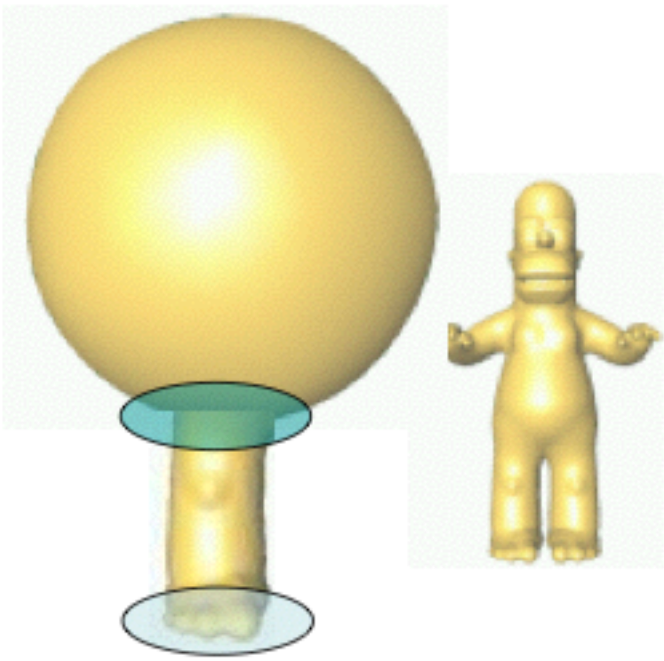
Gauss-Bonnet Theorem

Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



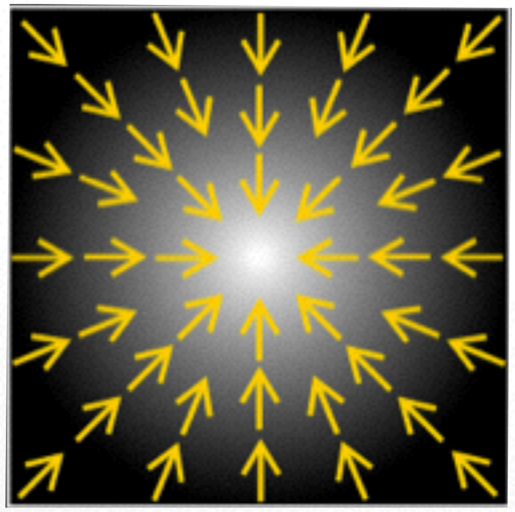
when sphere is deformed, new
positive and negative curvature cancel out

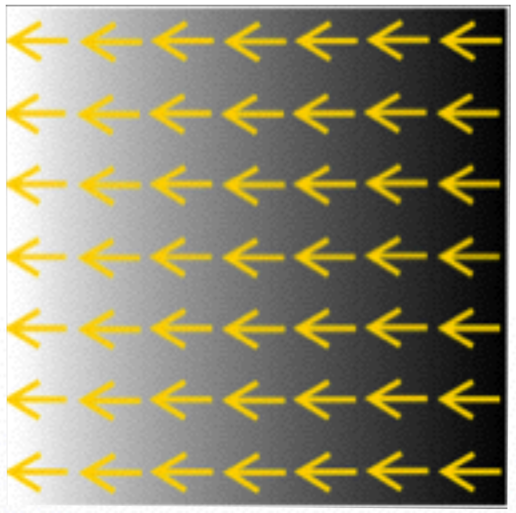
Differential Operators

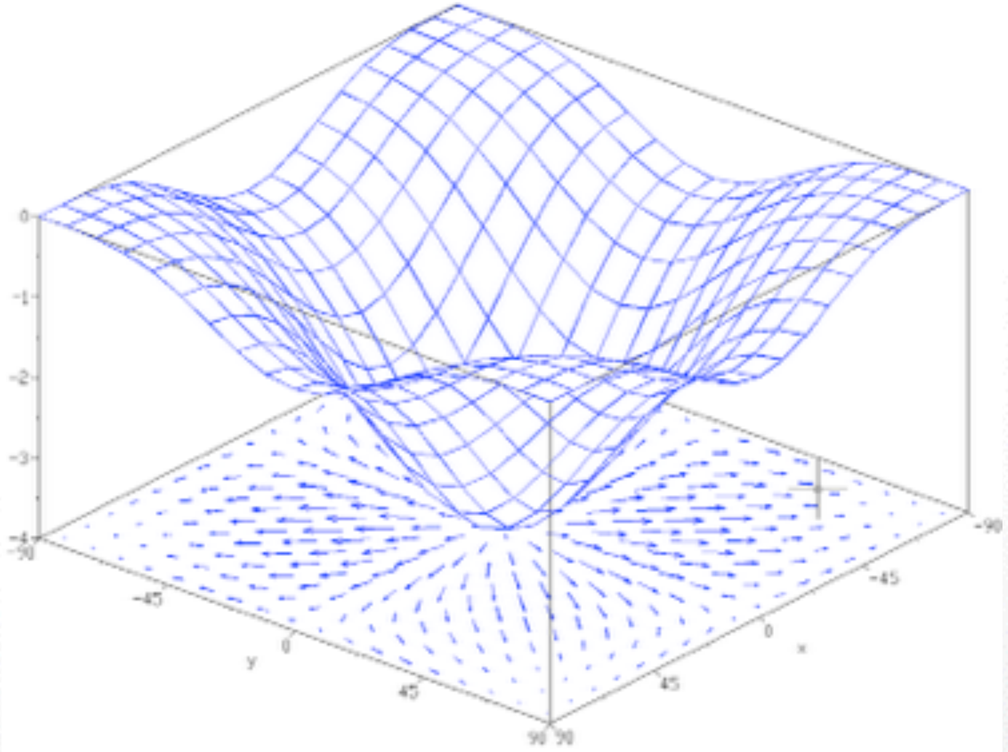
Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of the steepest ascend







Differential Operators

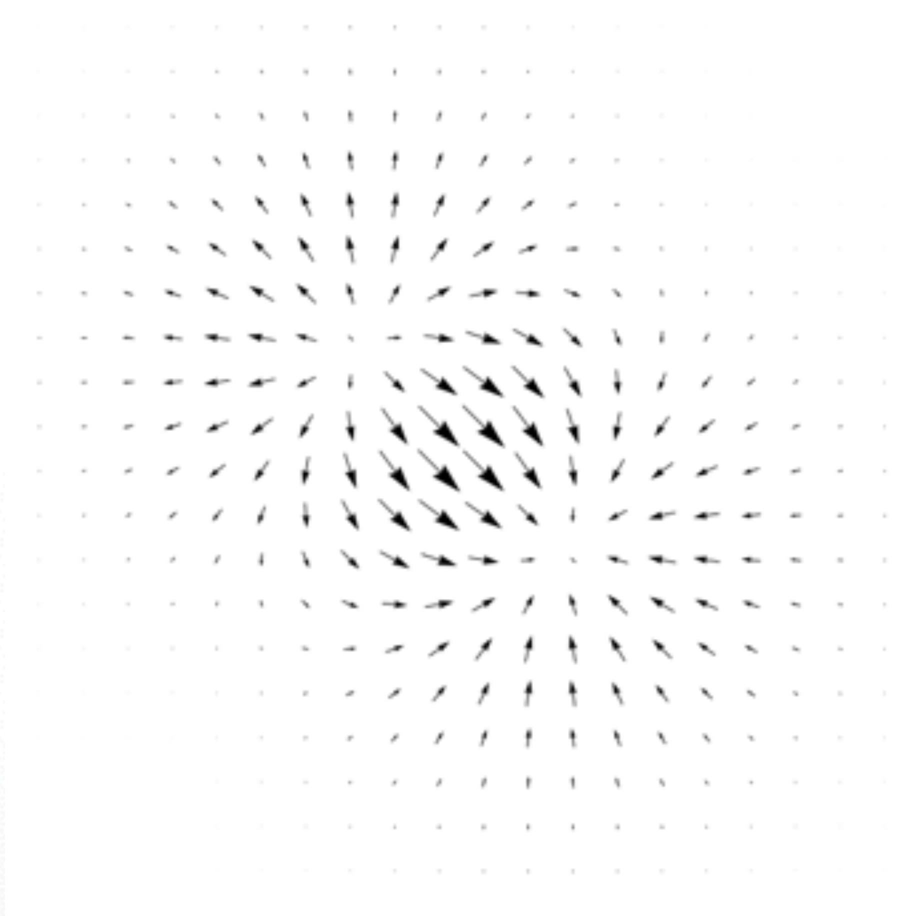
Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

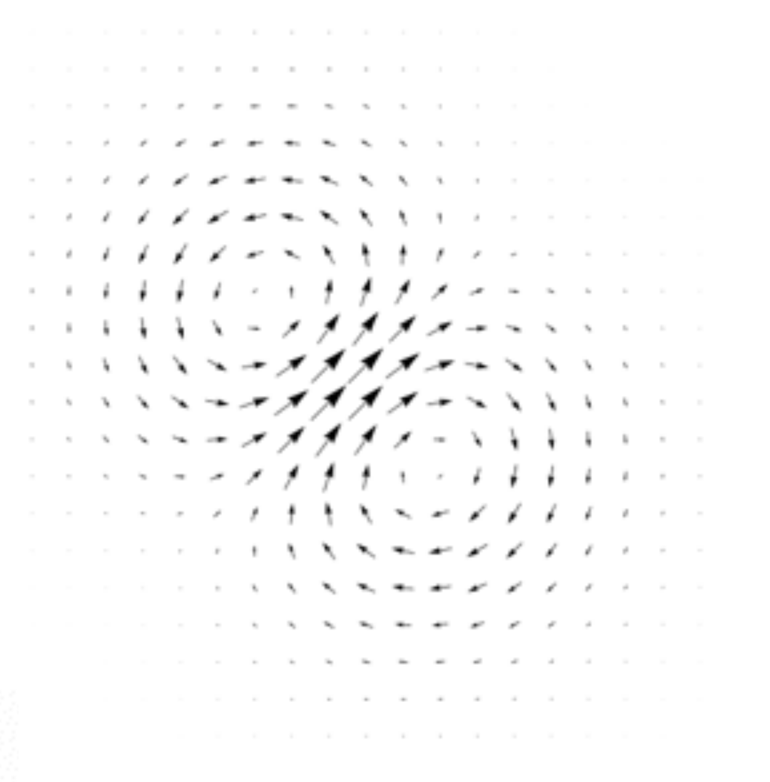
- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: incompressible fluid
  - velocity field is divergence-free

Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

Laplace Operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

Extension of Laplace fo functions on manifolds

Laplace-Beltrami

gradient operator

…of the surface

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} f$$

function on manifold S

divergence operator

Laplace on the surface

Laplace-Beltrami Operator

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

surface normal

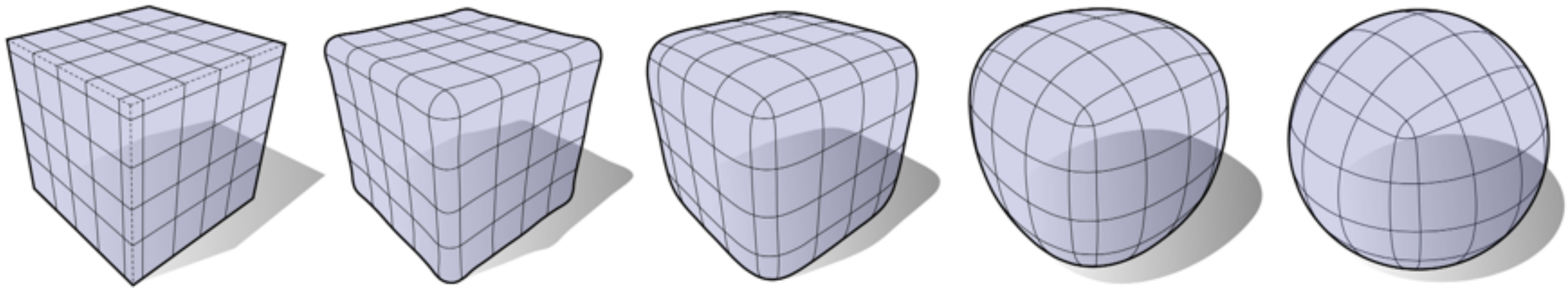
function on manifold S

divergence operator

Literature

- M. Do Carmo: **Differential Geometry of Curves and Surfaces, Prentice Hall**, 1976
- A. Pressley: **Elementary Differential Geometry**, Springer, 2010
- G. Farin: **Curves and Surfaces for CAGD**, Morgan Kaufmann, 2001
- W. Boehm, H. Prautzsch: **Geometric Concepts for Geometric Design**, AK Peters 1994
- H. Prautzsch, W. Boehm, M. Paluszny: **Bézier and B-Spline Techniques**, Springer 2002
- ddg.cs.columbia.edu
- http://graphics.stanford.edu/courses/cs468-13-spring/schedule.html

Next Time



Discrete Differential Geometry

http://cs599.hao-li.com

Thanks!