Fall 2018

CSCI 420: **Computer Graphics**

13.2 Physically Based Simulation II

Mass-Spring Systems

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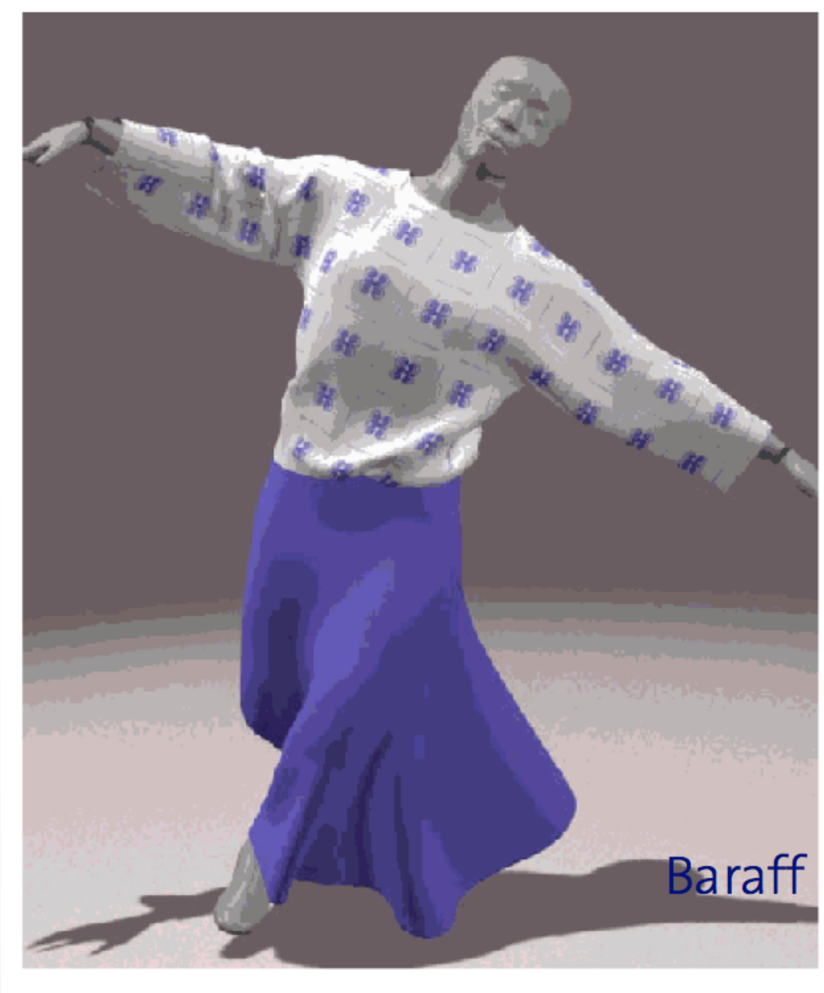
Mass-Spring Systems

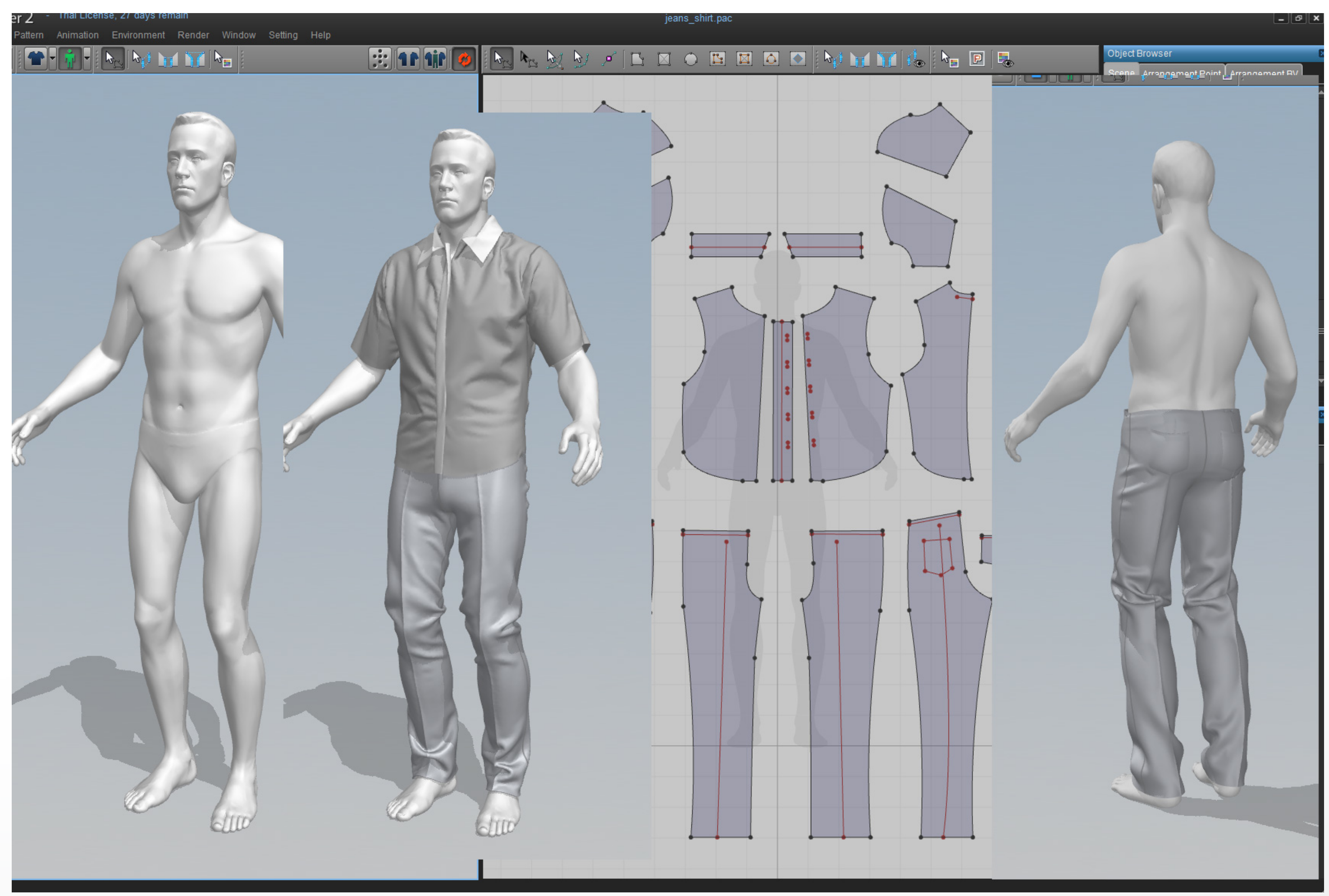
- **The 101 of Physics Simulation**

- What do we want to simulate? **Deformable Objects**
- Design a model. **Mass points + springs.**
- Write differential equations. **Newton's 2nd Law (Hooke)**
- Discretize equations. **Integration methods for ODEs**
- Add interaction. **Collision detection + response**
- Simulate!

Mass-Spring Systems

- Simulation of cloth based on deformable surfaces (Polygonal mesh)
- Realistic simulation of cloth with different fabrics such as wool, cotton, or silk for garment design





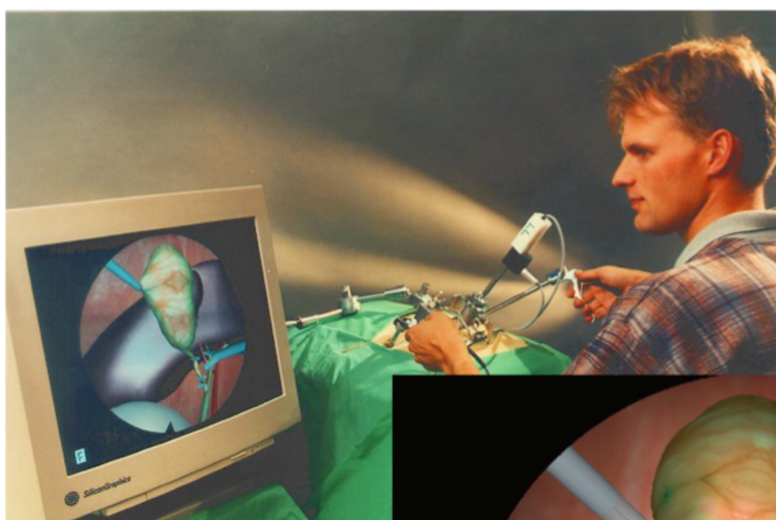
Facial Animation

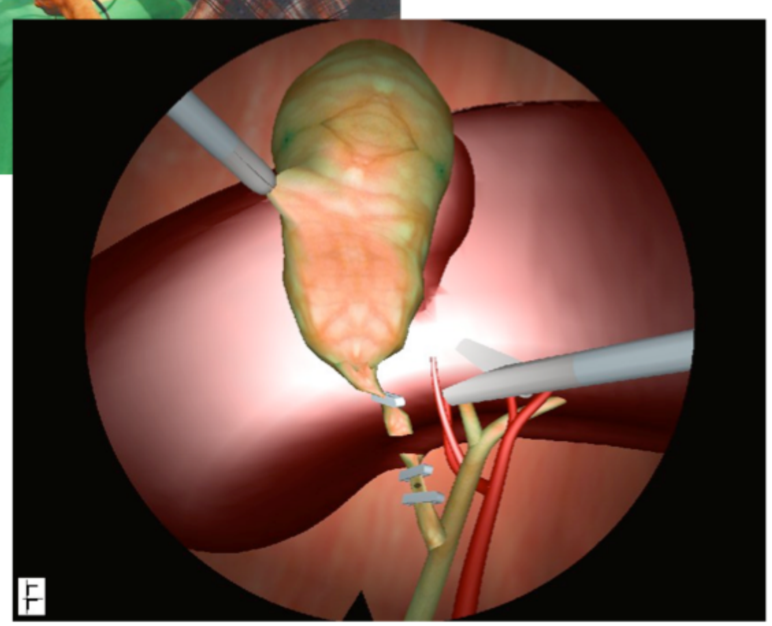
- Simulation of facial expressions based on deformable surfaces/volumes/muscles
- Animation of face models from speech and mimic parameters

Thalmann

Medical Simulation

- Simulation of deformable soft tissue
- Surgical planning
- Medical training





Virtual endoscopy

Kuehnapfel

Teschner

Prediction of the surgical outcome in craniofacial surgery

Overview

- Model and Physics
- Implementation Hints
- Time-Discretization
- Collision Response
- (Simulation Loop)

Mass-Point System

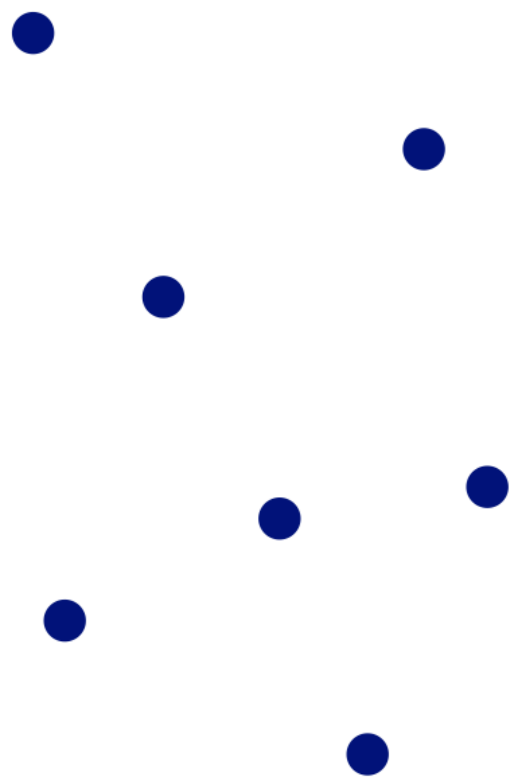
- Discretization of an object into mass points (gas, fluid, elastic object, inelastic object)
- System with multiple mass centers (Planetary System)
- Interaction between points i and j≠i based on internal forces $\mathbf{F}_{ij}^{int}$
- All other forces at point i are external forces $\mathbf{F}_{i}^{ext}$
- Overall force $\mathbf{F}_i = \mathbf{F}_{ij}^{int} + \mathbf{F}_{i}^{ext}$

$$\mathbf{F}_{ij}^{int} = -\mathbf{F}_{ji}^{int} \qquad \sum_i \sum_j \mathbf{F}_{ij}^{int} = 0$$

Mass-Point System

- Discretization of an object into mass points
- Representation of forces between masses by springs
- Computation of dynamics

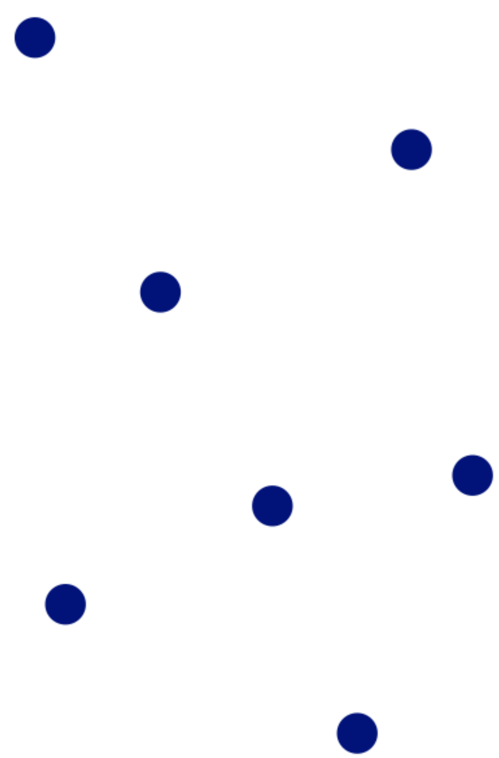
Mass-Points



Object sampled using mass points
Mass of object: M
Number of points: n
Mass of each point: $m=M/n$
(if uniformly distributed)

Simulate the motion of each mass point

Physically-based Equations



Equations that describe the behavior of the system (i.e. the mass points)

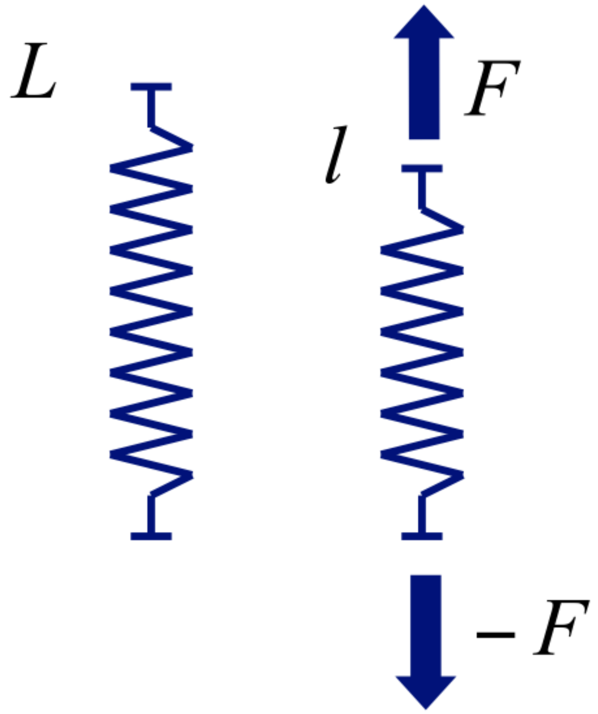
Physically-based model:

Newton's 2nd Law

$$\sum \mathbf{F}_i^{\mathbf{int}} + \mathbf{F}_i^{\mathbf{ext}} = m\mathrm{a}_i$$

Next: Model the forces

Elastic Forces: Springs



Spring stiffness is denoted as *k*
Initial spring length *L*
Current spring length *l*

Deformation linear w.r.t. force:

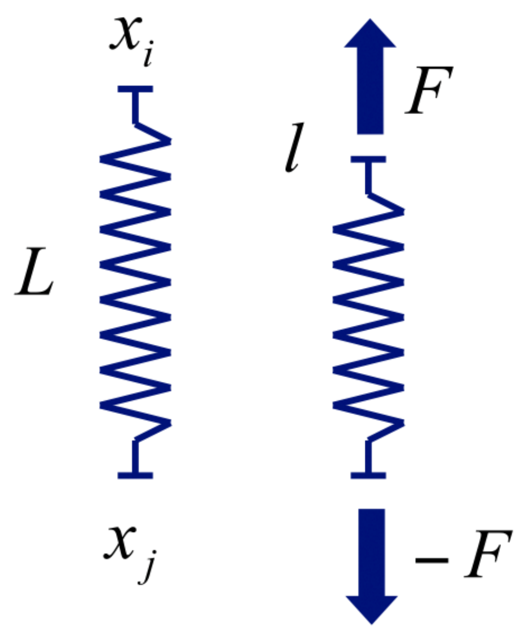
$$F = -k\left(l - L\right)$$

Hooke's Law

Elasticity: Ability of a spring to return to its initial form when the deforming force is removed.

Simple mechanism for internal forces.

Elastic Energies



Elastic energy:

$$E = \frac{1}{2}k(l-L)^2$$

Force = - Partial Derivative (Gradient)

$$F_i = -\frac{\partial E}{\partial x_i}$$

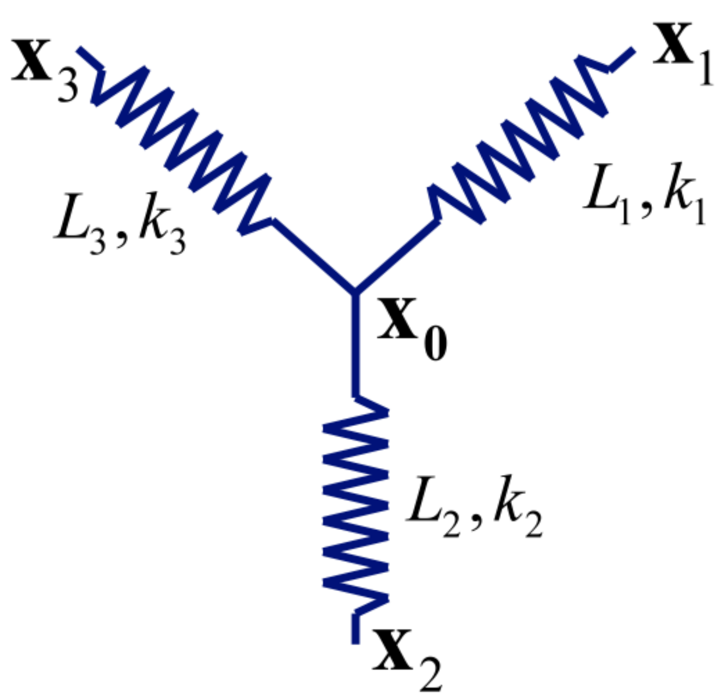
Force in vector notation:

$$F_i = -k(l-L)\frac{\mathbf{x}_i - \mathbf{x}_j}{l}$$

Force-centered view versus energy-centered view

Forces at a Mass Point

Internal forces $\mathbf{F}^{\mathrm{int}}$



$$\mathbf{F}_0^{\mathrm{int}} = -\sum_{i \mid i \in \{1,2,3\}} k_i (l_i - L_i) \frac{\mathbf{x_i} - \mathbf{x_0}}{l_i}$$

External forces $\mathbf{F}^{\mathrm{ext}}$

Gravity
Contact forces
All forces that are
not caused by springs

Resulting force at point i

$$\mathbf{F_i} = \mathbf{F_i^{int}} + \mathbf{F_i^{ext}}$$

Dissipative Forces

Dissipative forces

Damping
Friction

$$\mathbf{F}^{damping}(t) = -\gamma \cdot \mathbf{v}(t)$$

System Equations

Equation of Motion for one mass point (3 eqs.)

$$m_i \frac{d^2 \mathbf{x}_i(t)}{dt^2} = \mathbf{F}_i^{\mathbf{int}}(t) + \mathbf{F}_i^{\mathbf{ext}}(t)$$

Equation of Motion for a system of mass points (3n eqs.)

$$\mathbf{M} \frac{d^2 \mathbf{X}(t)}{dt^2} = \mathbf{F}^{\mathbf{int}}(t) + \mathbf{F}^{\mathbf{ext}}(t)$$

M is a diagonal matrix

System Equations

Incorporation of **damping**

$$\mathbf{M}\frac{d^2\mathbf{X}(t)}{dt^2} + \mathbf{D}\frac{d\mathbf{X}(t)}{dt} = \mathbf{F}^{\mathrm{int}}(t) + \mathbf{F}^{\mathrm{ext}}(t)$$

Overview

- Model and Physics
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Elastic Spring

```
class SPRING
{
  public:
    POINT *point1;
    POINT *point2;
    float  stiffness;      // k
    float  initialLength; // L
    float  currentLength; // l

    . . .
}
```

Mass Point

```
class POINT
{
   public:
       float mass;
       float position[3];
       float velocity[3];
       float force[3];
       float damping;

      .   .   .
}
```

Force Computation

```
for all points
     point_i.ClearForce()
     point_i.AddGravityForce()

//Add other external forces

for all springs
     spring_i.ComputeElasticForce()
     spring_i.AddForceToEndPoints()
```

Overview

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System Equations

System of $3n$ 2nd order Ordinary Differential Equations (ODE)

$$\mathbf{M}\frac{d^2\mathbf{X}(t)}{dt^2} + \mathbf{D}\frac{d\mathbf{X}(t)}{dt} = \mathbf{F}^{\mathbf{int}}(t) + \mathbf{F}^{\mathbf{ext}}(t)$$

One 2nd order ODE (1-dimensional problem)

$$m\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} = F(t)$$

Initial value problem: $x(0)$ and $v(0)$ are known

Solution

a) Analytical solution (if we care about the exact state at time *t*)

b) **Discrete** solution
- Graphics: the goal is to **display** the state at t_i
- Find solution at discrete time instants t_i, assuming that we know previous solutions t_{i-1}, t_{i-2}, etc.
- We do not care about the steady state error, but we want **plausible behavior** and **response to external forces**

Problem

- We have:
  - Initial position x
  - Initial velocity v
  - 2nd derivative of position x with respect to time

$$\boxed{\frac{d^2\mathbf{x}_\mathbf{i}(t)}{dt^2} = \frac{\mathbf{F}_\mathbf{i}(t) - \gamma\mathbf{v}_\mathbf{i}(t)}{m_i}}$$

- **Goal**: Computation of position x over time

Numerical Integration Methods

- Explicit Integration
  - Euler
  - Leapfrog
  - Heun
  - Midpoint
  - Runge-Kutta methods
- Implicit Integration
  - Backward Euler
- Predictor-Corrector methods
  - Gear
- Methods for higher order ODEs
  - Verlet
  - Beeman
- Variable time-step methods

Numerical Integration Methods

- Reduction of a second-order ODE to two coupled first-order ODEs.

$$m_i \frac{d^2\mathbf{x}_\mathrm{i}(t)}{dt^2} + \gamma \frac{d\mathbf{x}_\mathrm{i}(t)}{dt} = \mathbf{F}_\mathrm{i}$$

$$\frac{d\mathbf{x}_\mathrm{i}(t)}{dt} = \mathbf{v}_\mathrm{i}(t) \qquad \frac{d\mathbf{v}_\mathrm{i}(t)}{dt} = \frac{\mathbf{F}_\mathrm{i}(t) - \gamma\mathbf{v}_\mathrm{i}(t)}{m_i}$$

velocity

acceleration

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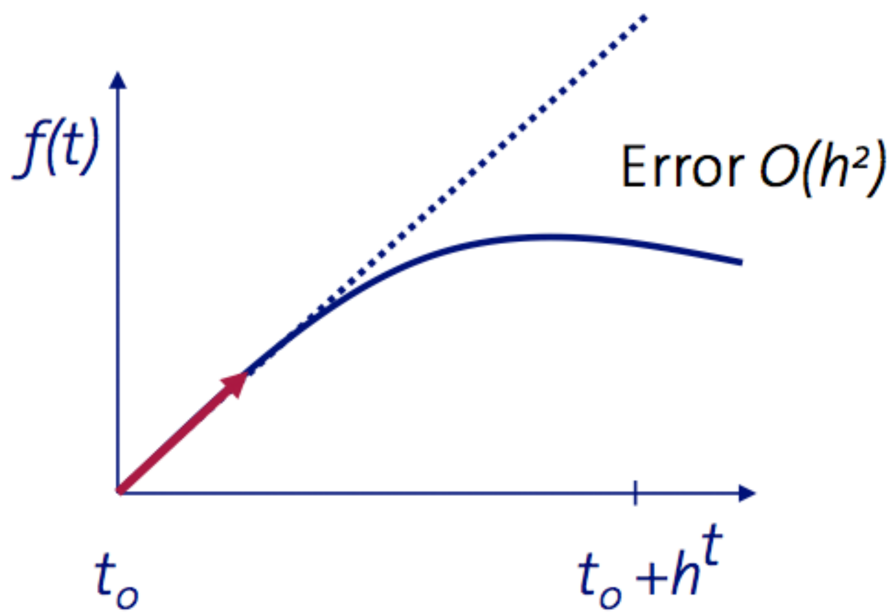
Explicit Integration



Euler Method

Leonard Euler:
1707 (Basel) – 1783 (St. Petersburg)

- Initial value $f(t_o)$
- Compute the derivative at t_o
- Move from t_o to t_o+h using the derivative at t_o

Explicit Integration

$$f(t_0 + h) = f(t_0) + h \cdot f'(t_0) + \frac{h^2}{2} f''(t_0) + \ldots$$

Taylor series

$$f(t_0 + h) = f(t_0) + h \cdot f'(t_0) + O(h^2)$$

$$f(t_0 + h) \cong f(t_0) + h \cdot f'(t_0)$$

Euler method

Explicit Integration

$$\mathbf{x}'(t) = \mathbf{v}(t) \qquad \mathbf{v}'(t) = \frac{\mathbf{F}(t) - \gamma\mathbf{v}(t)}{m}$$

Start with initial values	$\mathbf{x}(t_0) = \mathbf{x}_0 \qquad \mathbf{v}(t_0) = \mathbf{v}_0$
Compute	$\mathbf{v}'(t_0) \qquad \mathbf{x}'(t_0)$
Assume	$\mathbf{v}'(t) = \mathbf{v}'(t_0) \qquad \mathbf{x}'(t) = \mathbf{x}'(t_0) \qquad t_0 \leq t \leq t_0 + h$
Compute	$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + h\mathbf{x}'(t_0) = \mathbf{x}(t_0) + h\mathbf{v}(t_0)$
Compute	$\mathbf{v}(t_0 + h) = \mathbf{v}(t_0) + h\mathbf{v}'(t_0) = \mathbf{v}(t_0) + h\frac{\mathbf{F}(t_0) - \gamma\mathbf{v}(t_0)}{m}$

F*(t)* is computed from x*(t)* and external forces!

Error Accumulation

$$\mathbf{x}'(t) = \mathbf{v}(t) \qquad \mathbf{v}'(t) = \frac{\mathbf{F}(t) - \gamma\mathbf{v}(t)}{m}$$

Euler step from t_o to t_o+h

$$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + h\mathbf{v}(t_0) \qquad \mathbf{v}(t_0 + h) = \mathbf{v}(t_0) + h\frac{\mathbf{F}(t_0) - \gamma\mathbf{v}(t_0)}{m}$$

$$\mathbf{x}(t_0 + 2h) = \mathbf{x}(t_0 + h) + h\mathbf{v}(t_0 + h)$$

Euler step
from t_o+h to t_o+2h

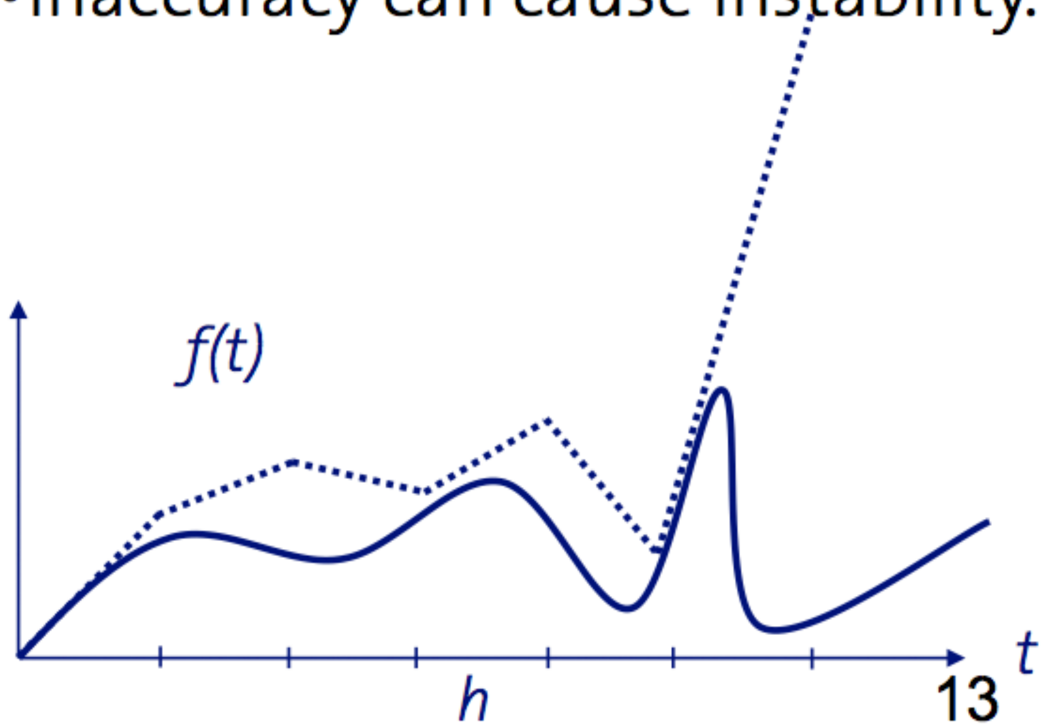
$$\mathbf{v}(t_0 + 2h) = \mathbf{v}(t_0 + h) + h\frac{\mathbf{F}(t_0 + h) - \gamma\mathbf{v}(t_0 + h)}{m}$$

Problems

- Numerical integration is inaccurate.

$$\underbrace{f(t+h) = f(t) + f'(t)h}_{\text{Euler step}} + \underbrace{\boxed{O(h^2)}}_{\text{Error}}$$

- Inaccuracy can cause instability.



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Error

$$0 \le e < \frac{h^2}{2} \cdot f''(t_e), \quad t_e \in [t, t+h]$$

Improving Accuracy - Leap Frog

$$\mathbf{v}(t+h/2) = \mathbf{v}(t-h/2) + h\cdot\mathbf{a}(t)$$

$$\mathbf{x}(t+h) = \mathbf{x}(t) + h\cdot\mathbf{v}(t+h/2)$$

Error $O(h^3)$

time step *h is* significantly larger compared to Euler

Implementation

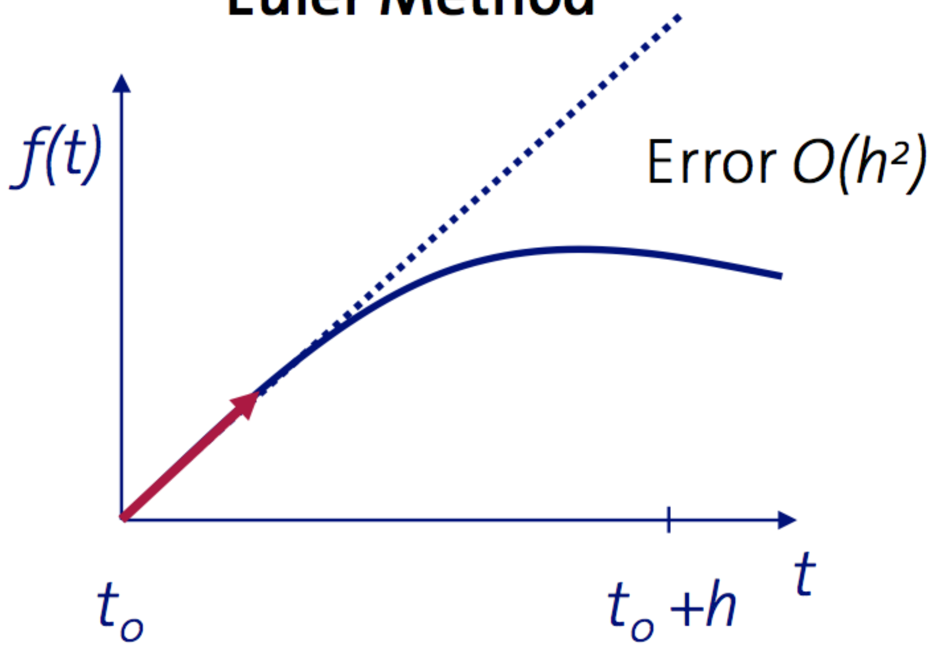
Euler	Leapfrog
... *addForces();* *// F(t)* *positionEuler(h);* *// x=x(t+h)=x(t)+hv(t)* *velocityEuler(h);* *// v=v(t+h)=v(t)+ha(t)* ...	*initV() // v(0) = v(0) – h/2a(0)* ... *addForces(h);* *// F(t)* *velocityEuler(h);* *// v=v(t+h)=v(t)+ha(t)* *positionEuler(h);* *// x=x(t+h)=x(t)+hv(t+h)* ...

(In practice, it is irrelevant that velocities are computed at mid time steps)

Improving Accuracy - Runge Kutta

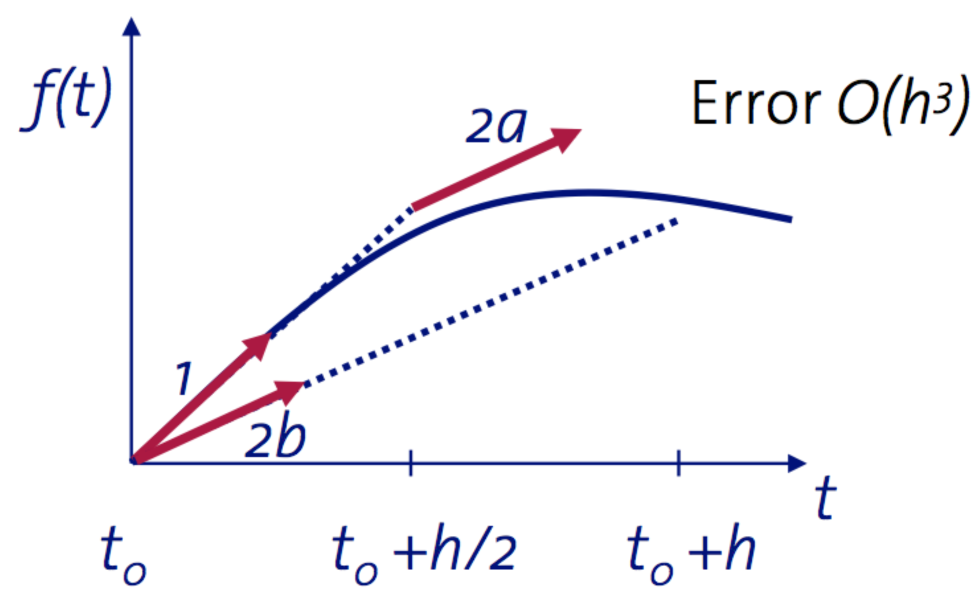
2nd order (midpoint method)

Euler Method



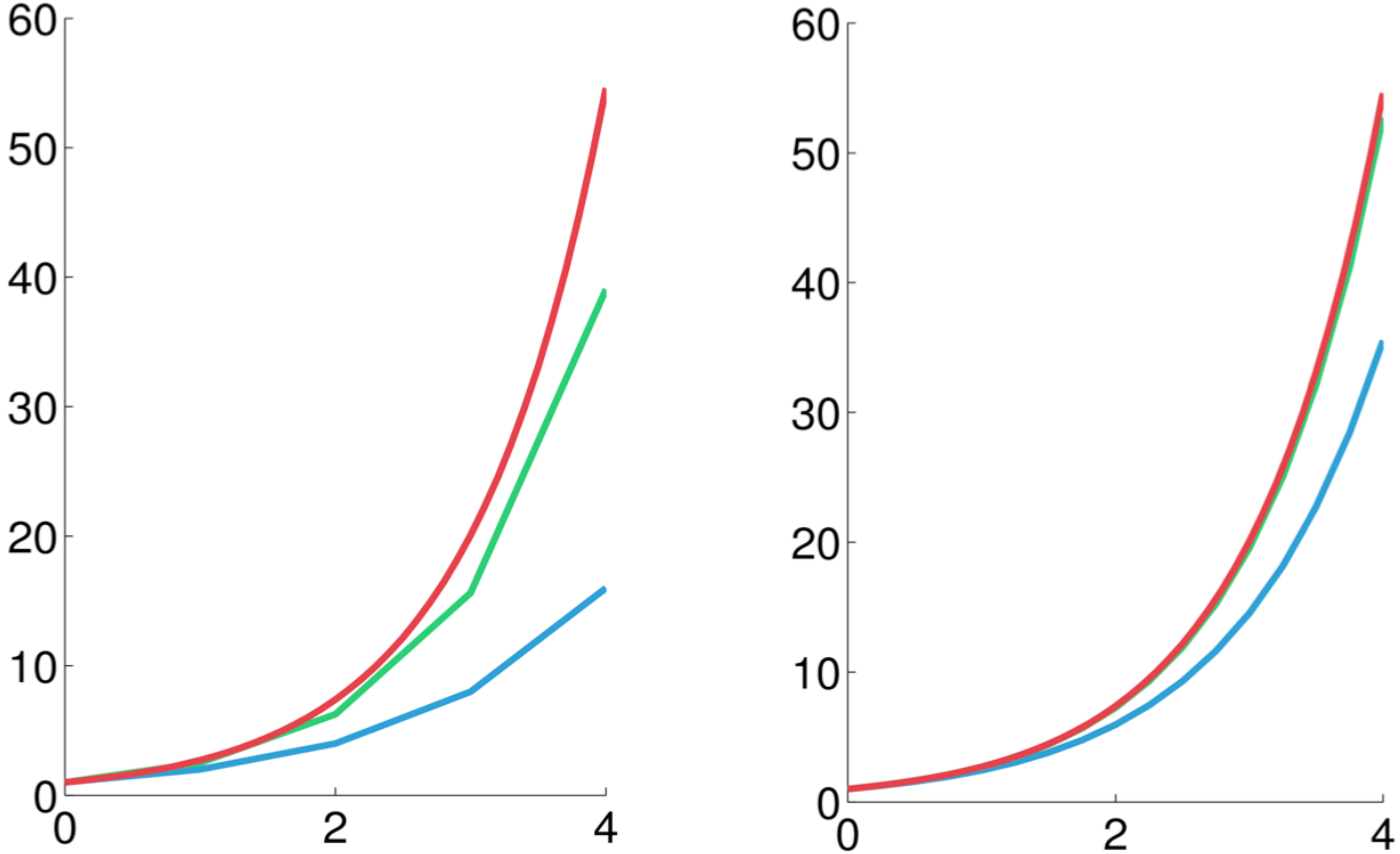
- Compute the derivative at t_o
- Move from t_o to $t_o + h$ using the derivative at t_o

Runge-Kutta Methods



- Compute the derivative at t_o
- Move to $t_o + h/2$
- Compute the derivative at $t_o + h/2$
- Move from t_o to $t_o + h$ using the derivative at $t_o + h/2$
- Second order R-K also called “midpoint”

Midpoint vs Euler



- Green = Midpoint
- Blue = Euler
- h=1 vs. h=1/4

Implementation

Euler Method

Straightforward:

- Compute spring forces
- Add external forces
- Update positions
- Update velocities

Runge-Kutta Methods

- Compute spring forces
- Add external forces
- Compute **auxiliary** positions and velocities
  - once for second-order
  - three times for fourth-order
  - requires **additional data copies**
- Update positions
- Update velocities

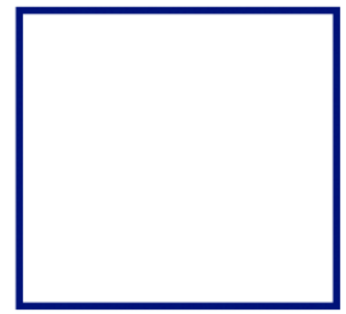
Avoiding Instability

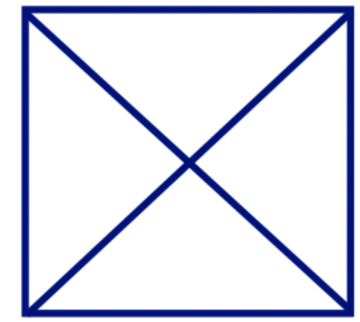
- No general solution to avoid instability for complex mass-point systems.
- A smaller time step increases the chance for stability.
- A larger time step speeds up the simulation.
- Parameters and topology of the mass-point system, and external forces influence the stability of a system.
- Increasing damping does not always help.

Topology and Stability

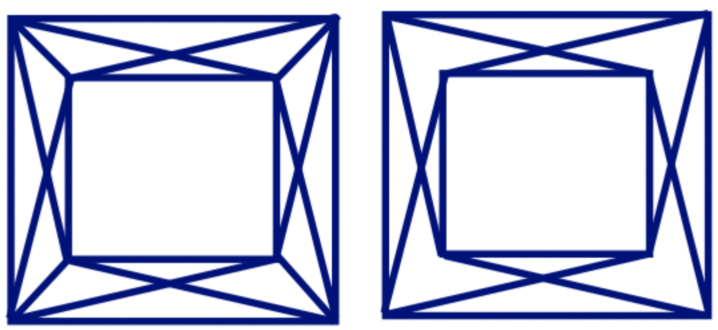
- Stable model topologies with respect to deformation

not stable





stable but not general

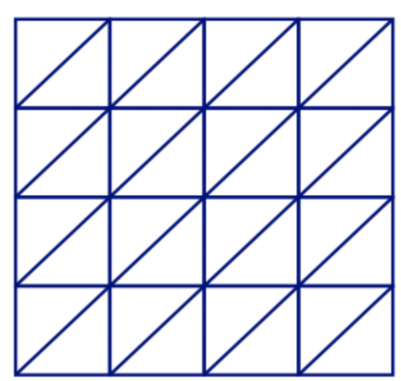


stable

can be generated automatically by copying the surface to an inner layer and connecting both – **layered model**

in the extreme case, consider the inner layer to be just a point

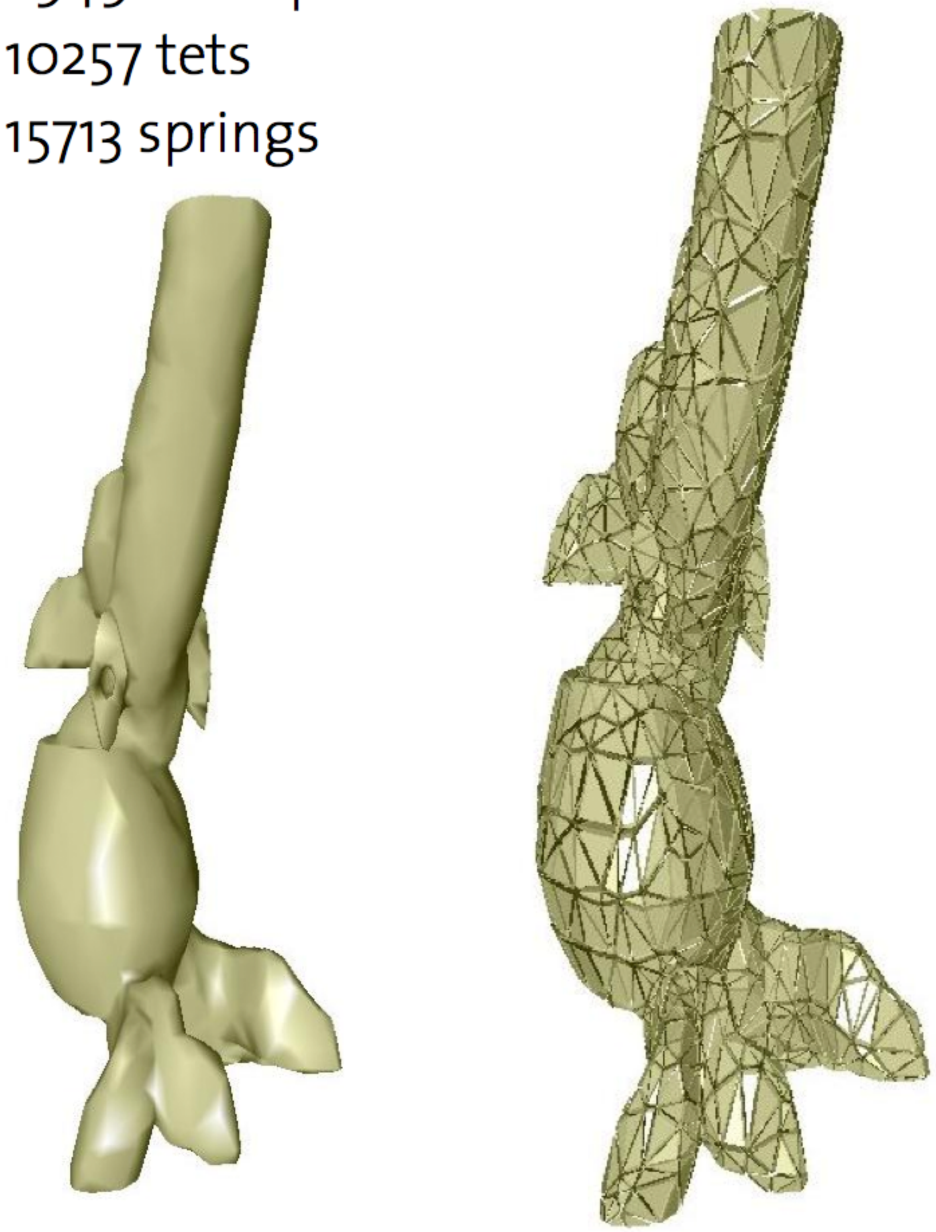
- Design problem

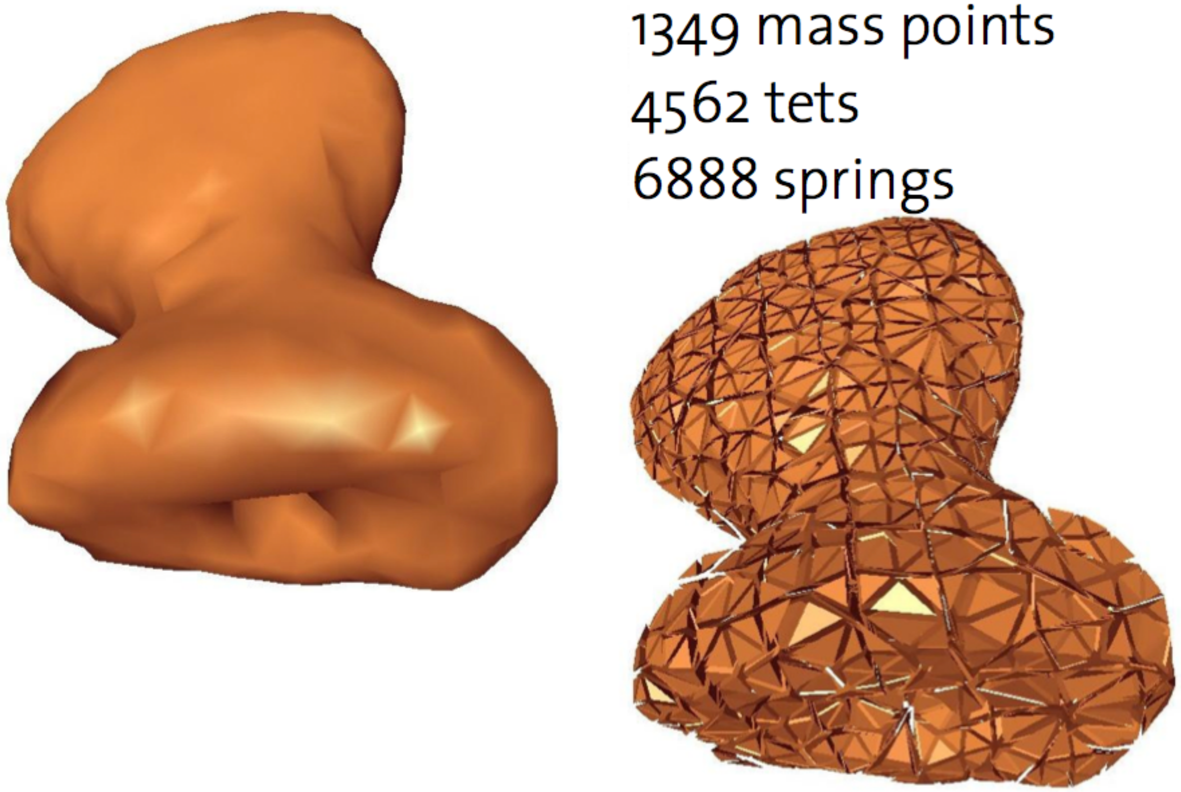


much more resistant in ↗ direction than in ↘ direction.

Volumetric Models - Tet Meshes

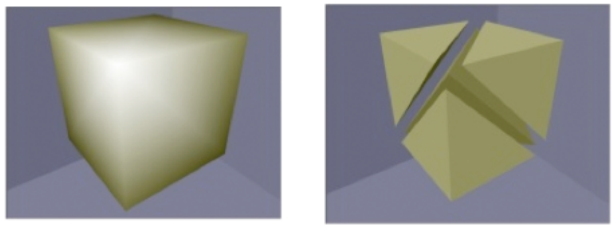
2949 mass points
10257 tets
15713 springs





1349 mass points
4562 tets
6888 springs

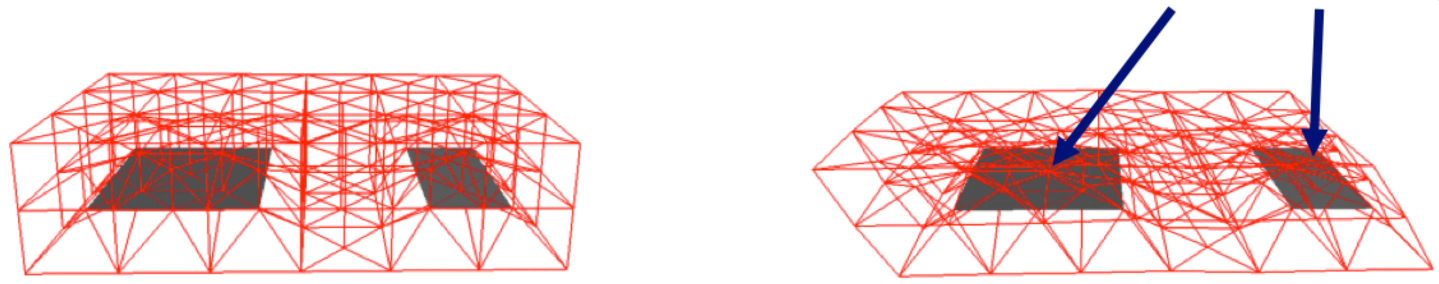
8 mass points
5 tets



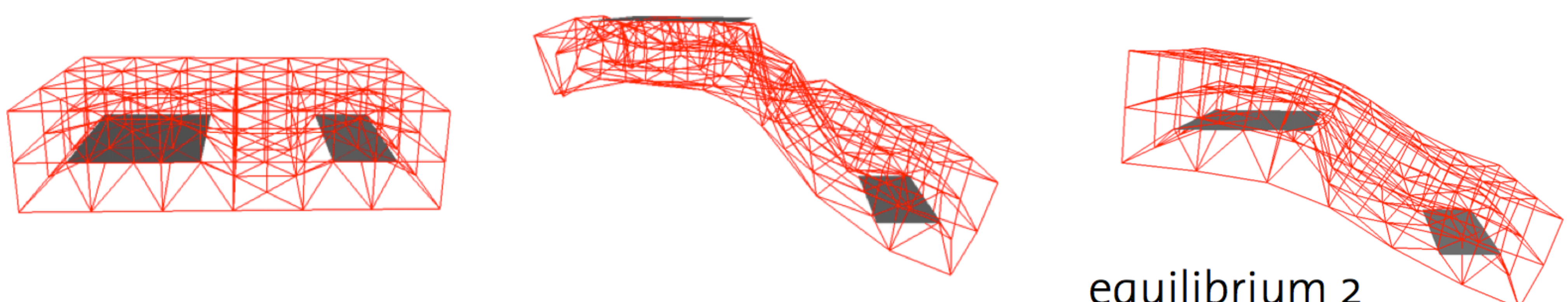
Topology Ambiguity Problem

- Unappropriate topology without diagonal springs
- No force penalty for shearing

model is attached
to these two planes



- Appropriate topology with diagonal springs
- However, self-collision problem, springs have no notion of volume



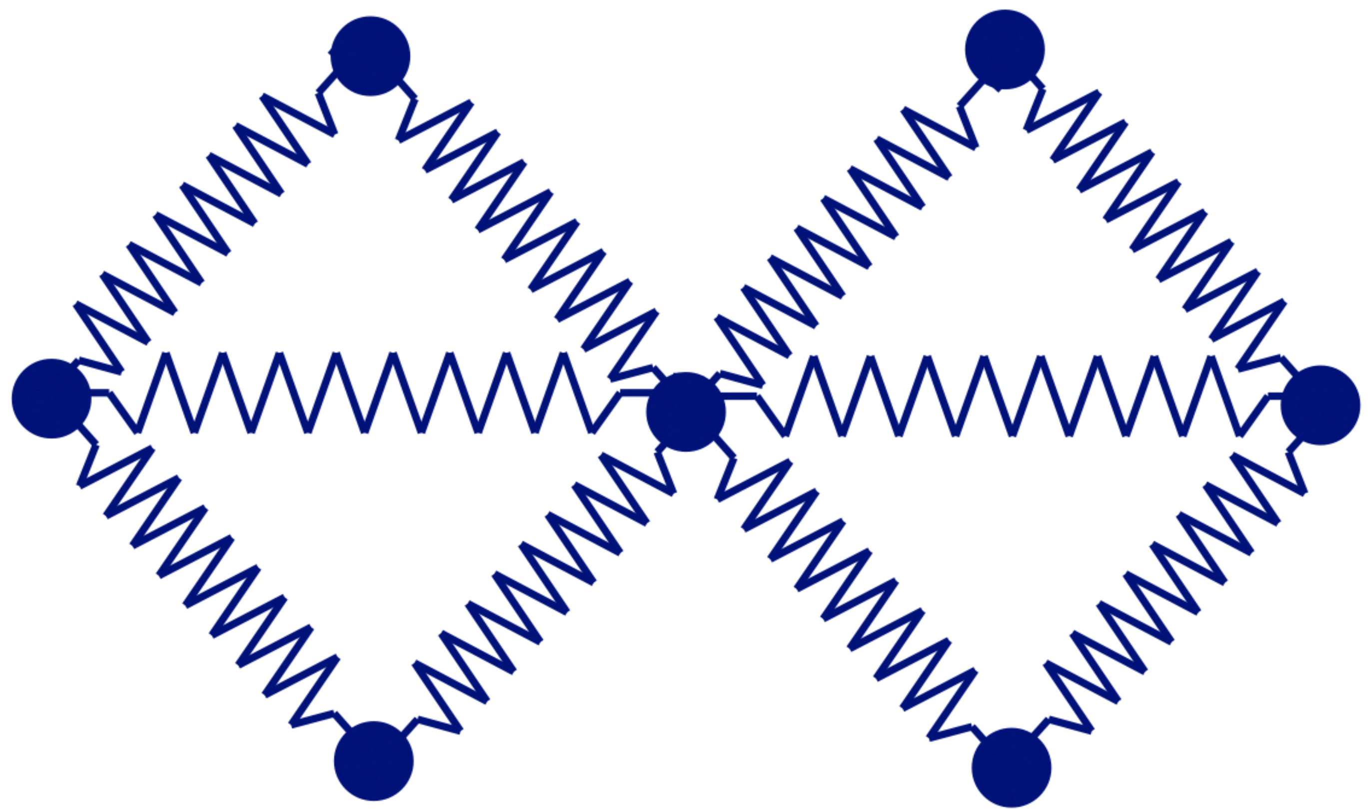
original

equilibrium 1

equilibrium 2

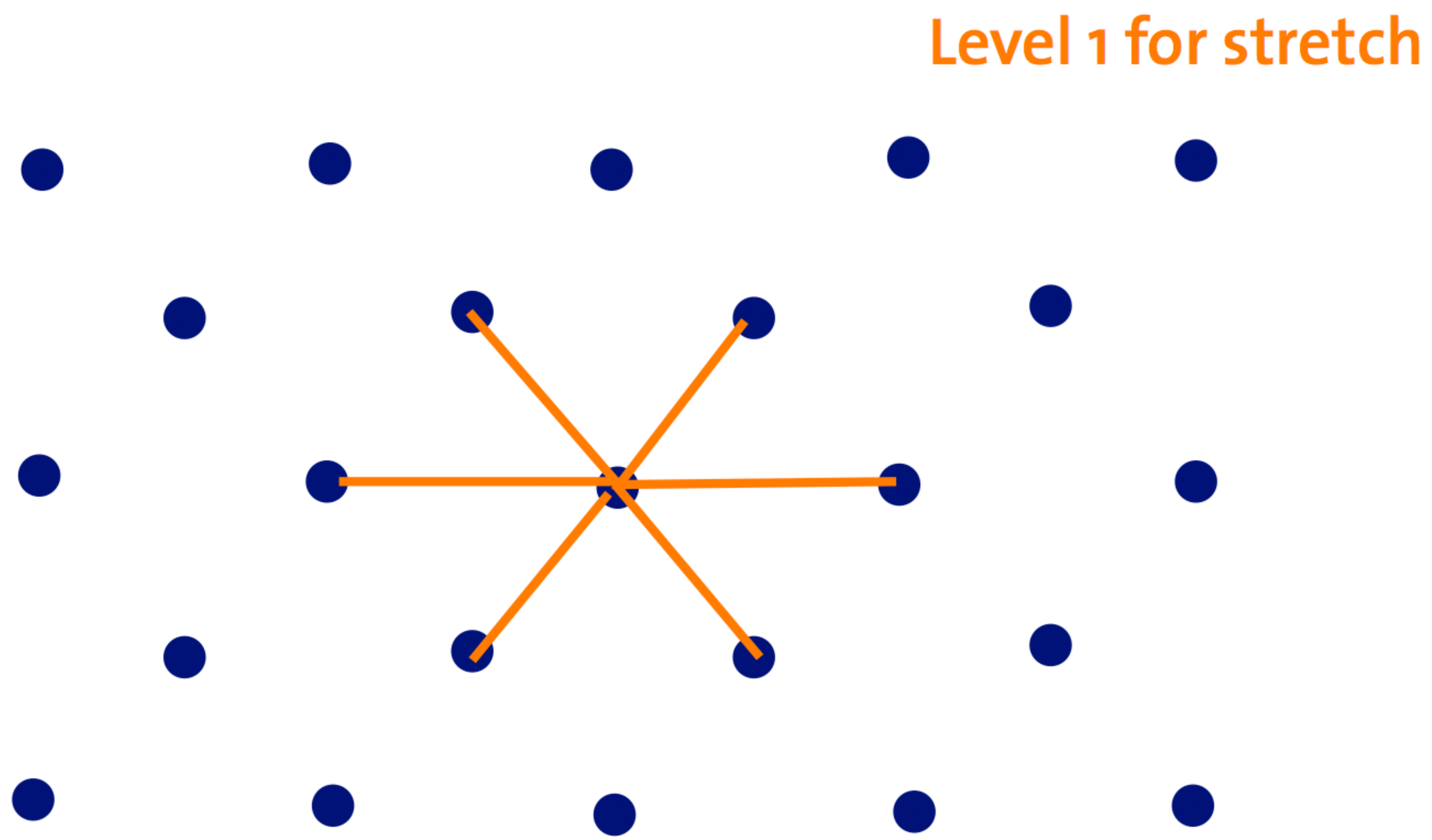
Cloth Forces

- Types of forces in cloth: **stretch**, **bending**, **shear**
- Bending cannot be modeled with a simple network of springs



Cloth Springs

- Combine level-1 and level-2 springs



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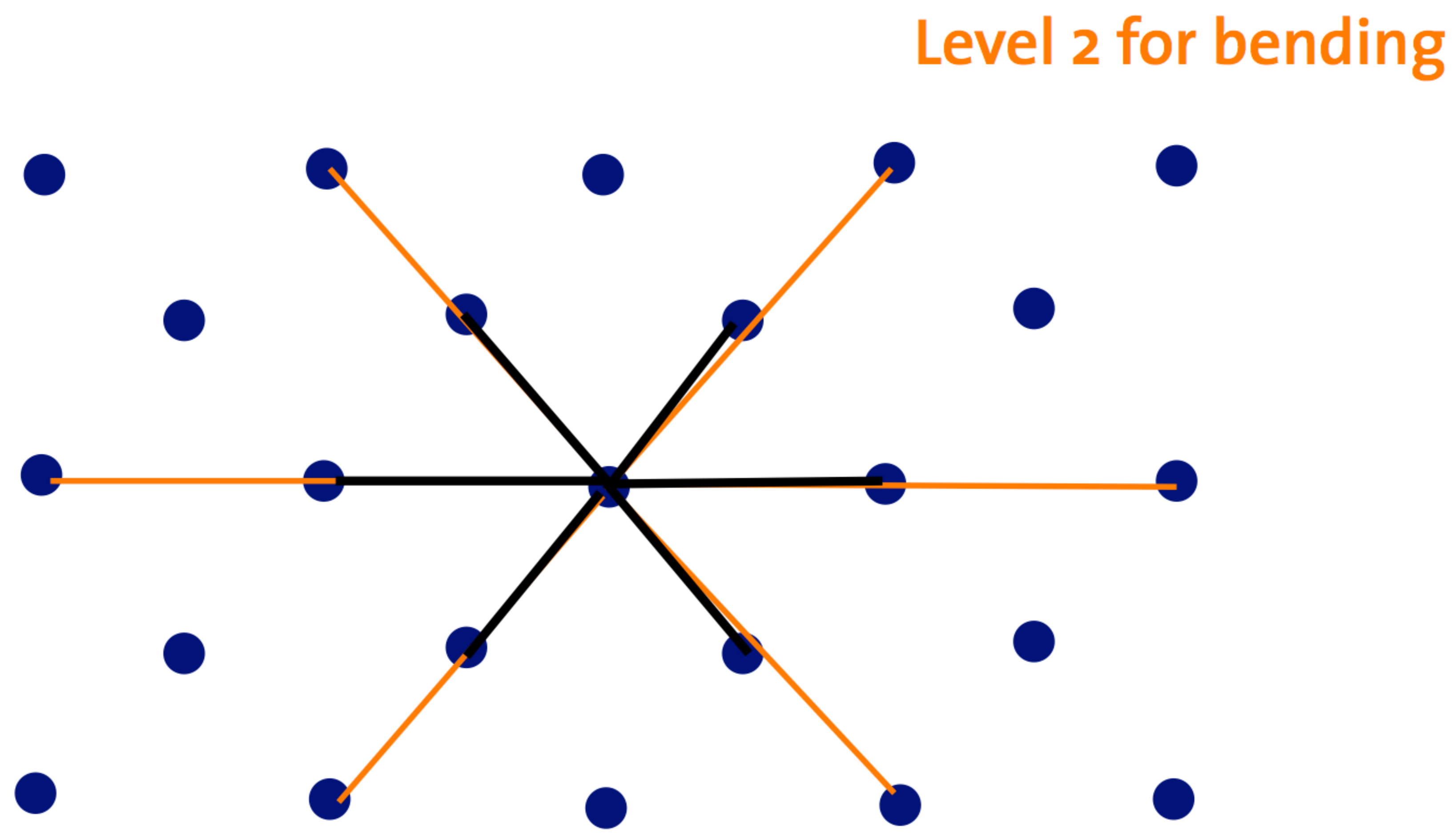
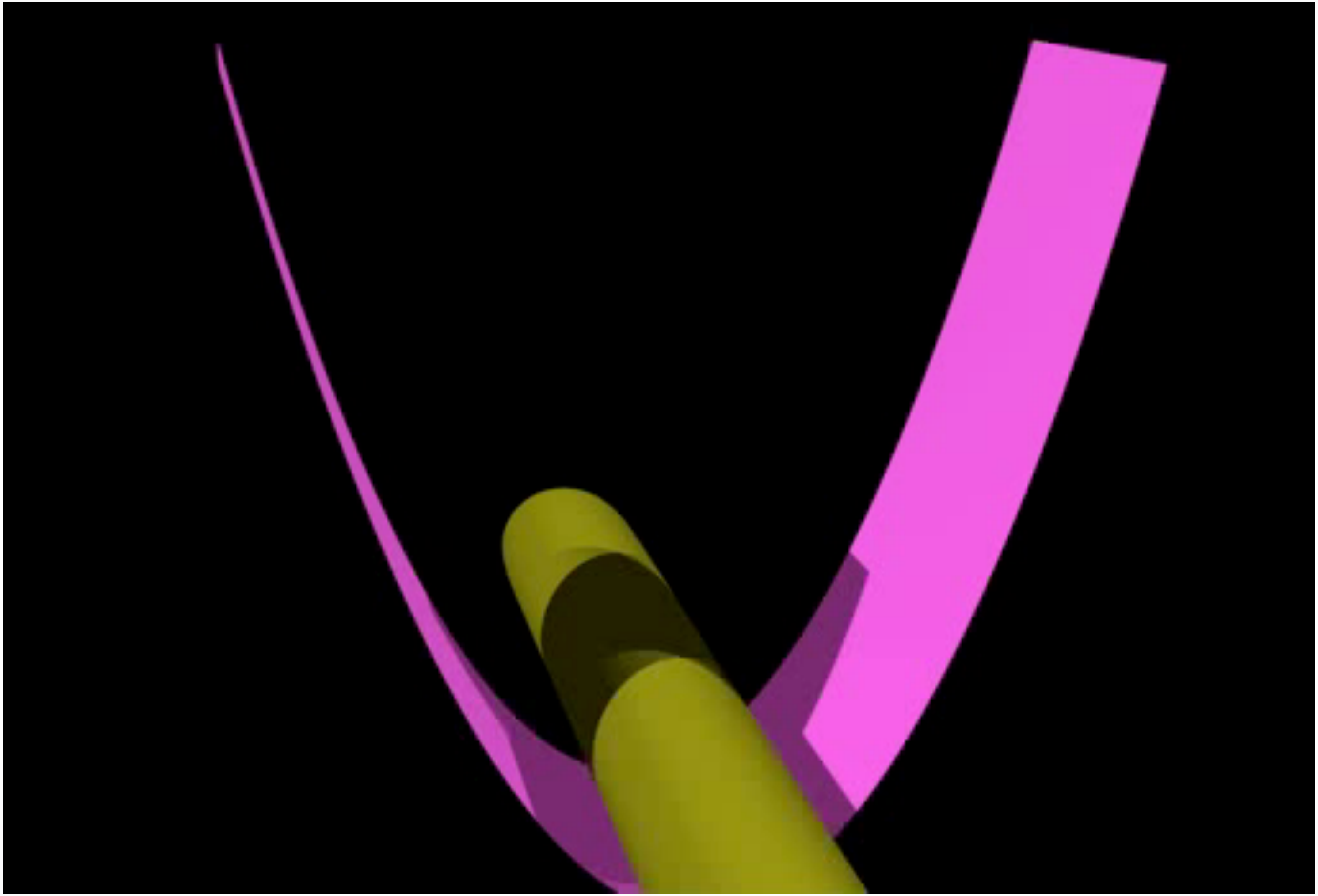


Cloth Springs



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Thanks!